Classical W Tensors

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ABSTRACT

It has been recently shown that classical W transformations (w-morphisms) have a simple geometric interpretation as deformations of constant energy surfaces in a two dimensional phase space. We pursue this approach to construct geometric representations of the w-morphism algebra. These representations are obtained through the tensor product of some basic representations which can be understood as the W analogs of vectors and their duals. We also give convincing evidence that this classical W-tensors have their natural counterpart in conformal field theories enjoying W-symmetry.
This is a rather old unfinished preprint. It was written originally in 1993. It has not been touched since.

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§1 Introduction and Motivation

What is a W-primary field? One of the purposes of this introduction is to try and impress upon the reader the fact that this question is important—arguably the single most important unsolved question in W-algebras—both for practical and formal reasons, and also that its answer is intimately linked with the notion of W-geometry. We also hope to impress upon the reader the fact that there are immediate and fruitful applications for the answer to this question: W-covariant CFT, W-bootstrap,... We don't pretend that direct construction (at least via the bootstrap) is going to solve the classification problem for W-algebras, but it is unquestionable that out of the relatively few cases that have been so constructed, there has come out valuable insight into the problem and considerable help in forming conjectures.

Here something about other approaches...

To start, let us review the notion of a Virasoro primary field in CFT. According to the BPZ axioms [**1**], for every Vir representation $\mathcal{H}_h$—necessarily of highest weight h—there exists a field $\phi_h(z)$ such that

$$\lim_{z\to 0} \phi_h(z)|0\rangle = |h\rangle \ , \tag{1.1}$$

where $|0\rangle$ is the (unique) $SL(2,\mathbb{C})$-invariant vacuum and $|h\rangle$ is the highest-weight vector of $\mathcal{H}_h$, obeying

$$L_0|h\rangle = h|h\rangle \qquad \text{and} \qquad L_{n>0}|h\rangle = 0 \ . \tag{1.2}$$

In OPE language these conditions translate into[1]

$$T(z)\phi_h(w) = \frac{h\phi_h(w)}{(z-w)^2} + \text{lower order terms} \ . \tag{1.3}$$

In particular, the simple pole is not specified by (1.2). Hence the singular part of the OPE—which contains all the information needed to compute the correlation functions—is not uniquely determined from representation theory alone. Let us call any field satisfying (1.3) a *preprimary* field of weight h. In other words, representation theory gives us an equivalence class of preprimary fields out of which we must choose *by other means* an essentially unique

[1] Strictly speaking, this assumes that L_0 is diagonalizable in the space of fields. As has been evidenced recently in [**2**], this is not always possible. But for the present illustrative purposes this assumption will do.

representative: the primary field, obeying

$$T(z)\phi_h(w) = \frac{h\phi_h(w)}{(z-w)^2} + \frac{\partial\phi_h(w)}{z-w} + \text{reg.}\ . \tag{1.4}$$

This choice can be rationalized by analogy with the transformation properties of a classical $\mathsf{diff}(S^1)$ tensor

$$\delta_\varepsilon \phi_h = -\varepsilon\partial\phi_h - h\partial\varepsilon\phi_h\ , \tag{1.5}$$

if we require that the energy-momentum tensor be the generator of these transformations; that is, if we require that

$$\delta_\varepsilon\phi_h(w) = \oint_{C_w} \frac{dz}{2\pi i}\,\varepsilon(z)T(z)\phi_h(w)\ . \tag{1.6}$$

In other words, the choice (1.4) comes dictated by the geometry of the $\mathsf{diff}(S^1)$ transformations.

Given the notion of a primary field, we can then set up the conformal bootstrap. Let us recall the main points. Given the following initial data: a family of highest-weight representations $\{\mathcal{H}_h\}$ of the Virasoro algebra and a corresponding set of primary fields $\{\phi_h\}$, the conformal bootstrap sets up to determine the most general associative algebra obeyed by the local fields. The local fields are generated by the primary fields $\{\phi_h\}$ and the energy-momentum tensor T under the operations of normal ordered products and taking derivatives. In particular, since T is a local field, associativity of the operator product algebra implies that the operator product is covariant under the Virasoro algebra. This covariance is instrumental in setting up the bootstrap because it allows us to determine the OPE of any two local fields, from the one of the primary fields in their respective conformal families. Indeed, let $\{\phi_i\}$ denote some indexing of the primary fields where ϕ_i has conformal weight h_i. The OPE of two primary fields can then be written as

$$\phi_i(z)\phi_j(0) = \sum_{\text{primaries } k} C_{ij}^k z^{h_k-h_i-h_j}[\phi_k](z)\ , \tag{1.7}$$

where $[\phi_k](z)$ stands for the contribution coming from the conformal family of ϕ_k. We can write this down more explicitly as follows. The correspondence $|h\rangle \leftrightarrow \phi_h$ extends to a correspondence between the the Verma module $V(h,c)$ and the descendent fields of ϕ_h. To every vector $L_{-n_1}\cdots L_{-n_k}|h\rangle$ we associate the field $\phi_h^{\{n_1,\ldots,n_k\}}$ defined by:

$$\phi_h^{\{n_1,\ldots,n_k\}}(w) = \widehat{L}_{-n_1}\cdots\widehat{L}_{-n_k}\phi_h(w)\ , \tag{1.8}$$

where, if ϕ is any local field, we define $\widehat{L}_{-n}\phi(w)$ as the coefficient of $(z-w)^{n-2}$ in the OPE of $T(z)$ with $\phi(w)$. We shall find it convenient to introduce the following abbreviation for the basis of the Verma module: if $I = \{i_1, \ldots, i_k\}$ is a multi-index of length $|I| = k$ we define $\mathcal{L}_{-I}|h\rangle \equiv L_{-i_1} \cdots L_{-i_k}|h\rangle$ and we will let ϕ_h^I denote the corresponding descendent field. In this notation, we can write

$$[\phi_k](z) = \sum_{N\geq 0} z^N \sum_{\substack{I \\ |I|=N}} \beta_{ijk}^I \phi_k^I(0) \ , \tag{1.9}$$

where β_{ijk}^I are coefficients that can be determined from conformal covariance by a well-known procedure described in [**1**] (Appendix B). As described for example in [**3**], the general form of the β's is obtained as follows:

$$\beta_{ijk}^I = \sum_{\substack{J \\ |J|=|I|}} \mathcal{M}^{IJ} f_{ijk}^J \ , \tag{1.10}$$

where $\mathcal{M}^{IJ}$ are the matrix elements of the inverse of the Shapovalov form of the Virasoro Verma module $V(h_k, c)$ and the f's—polynomials in the conformal weights h_i, h_j, h_k with integer coefficients—can be obtained from the three-point function $\langle\phi_k \phi_i^J \phi_j\rangle$ (*cf.* [**3**]). This three-point function is calculated from the primary three-point function $\langle\phi_k \phi_i \phi_j\rangle$ via the conformal Ward identities for which we only need the singular part of the OPE between T and the relevant fields. If the $\{\phi_i\}$ were preprimary fields, not necessarily primary, then this procedure would be unworkable since we would have to know what the effect of the insertion of the operator $\widehat{\mathcal{L}}_{-I}$ is in a correlator. Now all these operators can be written as monomials involving only $\widehat{L}_{-1}$ and $\widehat{L}_{-2}$. By definition, this latter operator corresponds to the operation of taking normal-ordered product with T. But we still need to know what $\widehat{L}_{-1}$ corresponds to. Without the geometric input that says that this is simply the derivative, the conformal Ward identity would generate more and more unknown correlators which would render a finite recursive calculation impossible.

This is exactly what happens in the case of W_3. In this case, we have a family of highest-weight representations $\{\mathcal{H}_{h,\omega}\}$ of the W_3 algebra. By the BPZ axioms, there is a local field $\phi_{h,\omega}$ obeying

$$\lim_{z\to 0} \phi_{h,\omega}(z)|0\rangle = |h,\omega\rangle \ , \tag{1.11}$$

where $|h,\omega\rangle$ is a highest-weight vector—that is,

$$L_0|h,\omega\rangle = h|h,\omega\rangle \qquad \text{and} \qquad L_{n>0}|h,\omega\rangle = 0 \ ,$$
$$W_0|h,\omega\rangle = \omega|h,\omega\rangle \qquad \text{and} \qquad W_{n>0}|h,\omega\rangle = 0 \ .$$

These conditions imply that $\phi_{h,\omega}$ is a Virasoro primary field of weight h and, in addition, that it satisfies the following OPE

$$W(z)\phi_{h,\omega}(w) = \frac{\omega\phi_{h,\omega}(w)}{(z-w)^3} + \text{lower order terms} \,, \tag{1.12}$$

where again we assume that W_0 is diagonal. In particular, notice that neither the second order nor the first order poles are specified. The coefficients of these poles are, by definition, $\widehat{W}_{-1}\phi_{h,\omega}$ and $\widehat{W}_{-2}\phi_{h,\omega}$. In trying to set up the bootstrap one would hope that the operator product of any two fields in the local algebra would be determined in terms of the operator product of the W_3-primary fields. But this requires a knowledge of the analogous coefficients β or, equivalently, a knowledge of the f's. Without the geometric input that would allow us to derive useful Ward identities for the insertion of $\widehat{W}_{-1}$ and $\widehat{W}_{-2}$ in a correlator[2], we are unable to compute the f's and we are forced to work in a formalism where W_3-covariance is not manifest.

This state of affairs is clearly unsatisfactory. For suppose that we would be interested in studying extensions of, say, W_3. In a W_3-covariant formalism we would simply have to consider four-point functions involving only W_3-primary fields when solving the associativity constraints. On the contrary, were W_3-covariance not manifest, we would have to check all correlators involving Vir-primary fields—an impractical task, since a generic representation of W_3 contains an infinite number of Vir primaries. In other words, a manifestly W-covariant formalism would allow us to treat infinitely-generated extensions of Vir simply as finite extensions of a W-algebra in much the same way that a CFT with an infinite number of primaries can be rendered rational by extending the chiral algebra.

The need to derive Ward identities for $\widehat{W}_{-1}$ and $\widehat{W}_{-2}$ insertions in correlators, is directly related to the existence of an infinite number of Virasoro primaries in a generic W_3 representation. This can be seen already from (1.12) above, where a simple counting of the degrees of freedom which can be accounted for by the conformal family of ϕ_h reveals that we have new primary fields appearing at the first and second order poles—the former one being a linear combination of the two possible Virasoro primaries at that level in the W_3-family of ϕ_h. Demanding that these primaries do not appear, imposes constraints in the conformal and W-dimensions of the primary as well as, possibly, on the central charge of the theory. This can also be seen abstractly, but it will prove convenient to pay some attention to perhaps the simplest CFT with W_3-symmetry: the 3-state Potts model (see, for instance, [**4**]).

[2] Strictly speaking we only need to know what the effect of a $\widehat{W}_{-1}$ insertion is, since $\widehat{W}_{-2} = \left[\widehat{W}_{-1}\,,\,\widehat{L}_{-1}\right]$.

For the present purposes, the interesting subalgebra of the 3-state Potts model (at $c = \frac{4}{5}$) is the one generated by the identity I together with the following Vir-primary fields: W, $\phi_{\frac{7}{5}}$ and $\phi_{\frac{2}{5}}$, where the subscripts denote the conformal weights. Apart from the obvious fusion rules, the algebra obeys

$$\begin{aligned}
W \times \phi_{\frac{7}{5}} &= \phi_{\frac{2}{5}} \\
W \times \phi_{\frac{2}{5}} &= \phi_{\frac{7}{5}} \\
\phi_{\frac{7}{5}} \times \phi_{\frac{2}{5}} &= \phi_{\frac{2}{5}} + W \\
\phi_{\frac{2}{5}} \times \phi_{\frac{2}{5}} &= I + \phi_{\frac{7}{5}} \\
\phi_{\frac{7}{5}} \times \phi_{\frac{7}{5}} &= I + \phi_{\frac{7}{5}}
\end{aligned} \tag{1.13}$$

In particular, notice that $\phi_{\frac{7}{5}}$ and $\phi_{\frac{2}{5}}$ transform as a "doublet". Later on in this paper, we will exhibit a natural geometric representation of the classical algebra w_3 which has a structure very similar to this one. Moreover for the classical representations that we will consider we will be able to take the tensor product of these representations together. However, tensoring takes us away from the special values of the dimensions where the infinite tower of Vir-primaries are absent, and we will see explicitly how these new primaries appear.

The main motivation for looking at the classical limit of W-algebras is that in the past they have proven to contain all the essential features that typify their quantum analogues—chief among them the nonlinearity—while simplifying many of the calculations. One may be even tempted to think that, if anything, the quantum algebras obscure the geometry, to be found already in the very definition of the classical algebras as symmetries on the space of symbols of differential operators. More recently, in joint work with S. Stanciu [**5**], we interpreted classical W-transformations as a particular kind of canonical transformations in a two-dimensional phase space. It is precisely this simple set-up that we will use to address the problem of constructing geometrical representations of the classical W-algebras; that is, to construct classical W-tensors.

Classical W-algebras can be obtained as a $c \to \infty$ contraction of the operator product algebras appearing in conformal field theories which enjoy W symmetry. But they are also fascinating objects by themselves because of their connection to integrable hierarchies [**6**][**7**], topological field theories [**8**][**9**], and noncritical string theory with $c \leq 1$ [**10**], to name but a few of their most interesting applications.

This paper is organized as follows. In Section 2 we describe the algebraic approach to classical W-algebras in terms of an algebraization of the geometry on the space of symbols of differential operators in one dimension. In Section 3 we discuss the geometric interpretation of classical W-transformations in terms of deformations of constant "energy" surfaces in a two-dimensional phase space. We will see that unlike the usual geometric transformations which are parametrized by a "group", classical W-transformations seem to be parametrized by a homogeneous space. Section 4 contains the main results of the paper. There we look at the simplest classical W_3-tensors and we will see that the structure of the representation is very similar to that of the representations to be found in the critical 3-state Potts model. We will moreover be able to construct tensors out of this representation. Finally in Section 5 we end the paper with some further comments of a more speculative nature about the picture of W-geometry that seems to emerge from our results.

§2 Algebraic Approach to Classical W-Algebras

Classical W-algebras naturally appear as Poisson brackets structures in the space of symbols of pseudodifferential operators of the classical type [**11**]. Here we will circumvent all that machinery by defining our phase space M^n to be the space of polynomials in the abstract symbol p and taking the form

$$L = p^n + \sum_{j=1}^{n} u_j(q) p^{n-j}, \tag{2.1}$$

with the u's smooth functions on the circle, and n a positive integer.

From the definition of M^n it is clear that a complete analysis of these Poisson structures would require to do analysis in infinite dimensional manifolds, a difficult and tricky business. Rather than doing so we will try to extract the main algebraic features of Poisson structures in finite dimensional manifolds and try to generalize them directly to the infinite dimensional case, shortcutting through delicate analytical issues. We will thus follow Dickey's approach [**12**] both in spirit and in form.

Let us begin by considering a two-dimensional manifold Y, and let us denote by $\mathcal{F}$ the ring of smooth functions on Y. A Poisson bracket $\{\cdot\,,\,\cdot\}$ on Y is an antisymmetric map from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$, turning $\mathcal{F}$ into a Lie algebra, and enjoying the derivation property

$$\{fg\,,\, h\} = f\,\{g\,,\, h\} + \{f\,,\, h\}\, g, \tag{2.2}$$

for any $f, g, h \in \mathcal{F}$. Because of (2.2) the Poisson bracket is bound to be of the form

$$\{f\,,\, g\} = \Omega^{ij} \partial_i f \partial_j g, \tag{2.3}$$

where Ω is an antisymmetric 2×2 matrix such that its Schouten bracket with itself is zero, which is tantamount to saying that the bracket defined by Ω fulfills the Jacobi identity. From (2.3) we can extract the geometrical objects required for the definition of the Poisson brackets. Besides the functions on Y, we have to be able to take gradients and, moreover, we also need a map Ω mapping gradients into vectors and such that Jacobi identities are obeyed. From now on we will say that a map is hamiltonian if enjoys these two properties. A convenient way to encode all this properties is in the framework of symplectic geometry.

A symplectic manifold is an even dimensional manifold endowed with a closed nondegenerate two form ω. We can use ω to give a map from gradients to vector fields as follows

$$\omega(H_f, \cdot) = -df, \tag{2.4}$$

or explicitly introducing local coordinates such that $\omega = \frac{1}{2}\Omega_{ij} dx^i \wedge dx^j$, we have

$$H_f^i = \Omega^{ij}\partial_j f, \tag{2.5}$$

with $\Omega^{ij}\Omega_{jk} = \delta^i_k$. It is now simple to show that $\{f\, ,\, g\} = \omega(H_f, H_g)$ defines consistent Poisson brackets. First of all,

$$\omega(H_f, H_g) = -df(H_g) = H_g \cdot f = \Omega^{ij}\partial_i f \partial_j g\ , \tag{2.6}$$

which is clearly antisymmetric. Moreover the Jacobi property is tantamount to the closedness of ω ($d\omega = 0$), as can be easily checked by a standard computation in local coordinates. This property can also be stated, in way that will be useful for our purposes, by using that the exterior derivative of a two form can be expressed in a coordinate independent manner as follows

$$d\omega(\xi_1, \xi_2, \xi_3) = \xi_1 \cdot \omega(\xi_2, \xi_3) - \omega([\xi_1\, ,\, \xi_2]\, , \xi_3) + \text{cyclic permutations.} \tag{2.7}$$

Therefore imposing that the RHS of (2.7) is zero for any three vectors becomes equivalent to the statement that the Jacobi identity is fulfilled.

Now we should come back to our infinite dimensional case and try to apply a similar algebraic approach to the one described above. To define functions and vector fields in M^n is a simple task. We are going to define Poisson brackets on function(al)s of the form

$$F[L] = \int f(u_i), \tag{2.8}$$

where $f(u_i)$ are differential polynomials in the u's, *i.e.* polynomial in the u's and their derivatives. Vector fields are parametrized by infinitesimal deformations $L \mapsto L + \epsilon A$ where $A = \sum_{j=1}^{n} a_j p^{n-j}$. We denote the space of such operators by $\mathcal{R}_n$. To such operator $A \in \mathcal{R}_n$ we associate a vector field as follows. If $F = \int f$ is a function(al) then

$$\begin{aligned} \partial_A F =& \frac{d}{d\epsilon} F(L + \epsilon A)|_{\epsilon=0} \\ =& \int \sum_{j=1}^{n} \sum_{i=0}^{\infty} a_j^{(i)} \frac{\partial f}{\partial u_j^{(i)}}. \end{aligned} \tag{2.9}$$

Integrating by parts we can write this as

$$\partial_A F = \int \sum_{j=1}^{n} a_j \frac{\delta f}{\delta u_j}, \tag{2.10}$$

where the Euler variational derivative is given by

$$\frac{\delta}{\delta u_j} = \sum_{i=0}^{\infty} (-\partial)^i \frac{\partial}{\partial u_j^{(i)}}. \tag{2.11}$$

Since vector fields are parametrized by $\mathcal{R}_n$, it is natural to think of 1-forms as parametrized by its dual space $\mathcal{R}_n^*$. This turns out to be given by Laurent polynomials in p with the dual pairing provided by the Guillemin symplectic trace [**13**]. Let us consider the space $\mathcal{S}$ of formal Laurent polynomial in p^{-1}, *i.e.* $S \in \mathcal{S}$ if $S = \sum_{j=-k}^{\infty} s_j(q) p^{-j}$ where k is an arbitrary integer. The (symplectic) trace is given by

$$\operatorname{Tr} S = \int \operatorname{res} S = \int s_1. \tag{2.12}$$

Notice that we can turn $\mathcal{S}$ into a Lie algebra by declaring its Lie bracket to be the Poisson bracket with respect the fundamental Poisson bracket[3] $\{p\,,\, q\} = 1$. With this in mind, the trace "appellation controle" is justified by the crucial property

$$\operatorname{Tr} \{T\,,\, S\} = 0 \tag{2.13}$$

for T and S arbitrary Laurent polynomials in p^{-1}. This property can be easily proved as follows. Because of the linearity of the trace we can restrict ourselves,

[3] The reader should not confuse this Poisson brackets in the two-dimensional phase space with coordinates p and q with the ones in M^n which will provide the definition of the classical W-algebras.

without loss of generality, to the case in which T and S are monomials of the form $T = ap^k$ and $S = bp^j$ for k and j arbitrary integers. Then (2.13) reads

$$\begin{aligned} \operatorname{Tr}\left\{ap^k\,,\,bp^j\right\} &= \operatorname{Tr}(kab' - ja'b)p^{k+j-1} \\ &= k\delta_{k+j,0}\int (ab)' = 0. \end{aligned} \tag{2.14}$$

We can now define the pairing between a vector ∂_A and a one-form X by

$$X(\partial_A) = \operatorname{Tr} XA. \tag{2.15}$$

It is clear from this definition that the dual space of $\mathcal{R}_n$ is given by polynomials in p^{-1} of the form

$$X = \sum_{j=1}^{n} x_j p^{j-n-1}. \tag{2.16}$$

This lets us define the gradient of a function by

$$dF(\partial_A) = \partial_A F, \tag{2.17}$$

which implies that

$$dF = \sum_{j=1}^{n} \frac{\delta F}{\delta u_j} p^{j-n-1}. \tag{2.18}$$

In analogy with the finite dimensional case, we should provide a map from one forms to vector fields in order to define the Poisson brackets. In order to do so we still require a little more of machinery. The required map is given by a suitable modification (in fact, contraction) of the standard Adler map [**14**], which reads

$$J(X) = \{L, X\}_+ L - \{L, (LX)_+\}, \tag{2.19}$$

where the + subindex stands for the projection on the polynomial part, *i.e.* if $S \in \mathcal{S}$, and is defined as before, then $S_+ = \sum_{j=-k}^{0} s_j(q)p^{-j}$ if k is a positive integer and zero otherwise. We also define $S_- = S - S_+$.

First notice that because of its definition $J(X)$ is a polynomial in p. Moreover if we write (2.19) as

$$J(X) = -\{L, X\}_- L + \{L, (LX)_-\} \tag{2.20}$$

it is clear that $J(X)$ is a polynomial of at most order $n-1$; whence $J(X) \in \mathcal{R}_n$ parametrizes a vector field in M^n.

Let Ω denote the map $X \mapsto \partial_{J(X)}$ from 1-forms to vectors fields induced by (2.19). In analogy with the finite dimensional case, it is convenient to introduce the symplectic form ω defined, on the image of the map Ω, by

$$\omega(\Omega(X), \Omega(Y)) = \mathrm{Tr}\, J(X)Y. \tag{2.21}$$

Notice that, in contrast with the usual case in classical mechanics, this 2-form is not defined for all vector fields since, in general, the map J will not be an isomorphism. It follows from the definition of ω that the Poisson brackets will be given by

$$\{F\,,\, G\}_{\mathsf{GD}} = \omega(\Omega(dF), \Omega(dG)) = \mathrm{Tr}\, J(dF)dG, \tag{2.22}$$

where we have introduced the suffix GD (for Gel'fand and Dickey) in order to avoid confusion with the canonical Poisson brackets in a finite-dimensional phase space used for the definition of the generalized Adler map.

It is now simple to check that this bracket is indeed antisymmetric. Explicitly,

$$\begin{aligned}\{F\,,\, G\}_{\mathsf{GD}} &= \mathrm{Tr}\, J(dF)dG \\ &= \mathrm{Tr}\left(\{L\,,\, dF\}_+\, LdG - \{L\,,\, (LdF)_+\}\, dG\right) \\ &= \mathrm{Tr}\left(-\{L\,,\, (LdG)_-\}\, dF + \{L\,,\, dG\}_-\, dF\right) \\ &= -\,\mathrm{Tr}\, J(dG)dF = -\{G\,,\, F\}_{\mathsf{GD}}\ ,\end{aligned} \tag{2.23}$$

where we have used $\mathrm{Tr}\, A_+B_+ = \mathrm{Tr}\, A_-B_- = 0$ for all A and B.

By analogy with the finite-dimensional case, we define $d\omega$ by

$$\begin{aligned}d\omega(\partial_{J(X)}, \partial_{J(Y)}, \partial_{J(Z)}) =& \partial_{J(X)}\omega(\partial_{J(Y)}, \partial_{J(Z)}) \\ &- \omega(\left[\partial_{J(X)}\,,\, \partial_{J(Y)}\right], \partial_{J(Z)}) + \text{c.p.}\end{aligned} \tag{2.24}$$

where c.p. is shorthand for cyclic permutations. But notice that the last term in (2.24) is not well defined unless $\mathrm{Im}\Omega$ forms a subalgebra of the vector fields. In fact, a direct computation shows that for any X and $Y \in \mathcal{R}_n^*$

$$\left[\partial_{J(X)}\,,\, \partial_{J(Y)}\right] = \partial_{J([\![X,Y]\!])}, \tag{2.25}$$

where

$$\begin{aligned}[\![X, Y]\!] =& \partial_{J(X)}Y - \partial_{J(Y)}X + + \{X\,,\, L\}_-\, Y + \{(LX)_-\,,\, Y\} + \\ & \{Y\,,\, L\}_+\, X + \{(LY)_+\,,\, X\}\end{aligned} \tag{2.26}$$

modulo the kernel of J.

With all of this in mind we can now state the main result of this section, which, as in the finite dimensional case, is equivalent to the fact that the brackets defined by (2.22) obey Jacobi identities.

For any three vector fields $\partial_{J(X)}$, $\partial_{J(Y)}$, and $\partial_{J(Z)}$ in $\mathrm{Im}\Omega$

$$d\omega(\partial_{J(X)}, \partial_{J(Y)}, \partial_{J(Z)}) = 0, \tag{2.27}$$

i.e. ω is a closed 2-form.

This can be checked by a long, straightforward and explicit computation of the left hand side of (2.24) that we will omit it in here.

We will finish this section by giving a convenient prescription for computing the fundamental Poisson brackets among the u_j's in the case that L is of a polynomial in p of order n. Although the "coordinates" u_j are not functions according to the definition we are using, we can still make sense of their Poisson bracket.

First notice that the classical Adler map is linear in X. This implies that $J(X)$ is necessarily of the form

$$J(X) = \sum_{i,j=1}^{n} (J_{ij} \cdot X_j) p^{n-j}, \tag{2.28}$$

where the J_{ij}'s are certain differential operators acting on the X_j's. The antisymmetry of the Poisson brackets is equivalent to the condition

$$J_{ij}^{\dagger} = -J_{ji}, \tag{2.29}$$

where the dagger stands for the standard adjoint operation on differential operators.

We now chose two linear functionals of the form

$$l_A = \mathrm{Tr}\, AL \quad \text{and} \quad l_B = \mathrm{Tr}\, BL, \tag{2.30}$$

with $A = a(q)p^{i-n-1}$ and $B = b(q)p^{j-n-1}$. Their gradients are given by

$$dl_A = ap^{i-n-1} \quad \text{and} \quad dl_B = bp^{j-n-1}, \tag{2.31}$$

which implies

$$\{l_A\,,\, l_B\}_{\mathsf{GD}} = \int (J_{ij} \cdot a) b. \tag{2.32}$$

It is obvious that we would have obtained the same result if we had declared our fundamental Poisson brackets among the u's to be

$$\left\{u_i(q)\,,\, u_j(q')\right\}_{\mathsf{GD}} = -J_{ij} \cdot \delta(q - q'), \tag{2.33}$$

and because of the Poisson property the brackets induced by (2.33) on arbitrary functionals are identical to the ones obtained via (2.22).

§3 Classical W-Transformations

The natural arena for a geometrical description of w-morphisms, as could have been guessed from the algebraic results of the previous section, is supplied by a two dimensional phase space Y. In order to fix ideas we will again consider Y to be the phase space which configuration space is the circle. If we denote by q a local coordinate on the circle it is possible to equip Y with local (Darboux) coordinates (q, p) such that $\{p\,,\,q\} = 1$.

It is natural to consider in Y the subgroup of diffeomorphisms $\mathsf{SDiff}(\mathsf{Y})$ preserving its canonical structure. These particular diffeomorphisms are called symplectomorphisms, although in 2-dimensions they are also frequently referred to as area-preserving diffeomorphims because in two dimensions the canonical symplectic form coincides with the area form. (We remind the reader that for symplectomorphisms which are polynomial in momentum, we obtain an algebra isomorphic to w_∞ [**15**].)

In order to define w-morphisms we will need some extra structure on Y, namely a "Hamiltonian" $H(p, q)$. Let us consider constant the constant energy surface Z defined by $L \equiv H(p, q) - \lambda = 0$, and let us denote by $\mathsf{SDiff}_L(\mathsf{Y})$ the subgroup of symplectomorphisms leaving Z invariant. We now can define w-morphisms subordinated to L by the quotient

$$\frac{\mathsf{SDiff}(\mathsf{Y})}{\mathsf{SDiff}_L(\mathsf{Y})}. \tag{3.1}$$

In what follows we will show that for particular choices of L we recover for infinitesimal transformations the ones induced by w_n, wB_n, and wC_n, thus justifying its name. But before getting into more technical matters we would like to point out that $\mathsf{SDiff}_L(\mathsf{Y})$ is not a normal subgroup of $\mathsf{SDiff}(\mathsf{Y})$, therefore the quotient given by (3.1) does not define a group but rather an homogeneous space.

It is clear that any function of the form FL, where F is any smooth function on Y, vanishes on Z. For generic values of λ the converse is also true and L generates the ideal $\mathcal{I}_\mathsf{Z}$ of functions vanishing on Z, in the sense that any function F which vanishes on Z can be written as $F = LG$ for some function G. Moreover, any function on Z extends to a function on all of Y and the difference of any two such extensions is a function vanishing on Z. In other words, there is a one-to-one correspondence between the functions $\mathcal{F}(\mathsf{Z})$ on Z and the quotient $\mathcal{F}(\mathsf{Y})/\mathcal{I}_\mathsf{Z}$. We let $\pi : \mathcal{F}(\mathsf{Y}) \to \mathcal{F}(\mathsf{Y})/\mathcal{I}_\mathsf{Z}$ denote the map which sends a function on Y to its equivalence class modulo $\mathcal{I}_\mathsf{Z}$. In the next section, and for the class of functions we shall consider, we exhibit an explicit model for this quotient.

We now investigate the effect of symplectomorphisms on the constant energy surface Z. We can analyze deformations of Z by looking at how the function L behaves on Z under symplectomorphisms.

Infinitesimal symplectomorphisms are locally generated by functions on Y. In fact, given a function S on Y, it gives rise to a vector field δ_S defined such that acting on a function F,

$$\delta_S F = \{S\,,\,F\}\ . \tag{3.2}$$

If S vanishes on Z, then δ_S is tangent to Z. In fact, such an S can be written as GL and hence

$$\delta_S L = \{GL\,,\,L\} = \{G\,,\,L\}\,L\ , \tag{3.3}$$

which vanishes on Z. (Physically this is nothing but energy conservation.) Therefore infinitesimal symplectomorphisms generated by functions in $\mathcal{I}_{\mathsf{Z}}$ do not change Z. In other words, nontrivial deformations of Z induced from symplectomorphisms are locally generated by $\mathcal{F}(\mathsf{Y})/\mathcal{I}_{\mathsf{Z}}$. Therefore, on Z, the function L transforms as

$$\delta_S L \equiv \pi(\{\pi(S)\,,\,L\})\ . \tag{3.4}$$

For a specific choice of hamiltonian, we will now see that (3.4) defines infinitesimal w-morphisms associated to the classical W-algebras: gd_n and its reduction w_n.

As our function L we choose one of the form $L(q,p) = p^n + \sum_{i=1}^n u_i(q) p^{n-i}$, where u_i are arbitrary smooth functions on the circle. Under a change of coordinates $(q,p) \to (Q,P)$,

$$L(q,p) = (Q')^n \left(P^n + \sum_{i=1}^n U_i(Q) P^{n-i} \right)\ , \tag{3.5}$$

where U_i and u_i are related by

$$u_i(q) = (Q')^i U_i(Q)\ . \tag{3.6}$$

Since $q \mapsto Q(q)$ is a diffeomorphism, Q' is nowhere vanishing, hence the submanifold Z which is defined as the zero locus of L in the coordinates (q,p) is defined, in the coordinates (Q,P), as the zero locus of the function $P^n + \sum_{i=1}^n U_i(Q) P^{n-i}$, which has the same form. Thus these constant-energy surfaces have an invariant geometric meaning.

In order to have an algebraic handle on the situation, we will work with symplectomorphisms generated by functions whose dependence on p is polynomial. Under a change of coordinates $(q,p) \to (Q,P)$, polynomials in p go over to polynomials in P. Let $\mathcal{E}$ denote those functions. Notice that L belongs to $\mathcal{E}$. We let $\mathcal{J}_{\mathsf{Z}}$ denote the ideal of $\mathcal{E}$ generated by L. Since $p^n = L - \sum_{i=1}^{n} u_i(q) p^{n-i}$, we notice that modulo $\mathcal{J}_{\mathsf{Z}}$ we can always reduce any function in $\mathcal{E}$ to one with at most $n-1$ powers of p. In other words, $\mathcal{E}/\mathcal{J}_{\mathsf{Z}}$ is in one-to-one correspondence with the functions of the form $\sum_{i=0}^{n-1} f_i(q) p^i$. We now give an explicit expression for this representative. For this we will have to introduce a formal inverse of L. Explicitly,

$$L^{-1} = p^{-n} \sum_{k=0}^{\infty} (-1)^k \left(\sum_{j=0}^{n-1} u_j(q) p^{j-n} \right)^k \quad . \tag{3.7}$$

Any element R of $\mathcal{E}$ is equivalent modulo $\mathcal{J}_{\mathsf{Z}}$ to a unique polynomial of order at most $n-1$ given by

$$\pi_L(R) = R - (RL^{-1})_+ L = (RL^{-1})_- L \ . \tag{3.8}$$

Notice that whereas L^{-1} is not defined in Z, in the above formula we only use the formal inverse defined in (3.7) whose coefficients are well defined in Z. For a careful treatment of this point we refer the reader to [**5**].

It is obvious that $\pi_L(R)$ is polynomial in p of order smaller than n and, moreover, $\pi_L(R) - R \in \mathcal{J}_{\mathsf{Z}}$. Uniqueness follows because the order of any function in $\mathcal{J}_{\mathsf{Z}}$ is equal or bigger than n.

This provides us with a concrete model for the equivalence space $\mathcal{E}/\mathcal{J}_{\mathsf{Z}}$—namely the space $\mathcal{E}_{<n}$ of functions polynomial in p with order strictly less than n.

We now have at our disposal all the ingredients to establish the link between the algebraic w-morphisms alluded to in the introduction and the deformation of constant-energy surfaces. To this effect, we compute (3.4) in this concrete example, where we now make use of our explicit projector π_L instead of π. Since

$$\pi_L(S) = ((SL^{-1})_- L)_+ \ , \tag{3.9}$$

it is natural to reparametrize w-morphisms by

$$X = (SL^{-1})_- \bmod p^{-n-1} \ , \tag{3.10}$$

with $\pi_L(S) = (XL)_+$. We can then write (3.4) as follows

$$\begin{aligned}\delta_X L \equiv \delta_S L &= \{(XL)_+\,,\,L\} - (\{(XL)_+\,,\,L\}\,L^{-1})_+ L \\ &= \{(XL)_+\,,\,L\} - \left\{(XL)_+ L^{-1}\,,\,L\right\}_+ L \\ &= \{(XL)_+\,,\,L\} - \{X\,,\,L\}_+ L\;, \end{aligned} \tag{3.11}$$

which is the classical limit of the Adler map (2.19) or, equivalently, of the Gel'fand–Dickey brackets—namely gd_n. This establishes the equivalence between the algebraic and geometric approaches to w-morphisms.

In order to obtain now the classical limit w_n of the W_n algebras, we need to restrict ourselves to functions L of the form

$$L(q,p) = p^n + \sum_{i=2}^{n} u_i(q) p^{n-i}\;. \tag{3.12}$$

One can always achieve this by a symplectomorphism of the form

$$\begin{aligned} p &\mapsto p - \frac{1}{n} u_1(q) \\ q &\mapsto q\;, \end{aligned} \tag{3.13}$$

which puts the coefficient of p^{n-1} to zero. Notice moreover that, under coordinate changes induced from diffeomorphisms of the circle, this form of L is preserved. It then follows that if we restrict ourselves to infinitesimal symplectomorphisms which preserve the constraint, (3.11) define w-morphisms associated with w_n [**7**].

Finally, if we restrict to functions L which are odd or even under the transformation $p \mapsto -p$, and we again only consider symplectomorphism preserving such property, (3.11) will induce w-morphisms associated with the wB or wC series, respectively.

<u>A Simple Example: w_3</u>

Consider now, as an example, the function

$$L(q,p) = p^3 + T(q)p + W(q)\;. \tag{3.14}$$

The associated classical W-algebra is the w_3-algebra:

$$\begin{aligned} \{T(x)\,,\,T(y)\}_{\mathsf{GD}} &= -\left[2T(x)\partial + T'(x)\right]\cdot\delta(x-y)\;, \\ \{W(x)\,,\,T(y)\}_{\mathsf{GD}} &= -\left[3W(x)\partial + W'(x)\right]\cdot\delta(x-y)\;, \end{aligned} \tag{3.15}$$

and

$$\{W(x)\,,\,W(y)\}_{\mathsf{GD}} = \left[\frac{2}{3}T(x)\partial T(x)\right]\cdot\delta(x-y)\;.$$

The algebraic w-morphisms generated by T and W under the above algebra are given by the usual formulas

$$\delta_\epsilon^{(T)} F(y) = \int dx\, \epsilon(x)\, \{T(x)\,,\, F(y)\}_{\mathsf{GD}} \ , \tag{3.16}$$

and

$$\delta_\alpha^{(W)} F(y) = \int dx\, \alpha(x)\, \{W(x)\,,\, F(y)\}_{\mathsf{GD}} \ . \tag{3.17}$$

With them we can compute the effect of w-morphisms on the generators themselves. We obtain

$$\begin{aligned} \delta_\epsilon^{(T)} T &= 2T\epsilon' + T'\epsilon \\ \delta_\epsilon^{(T)} W &= 3W\epsilon' + W'\epsilon \\ \delta_\alpha^{(W)} T &= 2W'\alpha + 3W\alpha' \\ \delta_\alpha^{(W)} W &= -\tfrac{2}{3}(\alpha T)'T \ . \end{aligned} \tag{3.18}$$

We now compute the deformation of the constant-energy surface Z defined by L using the geometric procedure introduced earlier. The most general infinitesimal symplectomorphism which yields a nontrivial deformation of Z is generated by functions of the form

$$\pi_L(S) = \alpha p^2 + \epsilon p + \beta \ . \tag{3.19}$$

Demanding that the symplectomorphism preserve the form (3.14) of L requires that $\beta = \frac{2}{3}\alpha T$. We can now compute (3.4) yielding

$$\delta_S L = \left(\delta_\epsilon^{(T)} T + \delta_\alpha^{(W)} T\right) p + \delta_\epsilon^{(T)} W + \delta_\alpha^{(W)} W \ , \tag{3.20}$$

with the variations given by (3.18).

§4 Classical W-Tensors

In the last section we showed how the "adjoint" representation for classical W-algebras can be given a geometrical interpretation in terms of symplectomorphisms on a phase space Y. w-morphisms were interpreted as particular diffeomorphisms (the ones induced by the two dimensional symplectomorphisms) in the space of "constant energy" surfaces defined by hamiltonians of the form (2.1). It is clear that this infinite dimensional space is into a one to one correspondence with M^n. From all of this the action of symplectomorphisms on TM^n should provide us with the simplest example of geometrical objects carrying a representation of the algebra of w-morphisms. Let us recall that because of equation (3.11) the variation of L under a w-morphism generated by $X \in \mathcal{R}_n^*$ is given by

$$\delta_X L = J(X) = \partial_{J(X)} L,$$

and this implies that

$$\delta_X \delta_Y - \delta_Y \delta_X = \delta_{[\![X,Y]\!]}, \tag{4.1}$$

where $[\![X,Y]\!]$ is the one given by (2.26).

Vector fields on TM^n are parametrized by elements of $\mathcal{R}_{n-1}$ and their variation under w-morphisms is given by

$$\delta_X \partial_V = \left[\partial_{J(X)}\,,\,\partial_V\right], \tag{4.2}$$

or in "components"

$$\delta_X V = \partial_{J(X)} V - \partial_V J(X). \tag{4.3}$$

It is clear from the above that if $V \in \mathcal{R}_{n-1}$, so does $\delta_X V$; whence (4.2) defines a consistent transformation rule. Moreover

$$\begin{aligned} \left[\delta_X\,,\,\delta_Y\right]\partial_V &= \left[\partial_{J(X)}\,,\,\left[\partial_{J(Y)}\,,\,\partial_V\right]\right] - \left[\partial_{J(Y)}\,,\,\left[\partial_{J(X)}\,,\,\partial_V\right]\right] \\ &= \left[\left[\partial_{J(X)}\,,\,\partial_{J(Y)}\right]\,,\,\partial_V\right] = \delta_{[\![X,Y]\!]}\partial_V \end{aligned} \tag{4.4}$$

so it carries a representation of w-morphisms as expected. The reader should notice that this procedure is consistent even when we impose some constraint in the form of L, as long as we consider symplectomorphisms which preserve such constraint (they obviously form a subalgebra).

It would be natural from the point of wiew of field theory to restrict ourselves to vector fields which do not depend on L. If that would be possible, the "transport" term $\partial_{J(X)} V$ will drop out of the transformation rule. Unfortunately this is not possible. Because of the nature of the transformations, even if we start with a L independent vector field the classical W transformations will generate terms depending on the u_j's. Notice that this is not a peculiarity of (4.2) but has to be true insofar as we want a representation of the algebra defined by (4.1). This is due to the fact that the structure constants of the algebra are themselves L dependent. The next best thing one can do is to consider vector fields whose dependence on L is only through differential polynomials in the u_j's. Nevertheless, it is clear that the new nontrivial terms on the transformation are given by $\partial_V J(X)$, so for the time being we will consider vector fields which are L independent. In this case,

$$\begin{aligned} \delta_X V = &J(\partial_V X) + \{V\,,\,X\}_+ L + \{L\,,\,X\}_+ V - \\ &\{V\,,\,(LX)_+\} + \{L\,,\,(VX)_+\}\,. \end{aligned} \tag{4.5}$$

(Notice also that because of the algebra relationships we cannot restrict ourselves to X's that are L independent either.)

It is also possible to define a representation in the dual of TM^n by imposing the invariance of the pairing under w-morphisms, *i.e.* if $Q \in T^*M^n$ then $\delta^*_X Q$ is defined trough the condition that for all $\partial_V \in TM^n$

$$(\delta^*_X Q)(\partial_V) = Q(\delta_X \partial_V). \tag{4.6}$$

In what follows for the clarity of the exposition we will restrict ourselves to the case of w_3 where explicit calculations are simple to perform and we can get some further intuition on the subject. Nevertheless the reader should keep in mind that all of the manipulations that follow can in principle be generalized to arbitrary gd_n or any of its reductions.

First of all let us compute the algebra of w_3-morphisms. A straightforward computation yields

$$\begin{aligned} \left[\delta^{(T)}_\epsilon\,,\,\delta^{(T)}_\eta\right] =& \delta^{(T)}_{\epsilon'\eta-\epsilon\eta'} \\ \left[\delta^{(T)}_\epsilon\,,\,\delta^{(W)}_\alpha\right] =& \delta^{(W)}_{2\epsilon'\alpha-\epsilon\alpha'} \\ \left[\delta^{(W)}_\alpha\,,\,\delta^{(W)}_\beta\right] =& \delta^{(T)}_{\frac{2}{3}(\alpha\beta'-\alpha'\beta)} \end{aligned} \tag{4.7}$$

The vector field V in this case has to be of the form

$$V = v_1 p + v_2.$$

If we consider v_1 and v_2 L independent its variation under diffeomorphisms with parameter ϵ and a pure w transformation with parameter α can be computed from (4.5) to give

$$\begin{aligned} \delta^{(T)}_\epsilon v_1 =& 2\epsilon' v_1 + \epsilon v_1', \\ \delta^{(T)}_\epsilon v_2 =& 3\epsilon' v_2 + \epsilon v_2', \\ \delta^{(W)}_\alpha v_1 =& 2\alpha v_2' + 3\alpha' v_2, \\ \delta^{(W)}_\alpha v_2 =& -\frac{2}{3}(\alpha v_1)' T - \frac{2}{3}(\alpha T)' v_1. \end{aligned} \tag{4.8}$$

At this point will be convenient to relax our pace and give some thought to some of the subtle points in (4.8). As stated before the transformation of V involve terms dependent in the u_j's and its derivatives (in this simple case it only depends on T and T') even when considering field independent parameters for our transformations.

Let us now compute using (4.8) the commutator of two pure w transformations on, for example,v_2:

$$\left[\delta_\alpha^{(W)}, \delta_\beta^{(W)}\right] v_2 = \frac{2}{3}(\beta'\alpha - \beta\alpha')Tv_2' + 3\left(\frac{2}{3}(\beta'\alpha - \beta\alpha')T\right)' v_2 \\ - \frac{4}{3}(\beta'\alpha - \beta\alpha')W'v_1 + 2\left((\beta'\alpha - \beta\alpha')Wv_1\right)' . \tag{4.9}$$

The first two terms of the transformation correspond to a standard diffeomorphism on v_2 with parameter $\frac{2}{3}(\beta'\alpha - \beta\alpha')T$ as was to be expected from the w-morphisms algebra, but moreover we are also obtaining terms which are dependent on W! This should not have come to much of a surprise if we would have taken a closer look at (4.5), a careful computation of the term $J(\partial_V X)$ yields the extra W-dependent terms in the above transformation.

Although perfectly consistent, the transformation rule (4.9) seems to cast some doubts into the relevance of these representations for conventional field theory; the argument runs as follows. In field theory, we expect the transformation rules to be generated via Poisson brackets with the associated conserved charges. If that would be the case,

$$\delta_\alpha v_2 = \int dx \, \{\alpha W , v_2\} , \tag{4.10}$$

where we have left the transformation parameter α inside the Poisson bracket anticipating the case in which the parameter becomes field-dependent. The Jacobi identities imply that

$$\left[\delta_\alpha^{(W)}, \delta_\beta^{(W)}\right] v_2 = \int dx \left\{\frac{1}{3}(\beta'\alpha - \beta\alpha')T^2 , v_2\right\} , \tag{4.11}$$

and it is clear that in this case the terms which are dependent on W are not generated. One may thus be tempted to conclude at this point that the representation (4.8) cannot be realized field-theoretically.

This opens the question if there is a representation of w_3 similar in structure to (4.8) and generated via Poisson brackets. We have checked using *Mathematica*^{TM} **[16]** that no Poisson brackets exist which reproduce a representation of w_3 with only two fields. Nevertheless, the similarity of the "doublet" structure of (4.8) with the representations appearing in the 3-state Potts model (*cf.* (1.13)) suggests otherwise. This seems somewhat paradoxical, for if the OPEs underlying (1.13) were to have a well-defined classical limit, we would expect them to yield a Poisson bracket.

The solution of this "paradox" goes through a careful observation of the structure of the transformations (4.3). For field-independent vectors $\delta_X V = -\partial_V J(X)$, the crucial observation is that $J(X)$ can itself be written in terms of Gel'fand-Dickey brackets (for field-independent X) as follows

$$J(X) = \sum_{ij=0}^{n-1} \int dx\; X_j \left\{u_j\,,\, u_i\right\}_{\mathsf{GD}} p^i. \tag{4.12}$$

And from here it follows that if we write $V = \sum_{k=0}^{n-1} v_k p^k$

$$\delta_X v_k = \sum_{j=0}^{n-1} \int dx X_j \left\{u_j\,,\, v_k\right\}_{\mathsf{GD}} + \sum_{j=0}^{n-1} \int dx X_j \left\{v_j\,,\, u_k\right\}_{\mathsf{GD}}. \tag{4.13}$$

While the first term in (4.13) reproduces the standard transformation rule via Poisson brackets the second term appears as a novel feature associated to this kind of representations. With all of this in mind, it is now simple to construct a field theory, based on a free field realization, which will provide us with an example that incorporates all these features. Let us introduced two classical fields ϕ_1 and ϕ_2 which obey the following Poisson algebra

$$\left\{\phi_1(x)\,,\, \phi_1(y)\right\} = \left\{\phi_2(x)\,,\, \phi_2(y)\right\} = \frac{2}{3}\partial \cdot \delta(x-y), \tag{4.14}$$

and

$$\left\{\phi_1(x)\,,\, \phi_2(y)\right\} = -\frac{1}{3}\partial \cdot \delta(x-y). \tag{4.15}$$

It is possible to construct a rep. of the w_3 using this "Miura" fields by defining

$$\begin{aligned} T =& \phi_1\phi_2 - (\phi_1+\phi_2)^2 \\ W =& -\phi_1\phi_2(\phi_1+\phi_2)^2. \end{aligned} \tag{4.16}$$

Notice that after the reduction of setting the term in p^2 of L to zero, and for X of the form $X = \alpha p^{-1} + \epsilon p^{-2} - 1/3\alpha T p^{-3}$, $J(X)$ can be written as

$$\begin{aligned} J(X) =& \int dy \left(\epsilon \left\{T(y)\,,\, T(x)\right\}_{\mathsf{GD}} + \alpha \left\{W(y)\,,\, T(x)\right\}_{\mathsf{GD}}\right) p \\ &+ \int dy \left(\epsilon \left\{T(y)\,,\, W(x)\right\}_{\mathsf{GD}} + \alpha \left\{W(y)\,,\, W(x)\right\}_{\mathsf{GD}}\right) . \end{aligned} \tag{4.17}$$

From this, following the same procedure as in (4.13), we get the transformations rules of v_1 and v_2 in terms of fundamental Poisson brackets.

$$\begin{aligned}
\delta_\epsilon v_1 &= \int dy\, \epsilon \left(\{T(y)\,,\, v_1(x)\}_{\mathsf{GD}} + \{v_1(y)\,,\, T(x)\}_{\mathsf{GD}}\right), \\
\delta_\epsilon v_2 &= \int dy\, \epsilon \left(\{T(y)\,,\, v_2(x)\}_{\mathsf{GD}} + \{v_1(y)\,,\, W(x)\}_{\mathsf{GD}}\right), \\
\delta_\alpha v_1 &= \int dy\, \alpha \left(\{W(y)\,,\, v_1(x)\}_{\mathsf{GD}} + \{v_2(y)\,,\, T(x)\}_{\mathsf{GD}}\right), \\
\delta_\alpha v_2 &= \int dy\, \alpha \left(\{W(y)\,,\, v_2(x)\}_{\mathsf{GD}} + \{v_2(y)\,,\, W(x)\}_{\mathsf{GD}}\right).
\end{aligned} \tag{4.18}$$

Of course the above expression is empty of significance unless we define which are the Poisson brackets of the v's with the generators of the w_3 algebra. This can be readily done by defining

$$\begin{aligned}
v_1 =& \phi_1 + \phi_2 \\
v_2 =& \frac{1}{3}(\phi_1 + \phi_2)^2 + \frac{2}{3}\phi_1\phi_2.
\end{aligned} \tag{4.19}$$

It is now a simple computation to check that with this definition (4.18) reproduce the w_3 transformations given by (4.8).

<u>Tensor Product of Representations</u>

It is now easy to construct tensor product representations of these representations. We should extend in the standard way the action of w-morphisms to $TM^n \otimes TM^n$; that is, if $\partial_V \otimes \partial_U \in TM^n \otimes TM^n$ then

$$\delta_X(\partial_V \otimes \partial_U) = (\delta_X \partial_V) \otimes \partial_U + \partial_V \otimes (\delta_X \partial_U). \tag{4.20}$$

It is now routine to check that (4.20) provides a representation of the algebra (4.7).

Before immersing ourselves in more generalities let us have a look again to the simple case of w_3. Then (4.20) can be written in "components" as

$$\begin{aligned}
\delta_\epsilon(v_2 \otimes u_2) =& (\epsilon v_2' + 3\epsilon' v_2) \otimes u_2 + v_2 \otimes (\epsilon u_2' + 3\epsilon' u_2) \\
\delta_\epsilon(v_2 \otimes u_1) =& (\epsilon v_2' + 3\epsilon' v_2) \otimes u_1 + v_2 \otimes (\epsilon u_1' + 2\epsilon' u_1) \\
\delta_\epsilon(v_1 \otimes u_2) =& (\epsilon v_1' + 2\epsilon' v_1) \otimes u_2 + v_1 \otimes (\epsilon u_2' + 3\epsilon' u_2) \\
\delta_\epsilon(v_1 \otimes u_1) =& (\epsilon v_1' + 2\epsilon' v_1) \otimes u_1 + v_1 \otimes (\epsilon u_1' + 2\epsilon' u_1),
\end{aligned} \tag{4.21}$$

and

$$\delta_\alpha(v_2 \otimes u_2) = -\frac{2}{3}\big((\alpha v_1)'T + (\alpha T)'v_1\big) \otimes u_2 - \frac{2}{3} v_2 \otimes \big((\alpha u_1)'T + (\alpha T)'u_1\big)$$

$$\delta_\alpha(v_2 \otimes u_1) = \left(-\frac{2}{3}(\alpha v_1)'T - \frac{2}{3}(\alpha T)'v_1\right) \otimes u_1 + u_2 \otimes \big(2\alpha u_2' + 3\alpha' u_2\big)$$
$$\delta_\alpha(v_1 \otimes u_2) = \big(2\alpha v_2' + 3\alpha' v_2\big) \otimes u_2 + v_1 \otimes \left(-\frac{2}{3}(\alpha u_1)'T - \frac{2}{3}(\alpha T)'u_1\right)$$
$$\delta_\alpha(v_1 \otimes u_1) = \big(2\alpha v_2' + 3\alpha' v_2\big) \otimes u_1 + v_1 \otimes \big(2\alpha u_2' + 3\alpha' u_2\big) \ , \tag{4.22}$$

where $V = v_1 p + v_2$ and $U = u_1 p + u_2$ are the elements of $\mathcal{R}_1$ parametrizing ∂_V and ∂_U respectively.

We now see one of the crucial differences between the transformation properties of the tensor product under diffeomorphisms and under pure w-transformations. Let us fix our attention on the transformation properties of, say, $v_1 \otimes u_1$. For diffeomorphisms,

$$\delta_\epsilon(v_1 \otimes u_1) = 4\epsilon'(v_1 \otimes u_1) + \epsilon\partial(v_1 \otimes u_1) \ , \tag{4.23}$$

where we are taking the tensor product to be linear over the functions, and taking the action of the derivative on the tensor product to be the standard one: ∂ acts by $\partial \otimes 1 + 1 \otimes \partial$. We can then confuse $v_1 \otimes u_1$ with the product $v_1 \otimes u_1$ and write the usual

$$\delta_\epsilon(v_1 u_1) = 4\epsilon' v_1 u_1 + \epsilon(v_1 u_1)' \ . \tag{4.24}$$

On the other hand, in the case of pure w-transformations it is impossible to write $\delta_\alpha(v_1 \otimes u_1)$ in the form (4.23) with the standard action of ∂ on the tensor product. Indeed, the naive attempt to write the variation as

$$(3\alpha' + 2\alpha\partial)(v_2 \otimes u_1 + v_1 \otimes u_2), \tag{4.25}$$

produces unwanted terms proportional to $v_2 \otimes u_1'$ and $v_1' \otimes u_2$. If as before we want to confuse the tensor product with the product (that is to say, to be able to work with the components of the tensors), we have to pay the price of introducing fields of the form $v_2 u_1'$ and $v_1' u_2$. In this concrete realization of the tensor product representation, these fields are obtained from the fundamental ones, but as an abstract representation of w_3 they are indeed new. This was to be expected since as we mentioned in the introduction the generic representation of W_3 (and hence of its classical limit) breaks up into an infinite number of different Vir-tensors except for very particular values of the conformal and W-dimensions and of the central charge.

§5 Final Comments

We have seen how the geometrical interpretation [**5**] of classical W-transformations provides us with examples of classical W-tensors. Although far from providing us with a complete understanding of the subject, these geometric realizations do share many of the interesting (and sometimes puzzling) properties of the representations of W-algebras appearing in conformal field theory. In particular, it provides the necessary geometric input to guess that the classical W-transformations are generated via the Poisson action given by (4.9). Moreover we have seen that at least for w_3 these representations have a structure similar to those appearing in conformal field theory, *e.g.*, the three-state Potts model at criticality.

The analogy with the Potts model, however, should be taken with a grain of salt, since we don't have at this point a clear grasp of the classical limit of these representations. The OPEs underlying (1.13) are of course only valid for $c = \frac{4}{5}$ and we have no reason to expect them to survive the classical limit, which involves running the central charge. Nevertheless, we can ask for the existence of representations of W_3 with the same "doublet" structure. In other words, we can ask for what values of the free parameters of the theory, can we have the following fusion rules

$$W \times \phi_h = \phi_{h+1} + \cdots \qquad \text{and} \qquad W \times \phi_{h+1} = \phi_h + \cdots \tag{5.1}$$

where the $\cdots$ represent contributions that would not be present in the singular part of the operator product expansion or which come from null fields. Evidently, this says that ϕ_h is a W_3-primary field of weight $(h, 0)$ and ϕ_{h+1} is a Vir-primary of weight $h+1$. Conformal covariance alone fixes the first OPE to be of the form

$$W(z)\phi_h(w) = \frac{\phi_{h+1}(w)}{(z-w)^2} + \frac{\mu\partial\phi_{h+1}}{z-w} + \text{lower order terms} \; , \tag{5.2}$$

where $\mu = \frac{2}{1+h}$, which says that the Vir primary $\phi_{h+2} \equiv \widehat{W}_{-2}\cdot\phi_h - \mu\partial(\widehat{W}_{-1}\cdot\phi_h)$ is null. Since it is already a Vir-primary, it is null if and only if it is annihilated by $\widehat{W}_1$. A quick calculation gives us the following relation between c and h: $ch + 2c + 8h^2 - 18h + 4 = 0$. As a check, notice that it is satisfied for $(h = \frac{2}{5}, c = \frac{4}{5})$. But more importantly, it shows that there are continuous values of the parameters of the CFT for which W_3 has "doublet" representations like the ones we have found using our geometrical interpretation for classical w_3 transformations.

About W-(super)space

One successful approach to W-geometry would certainly be to define a W-space in which W-transformations would be naturally realized as moving the points. W-tensors would be then defined as the natural functions on this space, which inherit the transformation properties in the usual way. It has long been realized that the natural candidate for a (chiral) W_3-space consists of an extension of the circle by a new variable of weight 2. The rationale is that we expect the derivative relative to the new coordinate should commute with ∂ and the natural such operator is W_{-2}. Nevertheless, the fact that w-morphisms and w-tensors can be given a geometrical interpretation based on objects living in a two-dimensional phase space Y seems to suggest that perhaps this bias should be abandoned. Instead we should perhaps think of Y as the natural candidate for the W-space with the constrained symplectomorphisms parametrized by (3.1) as the natural transformations. Comparing with the familiar case of the (1|1) superspace with coordinates (x, θ), we propose the following tentative "dictionary":

$$\begin{aligned} (1|1)\text{ superspace } (x,\theta) &\leftrightarrow \mathsf{W}\text{-space } (x,p) \\ \theta^2 = 0 &\leftrightarrow L = 0 \\ \text{superderivative} &\leftrightarrow \text{Poisson bracket} \\ \text{superVirasoro transformations} &\leftrightarrow \text{symplectomorphisms} \end{aligned}$$

In fact, this conclusion supported by the fact that w_∞ transformations can be given an easy geometric interpretation simply as the Poisson bracket. Indeed, $\left\{u_k p^k\,,\, u_j p^j\right\} = [u_k\,,\, u_j]_S\, p^{j+k-1}$ defines the (symmetric) Schouten bracket, which is the dual of w_∞. **references to Chris Hull?**

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