HOMOLOGICAL APPROACH TO SYMPLECTIC REDUCTION

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ABSTRACT

We discuss symplectic reduction from a homological point of view. Using standard techniques in homological algebra we construct a homology theory which, in dimension zero, is the algebraic analog of symplectic reduction. We study the rest of the homology and prove that the homology vanishes in all negative dimensions and that the rest is isomorphic (as a Poisson superalgebra) to the cohomology of vertical differential forms of a given fibration.
This is a rather old unpublished preprint. It was originally circulated in 1988 and then with some minor revisions in 1991. It has not been touched since.

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§1 INTRODUCTION

Let (M,Ω) be a $2n$-dimensional symplectic manifold and let G be a connected Lie group acting on M via symplectomorphisms. To each element X in the Lie algebra $\mathfrak{g}$ of G we associate a Killing vector field (which we shall denote by X) on M which is symplectic. To each symplectic vector field there is associated a closed 1-form on M: $i(X)\,\Omega$. If this form is also exact then the vector field is called hamiltonian. If all the Killing vector fields are hamiltonian the G-action is called hamiltonian. In this case we can associate to each vector $X \in \mathfrak{g}$ a hamiltonian function ϕ^X such that $d\phi^X + i(X)\,\Omega = 0$. Dual to this construction is the moment mapping $\Phi : M \to \mathfrak{g}^*$ defined by $\langle \Phi(m), X\rangle = \phi^X(m)$, for all $m \in M$. If a certain cohomological obstruction is overcome the hamiltonian functions $\{\phi^X\}$ close under Poisson bracket in the following way: $\left[\phi^X\,,\,\phi^Y\right] = \phi^{[X\,,Y]}$. If this is the case, the moment map is equivariant: intertwining between the G-actions on M and the coadjoint action on $\mathfrak{g}^*$.

Assume that we have a hamiltonian G-action on M giving rise to an equivariant moment map and furthermore suppose that $\mathbf{0} \in \mathfrak{g}^*$ is a regular value of the moment map. Denote $\Phi^{-1}(\mathbf{0})$ by M_o. Then M_o is a closed imbedded coisotropic submanifold of M with trivial normal bundle. Let $m \in M_o$. Then T_mM_o is a coisotropic subspace of T_mM and $(T_mM_o)^\perp$ is precisely the subspace spanned by the Killing vectors. In particular, the Killing vectors form an integrable distribution giving rise to a foliation on M_o whose leaves are just the orbits of the G-action. If the G-action on M_o is free and proper then the space of orbits inherits the structure of a smooth symplectic manifold. If $i : M_o \to M$ denotes the natural inclusion and $\pi : M_o \to M_o/G$ denotes the natural projection then the symplectic form $\widetilde{\Omega}$ on M_o/G is the unique form on M_o/G such that $\pi^*\widetilde{\Omega} = i^*\Omega$. If the G-action on M_o is not free but only locally free (*i.e.*, the isotropy is discrete) then the space of orbits is a symplectic orbifold.

The procedure we have just described is a special kind of symplectic reduction. A more general situation arises in the Dirac theory of constraints. Let $\{\phi_i\}_{i=1}^k$ be a set of first class constraints. That is, the $\{\phi_i\}$ are smooth functions on M such that the (multiplicative) ideal $J \subset C^\infty(M)$ they generate is a Poisson subalgebra of $C^\infty(M)$: $[J\,,\,J] \subset J$. We may assemble them together into a smooth function $\Phi : M \to \mathbb{R}^k$. Assuming that $\mathbf{0} \in \mathbb{R}^k$ is a regular value of Φ, the zero set $\Phi^{-1}(\mathbf{0}) \equiv M_o$ is a (closed imbedded) coisotropic submanifold. The null foliation $\mathcal{F}$ in this case arises from the integrable distribution spanned by the hamiltonian vector fields $\{X_i\}$ associated to the constraints via $i(X_i)\,\Omega + d\phi_i = 0$. If the foliation is also a fibration, the space of leaves $M_o/\mathcal{F}$ is a manifold and inherits, as before, a symplectic structure. Notice that if the linear span of the constraints closes under Poisson brackets we recover the case of the $\mathfrak{g}$-action.

It is interesting to notice that the reduced symplectic manifold is in either case a subquotient of M. That is, first we restrict to a submanifold (M_o) and then we quotient this submanifold by the null foliation. This is to be compared with homology which is also a subquotient of the chains: first we restrict to the subspace of the cycles and then we quotient by the boundaries. In this paper we make this heuristic observation precise by constructing a homology theory which becomes an algebraic analogue of symplectic reduction.

This homology theory has a relatively long history in the physics literature and goes under the name of "BRST cohomology." In this paper we present this construction in what we think is its natural mathematical setting emphasizing, on the one hand, the beautiful interplay between the algebraic and geometric aspects of this formalism; and, on the other hand, its Poisson algebraic structure.

In the case where the coisotropic submanifold is the zero set of an equivariant moment mapping and the null foliation is the one generated by a group action, BRST cohomology goes back to the work of Batalin and Vilkovisky [**1**] and, in the context of Yang-Mills theory, to the work of McMullen [**2**]. This latter work was later elaborated upon in [**3**] where the conditions for BRST cohomology to recover the Poisson algebra of smooth functions of the symplectic quotient were, however, only vaguely sketched; although we would be surprised if the precise conditions were not known by the authors. For the more general case where the coisotropic submanifold is the zero set of some first class constraints the study of this formalism (with the Koszul resolution as its center piece) was initiated in [**4**], borrowing from the seminal work of Henneaux [**5**], and continued in [**6**]. More recently Stasheff [**7**] has given a completely algebraic description of BRST cohomology in the context of constrained Poisson algebras. This direction seems to offer a very fruitful perspective in the quantization of systems in which the symplectic reduction cannot be effected due to the singularities present in the reduced manifold [**8**].

Since, as described above, symplectic reduction is a geometric subquotient, in order to construct its homological equivalent, it is first necessary to describe geometric objects algebraically. Dual to a differentiable manifold M we have the commutative algebra $C^\infty(M)$ of its smooth functions, which characterize it completely. The correspondence is well known and goes roughly as follows. To every point $p \in M$ there corresponds an ideal $I(p)$ of $C^\infty(M)$ consisting of those functions vanishing at p. Moreover this ideal is maximal and, with respect to any topology that makes the evaluation map continuous, it is closed. It turns out that these are all the maximal closed ideals of $C^\infty(M)$. Therefore there is a set isomorphism between M and the set of maximal closed ideals of $C^\infty(M)$, which can be suitably topologized and given a differentiable structure in such a way that the above isomorphism is actually a diffeomorphism. Similarly if $i : M_o \hookrightarrow M$ is a submanifold, it can be described by an ideal $I(M_o)$ consisting of all smooth functions vanishing at M_o. For a special class of submanifolds this ideal is finitely generated. This corresponds to submanifolds that can be described as the regular zero locus of a set of smooth functions. This will be the case of interest in this paper. The role of the submanifold M_o will be played by the zero locus of a set of regular first class constraints.

An algebraization of symplectic reduction—albeit not a homological one—was given by Śnyaticki and Weinstein [**8**] while attempting to set up a formalism to treat the case of singular moment mappings. We briefly review their formalism. We work in the general case of a set of first class constraints with regular zero locus M_o. Unless otherwise stated we work with smooth objects.

Any function on M restricts to a function on M_o and, conversely, since M_o is a closed and embedded submanifold of M, any function on M_o extends to a function on M. This extension is, of course, not unique but the difference of two such extensions must always vanish on M_o. Therefore, we have an algebra isomorphism

$$C^\infty(M_o) \cong C^\infty(M)/I(M_o) \ . \tag{1.1}$$

One can show (see section 2 for an indirect proof) that $I(M_o)$ is precisely the ideal J generated by the constraints, whence one has the algebra isomorphism

$$C^\infty(M_o) \cong C^\infty(M)/J \ . \tag{1.2}$$

This relates functions on M_o to functions on M. We now relate functions on $\widetilde{M}$ to functions on M_o. A function f on M_o projects to a function on $\widetilde{M}$ if and only if it is constant along the leaves of the null foliation. Let us denote all elements in $C^\infty(M_o)$ which are constant on the leaves by $C^\infty(M_o)^{\mathcal{F}}$. We therefore have the algebra isomorphism

$$C^\infty(\widetilde{M}) \cong C^\infty(M_o)^{\mathcal{F}} \ . \tag{1.3}$$

These two steps can be combined to relate functions on $\widetilde{M}$ directly to functions on M and at the same time show that the Poisson structure is inherited. From (1.2) and (1.3) it follows that functions on $\widetilde{M}$ are in bijective correspondence to those functions on M whose restriction to M_o is constant along the leaves of the null foliation. Put differently and since the leaves are connected, a function f on M yields a function on $\widetilde{M}$ if and only if for all $i = 1, \ldots, k$, $X_i(f) \in J$ which, using the Poisson structure on $C^\infty(M)$, can be rewritten as

$[\phi_i\,,\,f] \in J$. Thus we see that the f is in the normalizer $N(J)$ of J in $C^\infty(M)$. Therefore, we have the following algebra isomorphism

$$C^\infty(\widetilde{M}) \cong N(J)/J. \tag{1.4}$$

Notice that $N(J)$ is naturally a Poisson subalgebra of $C^\infty(M)$ and J is a Poisson ideal of $N(J)$. Therefore, $N(J)/J$, and thus $C^\infty(\widetilde{M})$, naturally becomes a Poisson algebra. Alternatively, $C^\infty(\widetilde{M})$ inherits a Poisson structure from the induced symplectic form $\widetilde{\Omega}$. Not surprisingly these two structures are the same. In fact, we have the following

Lemma 1.5. *Let $\tilde{f}$ be a function on $\widetilde{M}$ induced from a function f on M: i.e., $i^*f = \pi^*\tilde{f}$. Let X_{i^*f} denote the tangent vector field to M_o such that $i_*X_{i^*f} = X_f$ on M_o. Then the hamiltonian vector fields $X_{\tilde{f}}$ and X_{i^*f} are π-related. That is, $\pi_*X_{i^*f} = X_{\tilde{f}}$.*

Proof: Let $p \in M_o$, and $\widetilde{Y}$ a tangent vector field on $\widetilde{M}$. Lift $\widetilde{Y}$ to a vector field Y tangent to M_o such that $\pi_*Y = \widetilde{Y}$. Then

$$\begin{aligned}
\widetilde{\Omega}_{\pi(p)}(\pi_*X_{i^*f}, \widetilde{Y}) =& \widetilde{\Omega}_{\pi(p)}(\pi_*X_{i^*f}, \pi_*Y) \\
=& (\pi^*\widetilde{\Omega})_p(X_{i^*f}, Y) \\
=& (i^*\Omega)_p(X_{i^*f}, Y) \\
=& \Omega_p(X_f, i_*Y) \\
=& -df_p(i_*Y) \\
=& -(di^*f)_p(Y) \\
=& -(d\pi^*\tilde{f})_p(Y) \\
=& -d\tilde{f}_{\pi(p)}(\widetilde{Y}) \\
=& \widetilde{\Omega}_{\pi(p)}(X_{\tilde{f}}, \widetilde{Y}).
\end{aligned}$$

The non-degeneracy of $\widetilde{\Omega}$ concludes the proof. □

This leads to the following

Theorem 1.6. *Give $C^\infty(\widetilde{M})$ the Poisson structure it inherits from $\widetilde{\Omega}$. Then* (1.4) *extends to a Poisson algebra isomorphism.*

Proof: This follows easily from the lemma. In fact, let $\tilde{f}$ and $\tilde{g}$ be functions on $\widetilde{M}$ and let f and g be functions in $N(J)$ corresponding to them via (1.4): that is, $i^*f = \pi^*\tilde{f}$ and $i^*g = \pi^*\tilde{g}$. Then, for all $p \in M_o$,

$$\pi^*(\left[\tilde{f}\,,\,\tilde{g}\right])(p) = \widetilde{\Omega}_{\pi(p)}(X_{\tilde{f}}, X_{\tilde{g}})$$

$$
\begin{aligned}
&=\widetilde{\Omega}_{\pi(p)}(\pi_* X_{i^*f}, \pi_* X_{i^*g}) \\
&=(\pi^*\widetilde{\Omega})_p(X_{i^*f}, X_{i^*g}) \\
&=(i^*\Omega)_p(X_{i^*f}, X_{i^*g}) \\
&=\Omega_p(X_f, X_g) \\
&=i^*([f\,,\,g])(p).
\end{aligned}
$$

Clearly this does not depend on the choice of f and g. Since this is true for arbitrary $\tilde{f}, \tilde{g} \in C^\infty(\widetilde{M})$ the proof is finished. $\square$

The homological construction we discuss in this paper consists of three steps. The first is to turn (1.2) into an acyclic resolution of $C^\infty(M_o)$ as a $C^\infty(M)$-module. The second step, which is independent from the first, consists in a homological description of the isomorphism (1.3). Finally, the third step combines these two into a cohomology theory (BRST) which describes (in zero dimension) the isomorphism (1.4). Moreover, this can be done in such a way that the correspondence between the Poisson structures is manifest.

Thus this paper is organized as follows. In section 2 we study the first step of the subquotient: the restriction to the subspace. Suppose that $i : M_o \hookrightarrow M$ is a closed embedded submanifold of codimension k corresponding to the zero set (assumed regular) of a smooth function $\Phi : M \to \mathbb{R}^k$. We then define a Koszul-like complex associated to this embedding, which will provide us with a free acyclilc resolution of $C^\infty(M_o)$ as a $C^\infty(M)$-module. We give a novel proof of the acyclicity of this complex, in which we introduce a double complex analogous to the Čech-de Rham complex introduced by Weil **[9]** in his proof of the de Rham theorem. We call it the Čech-Koszul complex. In section 3 we tackle the second step of the subquotient. We define a cohomology theory associated to the foliation determined by the null distribution of $i^*\Omega$ on M_o. This is a de Rham-like cohomology theory of differential forms (co)tangent to the leaves of the foliation (vertical forms) relative to the exterior derivative along the leaves of the foliation (vertical derivative). If the foliation fibers onto a smooth manifold $\widetilde{M}$ the cohomology in zero dimension is naturally isomorphic to $C^\infty(\widetilde{M})$. We then lift this cohomology theory via the Koszul resolution obtained in section 2 to a cohomology theory (BRST) in a certain bigraded complex. The existence of this cohomology theory must be proven since the vertical derivative does not lift to a differential operator, *i.e.*, its square is not zero. However its square is chain homotopic to zero (relative to the Koszul differential) and the acyclicity of the Koszul resolution allows us to construct the desired differential. In section 4 we place BRST cohomology in a Poisson setting. In particular we show that the BRST cohomology can be computed from a differential that is an inner Poisson derivation. In this way, one can show that its cohomology naturally inherits the structure of a Poisson superalgebra. This, in turn, allows us to explicitly prove the isomorphism as Poisson superalgebras between the BRST cohomology and the cohomology of the vertical forms. Finally in section 5 we comment on possible generalizations and on directions for future research.

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§2 THE ČECH-KOSZUL COMPLEX

In this section we describe algebraically the restriction part of symplectic reduction. The key outcome of this section is a projective resolution of $C^\infty(M_o)$ as a $C^\infty(M)$-module. Hence we let M_o be the zero locus of k functions $\{\phi_i\}$ on M. We will now proceed to construct a differential complex (the Koszul complex)

$$\cdots \longrightarrow K^2 \longrightarrow K^1 \longrightarrow C^\infty(M) \longrightarrow 0 \ , \tag{2.1}$$

whose homology in positive dimensions is zero and in zero dimension is precisely $C^\infty(M_o)$. We shall refer to this fact as the acyclicity of the Koszul complex. It will play a fundamental rôle in all our constructions.

The Local Koszul Complex

We will first discuss the construction on $\mathbb{R}^m$ and later we will globalize to M. We start with an elementary observation.

Lemma 2.2. *Let $\mathbb{R}^m$ be given coordinates*

$$(y,x) = (y^1,\ldots,y^k,x^1,\ldots,x^{m-k}) \ .$$

Let $f:\mathbb{R}^m \to \mathbb{R}$ be a smooth function such that $f(\mathbf{0},x)=0$. Then there exist k smooth functions $h_i:\mathbb{R}^m\to\mathbb{R}$ such that $f=\sum_{i=1}^k \phi_i\, h_i$, where the ϕ_i are the functions defined by $\phi_i(y,x)=y^i$.

Proof: Notice that

$$\begin{aligned} f(y,x) &= \int_0^1 dt\, \frac{d}{dt} f(ty,x) \\ &= \int_0^1 dt \sum_{i=1}^k y^i\, (D_i\, f)(ty,x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} y^i \int_0^1 dt\,(D_i\, f)(ty,x) \\
&= \sum_{i=1}^{k} \phi_i(y,x) \int_0^1 dt\,(D_i\, f)(ty,x)\ ,
\end{aligned}
$$

where D_i is the i^{th} partial derivative. Defining

$$
h_i(y,x) \stackrel{\text{def}}{=} \int_0^1 dt\,(D_i\, f)(ty,x) \tag{2.3}
$$

the proof is complete. □

Therefore, if we let $P \subset \mathbb{R}^m$ denote the subspace defined by $y^i = 0$ for all i, the ideal of $C^\infty(\mathbb{R}^m)$ consisting of functions which vanish on P is precisely the ideal generated by the functions ϕ_i.

Definition 2.4. Let R be a commutative ring with unit. A sequence $(\phi_i)_{i=1}^k$ of elements of R is called regular if for all $j = 1,\ldots,k$, ϕ_j is not a zero divisor in R/I_{j-1}, where I_j is the ideal generated by $\phi_1,\ldots,\phi_j$ and $I_0 = 0$.

Proposition 2.5. *Let $\mathbb{R}^m$ be given coordinates*

$$
(y,x) = (y^1,\ldots,y^k,x^1,\ldots,x^{m-k})\ .
$$

Then the sequence (ϕ_i) in $C^\infty(\mathbb{R}^m)$ defined by $\phi_i(y,x) = y^i$ is regular.

Proof: First of all notice that ϕ_1 is not identically zero. Next suppose that $(\phi_1,\ldots,\phi_j)$ is regular. Let P_j denote the hyperplane defined by $\phi_1 = \cdots = \phi_j = 0$. Then by Lemma 2.2, $C^\infty(P_j) = C^\infty(\mathbb{R}^m)/I_j$. Let $[f]_j$ denote the class of a $f \in C^\infty(\mathbb{R}^m)$ modulo I_j. Then ϕ_{j+1} gives rise to a function $[\phi_{j+1}]_j$ in $C^\infty(P_j)$ which, if we think of P_j as coordinatized by

$$
(y^{j+1},\ldots,y^k,x^1,\ldots,x^{m-k})\ ,
$$

turns out to be defined by

$$
[\phi_{j+1}]_j\,(y^{j+1},\ldots,y^k,x^1,\ldots,x^{m-k}) = y^{j+1}\ . \tag{2.6}
$$

This is clearly not identically zero and, therefore, the sequence $(\phi_1\ \ldots,\phi_{j+1})$ is regular. By induction we are done. □

We now come to the definition of the Koszul complex **[10]**. Let R be a ring and let $\Phi = (\phi_1,\ldots,\phi_k)$ be a sequence of elements of R. We define a complex $K(\Phi)$ as follows: $K^0(\Phi) = R$ and for $p > 0$, $K^p(\Phi)$ is defined to be the free R module with basis $\{b_{i_1}\wedge\cdots\wedge b_{i_p} \mid 0 < i_1 < \cdots < i_p \leq k\}$.

Define a map $\delta_K : K^p(\Phi) \to K^{p-1}(\Phi)$ by $\delta_K b_i = \phi_i$ and extending to all of $K(\Phi)$ as an R-linear antiderivation. That is, δ_K is identically zero on $K^0(\Phi)$ and

$$\delta_K(b_{i_1} \wedge \cdots \wedge b_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} \phi_{i_j} \, b_{i_1} \wedge \cdots \wedge \widehat{b_{i_j}} \wedge \cdots \wedge b_{i_p} \ , \tag{2.7}$$

where a $\widehat{\ }$ adorning a symbol denotes its omission. It is trivial to verify that $\delta_K^2 = 0$, yielding a complex

$$0 \longrightarrow K^k(\Phi) \xrightarrow{\delta_K} K^{k-1}(\Phi) \longrightarrow \cdots \longrightarrow K^1(\Phi) \longrightarrow R \longrightarrow 0 \ , \tag{2.8}$$

called the Koszul complex.

The following theorem is a classical result in homological algebra whose proof is completely straight-forward and can be found, for example, in [**11**].

THEOREM 2.9. *If $(\phi_1, \ldots, \phi_k)$ is a regular sequence in R then the cohomology of the Koszul complex is given by*

$$H^p(K(\Phi)) \cong \begin{cases} \mathbf{0} & \text{for } p > 0 \\ R/J & \text{for } p = 0 \end{cases} \ , \tag{2.10}$$

where J is the ideal generated by the ϕ_i.

Therefore the complex $K(\Phi)$ provides an acyclic resolution (known as the Koszul resolution) for the R-module R/J. Therefore if $R = C^\infty(\mathbb{R}^m)$ and Φ is the sequence $(\phi_1, \ldots, \phi_k)$ of Proposition 2.5, the Koszul complex gives an acyclic resolution of $C^\infty(\mathbb{R}^m)/J$ which by Lemma 2.2 is just $C^\infty(P_k)$, where P_k is the subspace defined by $\phi_1 = \cdots = \phi_k = 0$.

Globalization: The Čech-Koszul Complex

We now globalize this construction. Let M be our original symplectic manifold and $\Phi : M \to \mathbb{R}^k$ be the function whose components are the first class constraints constraints, *i.e.*, $\Phi(m) = (\phi_1(m), \ldots, \phi_k(m))$. We assume that 0 is a regular value of Φ so that $M_o \stackrel{\text{def}}{=} \Phi^{-1}(0)$ is a closed embedded submanifold of M. Therefore for each point $m \in M_o$ there exists an open set $U \in M$ containing m and a chart $\Psi : U \to \mathbb{R}^m$ such that Φ has components $(\phi_1, \ldots, \phi_k, x^1, \ldots, x^{m-k})$ and such that the image under Φ of $U \cap M_o$ corresponds exactly to the points $(\underbrace{0, \ldots, 0}_{k}, x^1, \ldots, x^{m-k})$. Let $\mathcal{U}$ be an open cover for M consisting of sets like these. Of course, there will be some sets $V \in \mathcal{U}$ for which $V \cap M_o = \varnothing$.

To motivate the following construction let's understand what is involved in proving, for example, that the ideal J generated by the constraints coincides with the ideal $I(M_o)$ of smooth functions which vanish on M_o. It is clear that $J \subset I(M_o)$. We want to show the converse. That is, if f is a smooth function vanishing on M_o then there are smooth functions h^i such that $f = \sum_i h^i \phi_i$. This is always true locally. That is, restricted to any set $U \in \mathcal{U}$ such that $U \cap M_o \neq \varnothing$, Lemma 2.2 implies that there will exist functions $h^i_U \in C^\infty(U)$ such that on U

$$f_U = \sum_i \phi_i \, h^i_U \ , \tag{2.11}$$

where f_U denotes the restriction of f to U. If, on the other hand, $V \in \mathcal{U}$ is such that $V \cap M_o = \varnothing$, then not all of the ϕ_i vanish and the statement is also true. There is a certain ambiguity in the choice of h^U_i. In fact, if δ_K denotes the Koszul differential we notice that (2.11) can be written as $f_U = \delta_K h_U$, where $h_U = \sum_i h^i_U b_i$ is a Koszul 1-cochain on U. Therefore, the ambiguity in h_U is precisely a Koszul 1-cocycle on U, but by Theorem 2.9, the Koszul complex on U is quasi-acyclic and hence every 1-cocycle is a 1-coboundary. What we would like to show is that this ambiguity can be exploited to choose the h_U in such a way that $h_U = h_V$ on all non-empty overlaps $U \cap V$. This condition is precisely the condition for h_U to be a Čech 0-cocycle. In order to analyze this problem it is useful to make use of the machinery of Čech cohomology with coefficients in a sheaf. For a review of the necessary material we refer the reader to **[12]**; and, in particular, to their discussion of the Čech-de Rham complex.

Let $\mathcal{E}_M$ denote the sheaf of germs of smooth functions on M and let $\mathcal{K} = \bigoplus_p \mathcal{K}^p$ denote the free sheaf of $\mathcal{E}_M$-modules which appears in the Koszul complex: $\mathcal{K}^p = \bigwedge^p \mathbb{V} \otimes \mathcal{E}_M$, where $\mathbb{V}$ is the vector space with basis $\{b_i\}$. Let $C^p(\mathcal{U};\mathcal{K}^q)$ denote the Čech p-cochains with coefficients in the Koszul subsheaf $\mathcal{K}^q$. This becomes a double complex under the two differentials

$$\check{\delta} : C^p(\mathcal{U};\mathcal{K}^q) \to C^{p+1}(\mathcal{U};\mathcal{K}^q) \qquad \text{“Čech”}$$

and

$$\delta_K : C^p(\mathcal{U};\mathcal{K}^q) \to C^p(\mathcal{U};\mathcal{K}^{q-1}) \qquad \text{“Koszul”}$$

which clearly commute, since they are independent. We can therefore define the complex $CK^n = \bigoplus_{p-q=n} C^p(\mathcal{U};\mathcal{K}^q)$ and the differential $D = \check{\delta} + (-1)^p \delta_K$ on $C^p(\mathcal{U};\mathcal{K}^q)$. The total differential has total degree one $D : CK^n \to CK^{n+1}$ and moreover obeys $D^2 = 0$. Since the double complex is bounded, *i.e.*, for each n, CK^n is the direct sum of a finite number of $C^p(\mathcal{U};\mathcal{K}^q)$'s, there are two spectral sequences converging to the total cohomology. We now proceed to compute them. In doing so we will find it convenient to depict our computations graphically. The original double complex is depicted by the following diagram:

$C^0(\mathcal{U};\mathcal{K}^2)$	$C^1(\mathcal{U};\mathcal{K}^2)$	$C^2(\mathcal{U};\mathcal{K}^2)$	
$C^0(\mathcal{U};\mathcal{K}^1)$	$C^1(\mathcal{U};\mathcal{K}^1)$	$C^2(\mathcal{U};\mathcal{K}^1)$	
$C^0(\mathcal{U};\mathcal{K}^0)$	$C^1(\mathcal{U};\mathcal{K}^0)$	$C^2(\mathcal{U};\mathcal{K}^0)$	

Upon taking cohomology with respect to the horizontal differential (*i.e.*, Čech cohomology) and using the fact that the sheaves $\mathcal{K}^q$ are fine, being free modules over the structure sheaf $\mathcal{E}_M$, we get

$K^2(\Phi)$	0	0	
$K^1(\Phi)$	0	0	
$K^0(\Phi)$	0	0	

where $K^p(\Phi) \cong \bigwedge^p \mathbb{V} \otimes C^\infty(M)$ are the spaces in the Koszul complex on M. Taking vertical cohomology yields the Koszul cohomology

$H^2(K(\Phi))$	0	0	
$H^1(K(\Phi))$	0	0	
$H^0(K(\Phi))$	0	0	

Since the next differential in the spectral sequence necessarily maps across columns it must be identically zero. The same holds for the other differentials and we see that the spectral sequence degenerates at the E_2 term. In particular the total cohomology is isomorphic to the Koszul cohomology:

$$H^n_D \cong H^n(K(\Phi)) \ . \tag{2.12}$$

To compute the other spectral sequence we first start by taking vertical cohomology, *i.e.*, Koszul cohomology. Because of the choice of cover $\mathcal{U}$ we can use

Theorem 2.9 and Lemma 2.2 to deduce that the vertical cohomology is given by

0	0	0	
0	0	0	
$C^0(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	$C^1(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	$C^2(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	

where $\mathcal{E}_M/\mathcal{J}$ is defined by the exact sheaf sequence

$$0 \to \mathcal{J} \to \mathcal{E}_M \to \mathcal{E}_M/\mathcal{J} \to 0 \ , \tag{2.13}$$

where $\mathcal{J}$ is the subsheaf of $\mathcal{E}_M$ consisting of germs of smooth functions belonging to the ideal generated by the ϕ_i. Because of our choice of cover, Lemma 2.2 implies that $\mathcal{J}(U)$ agrees, for all $U \in \mathcal{U}$, with those smooth functions vanishing on $U \cap M_o$, and hence we have an isomorphism of sheaves $\mathcal{E}_M/\mathcal{J} \cong \mathcal{E}_{M_o}$, where $\mathcal{E}_{M_o}$ is the sheaf of germs of smooth functions on M_o. Next we notice that $\mathcal{E}_{M_o}$ is a fine sheaf and hence all its Čech cohomology groups vanish except the zeroth one. Thus the E_2 term in this spectral sequence is just

0	0	0	
0	0	0	
$C^\infty(M_o)$	0	0	

Again we see that the higher differentials are automatically zero and the spectral sequence collapses. Since both spectral sequences compute the same cohomology we have the following corollary.

Corollary 2.14. *If 0 is a regular value for $\Phi : M \to \mathbb{R}^k$ the Koszul complex $K(\Phi)$ gives an acyclic resolution for $C^\infty(M_o)$. In other words, the cohomology of the Koszul complex is given by*

$$H^p(K(\Phi)) \cong \begin{cases} 0 & \text{for } p > 0 \\ C^\infty(M_o) & \text{for } p = 0 \end{cases} , \tag{2.15}$$

where $M_o \stackrel{\text{def}}{=} \Phi^{-1}(0)$.

Together with Theorem 2.9 the above corollary implies the isomorphism (1.2). On the other hand, M_o being a closed embedded submanifold, we have the isomorphism (1.1). Hence these two isomorphisms yield, somewhat indirectly, the equality between the ideals $I(M_o)$ and J.

It may appear overkill to use the spectral sequence method to arrive at Corollary 2.14. In fact it is not necessary and the reader is urged to supply a proof using the "tic-tac-toe" methods in **[12]**. This way one gains some valuable intuition on this complex. In particular, one can show that way that the sequence Φ is regular in $C^\infty(M)$ and that $J = I(M_o)$ without having to first prove Corollary 2.14. Lack of spacetime prevents us from exhibiting both computations and the spectral sequence computation is decidedly shorter.

§3 BRST COHOMOLOGY

In this section we complete the construction of the algebraic equivalent of symplectic reduction by first defining a cohomology theory (vertical cohomology) that describes the passage of M_o to $\widetilde{M}$ and then, in keeping with our philosophy of not having to work on M_o, we lift it via the Koszul resolution to a cohomology theory (BRST) which allows us to work with $\widetilde{M}$ from objects defined on M. We shall assume for convenience that the foliation defining $\widetilde{M}$ is such that $\widetilde{M}$ is a smooth manifold and $\pi : M_o \twoheadrightarrow \widetilde{M}$ is a smooth surjection. In other words, the foliation is actually a fibration $M_o \xrightarrow{\pi} \widetilde{M}$ whose fibers are the leaves.

Vertical Cohomology

Since $\widetilde{M}$ is obtained from M_o by collapsing each leaf of the null foliation $\mathcal{F}$ to a point, a smooth function on $\widetilde{M}$ pulls back to a smooth function on M_o which is constant on each leaf. Conversely, any smooth function on M_o which is constant on each leaf defines a smooth function on $\widetilde{M}$. Since the leaves are connected (Frobenius' theorem) a function is constant on the leaves if and only if it is locally constant. Since the hamiltonian vector fields $\{X_i\}$ associated to the constraints $\{\phi_i\}$ form a global basis of the tangent space to the leaves, a function f on M_o is locally constant on the leaves if and only if $X_i f = 0$ for all i. In an effort to build a cohomology theory and in analogy to the de Rham theory, we pick a global basis $\{c^i\}$ for the cotangent space to the leaves such that they are dual to the $\{X_i\}$, *i.e.*, $c^i(X_j) = \delta^i_j$. We then define the vertical derivative d_V on functions as

$$d_V f = \sum_i (X_i f)\, c^i \qquad \forall\, f \in C^\infty(M_o)\ . \tag{3.1}$$

Let $\Omega_V(M_o)$ denote the exterior algebra generated by the $\{c^i\}$ over $C^\infty(M_o)$.

We will refer to them as vertical forms. We can extend d_V to a derivation

$$d_V : \Omega^p_V(M_o) \to \Omega^{p+1}_V(M_o) \tag{3.2}$$

by defining

$$d_V c^i = -\tfrac{1}{2} \sum_{j,k} f_{jk}{}^i \, c^j \wedge c^k \ , \tag{3.3}$$

where the $\{f_{ij}{}^k\}$ are the functions appearing in the Lie bracket of the hamiltonian vector fields associated to the constraints: $[X_i\, ,\, X_j] = \sum_k f_{ij}{}^k\, X_k$; or, equivalently, in the Poisson bracket of the constraints themselves: $[\phi_i\, ,\, \phi_j] = \sum_k f_{ij}{}^k\, \phi_k$.

Notice that the choice of $\{c^i\}$ corresponds to a choice of connection on the fiber bundle $M_o \xrightarrow{\pi} \widetilde{M}$. Let V denote the subbundle of TM_o spanned by the $\{X_i\}$. It can be characterized either as $\ker \pi_*$ or as $TM_o^{\perp}$. A connection is then a choice of complementary subspace H such that $TM_o = V \oplus H$. It is clear that a choice of $\{c^i\}$ implies a choice of H since we can define $X \in H$ if and only if $c^i(X) = 0$ for all i. If we let pr_V denote the projection $TM_o \to V$ it is then clear that acting on vertical forms, $d_V = \mathrm{pr}_V^* \circ d$, where d is the usual exterior derivative on M_o.

It follows therefore that $d_V^2 = 0$. We call its cohomology the vertical cohomology and we denote it as $H_V(M_o)$. As has been shown elsewhere [**13**], it can be computed in terms of the de Rham cohomology of the typical fiber in the fibration $M_o \xrightarrow{\pi} \widetilde{M}$. In particular, from its definition, we already have that

$$H^0_V(M_o) \cong C^\infty(\widetilde{M}) \ . \tag{3.4}$$

<u>The BRST Complex</u>

However this is not the end of the story since we don't want to have to work on M_o but on M. The results of the previous section suggest that we use the Koszul construction. Notice that $\Omega_V(M_o)$ is isomorphic to $\bigwedge \mathbb{R}^k \otimes C^\infty(M_o)$ where $\mathbb{R}^k$ has basis $\{c^i\}$. The Koszul complex gives a resolution for $C^\infty(M_o)$. Therefore extending the Koszul differential as the identity on $\bigwedge \mathbb{R}^k$ we get a resolution for $\Omega_V(M_o)$. We find it convenient to think of $\mathbb{R}^k$ as $\mathbb{V}^*$, whence the resolution of $\Omega_V(M_o)$ is given by

$$\cdots \longrightarrow \bigwedge \mathbb{V}^* \otimes \mathbb{V} \otimes C^\infty(M) \xrightarrow{\mathbf{1}\otimes\delta_K} \bigwedge \mathbb{V}^* \otimes C^\infty(M) \longrightarrow 0 \ . \tag{3.5}$$

This gives rise to a bigraded complex $K = \bigoplus_{c,b} K^{c,b}$, where

$$K^{c,b} \equiv \bigwedge{}^c \mathbb{V}^* \otimes \bigwedge{}^b \mathbb{V} \otimes C^\infty(M) \ , \tag{3.6}$$

under the Koszul differential $\delta_K : K^{c,b} \to K^{c,b-1}$. The Koszul cohomology of this bigraded complex is clearly zero for $b > 0$, and for $b = 0$ it is isomorphic to the vertical forms, where the vertical derivative is defined.

The purpose of the BRST construction is to lift the vertical derivative to K. That is, to define a differential δ_1 on K which commutes with the Koszul differential, which induces the vertical derivative upon taking Koszul cohomology, and which obeys $\delta_1^2 = 0$. This would mean that the total differential $D = (-1)^c \delta_K + \delta_1$ would obey $D^2 = 0$ acting on K and its cohomology would be isomorphic to the vertical cohomology. This is possible only in the case of a group action, *i.e.*, when the linear span of the constraints closes under Poisson bracket. In general this is not possible and we will be forced to add further δ_i's to D to ensure $D^2 = 0$.

We find it convenient to define $\delta_0 = (-1)^c \delta_K$ on $K^{b,c}$. We define δ_1 on functions and $\{c^i\}$ as the vertical derivative[1]

$$\begin{aligned} \delta_1 f &= \sum_i (X_i f)\, c^i \\ &= \sum_i [\phi_i\, ,\, f]\; c^i \end{aligned} \tag{3.7}$$

and

$$\delta_1 c^i = -\tfrac{1}{2} \sum_{j,k} {f_{jk}}^i\, c^j \wedge c^k\ . \tag{3.8}$$

We can then extend it as a derivation to all of $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$. Notice that it trivially anticommutes with δ_0 since it stabilizes $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$ where δ_0 acts trivially. We now define it on antighosts in such a way that it commutes with δ_0 everywhere. This does not define it uniquely but a convenient choice is

$$\delta_1 e_i = \sum_{j,k} {f_{kj}}^i\, \omega^j \wedge e_k\ . \tag{3.9}$$

Notice that $\delta_1^2 \neq 0$ in general, although it does in the case where the ${f_{ij}}^k$ are constant. However since it anticommutes with δ_0 it does induce a map in δ_0 (*i.e.*, Koszul) cohomology which precisely agrees with the vertical derivative d_V, which does obey $d_V^2 = 0$. Hence δ_1^2 induces the zero map in Koszul cohomology. This is enough (see algebraic lemma below) to deduce the existence of a derivation $\delta_2 : K^{c,b} \to K^{c+2,b+1}$ such that $\delta_1^2 + \{\delta_0\, ,\, \delta_2\} = 0$,

[1] Notice that the vertical derivative is defined on M_o and hence has no unique extension to M. The choice we make is the simplest and the one that, in the case of a group action, corresponds to the Lie algebra coboundary operator.

where $\{\,,\,\}$ denotes the anticommutator. This suggests that we define $D_2 = \delta_0 + \delta_1 + \delta_2$. We see that

$$D_2^2 = \delta_0^2 \oplus \{\delta_0\,,\,\delta_1\} \oplus (\delta_1^2 + \{\delta_0\,,\,\delta_2\}) \oplus \{\delta_1\,,\,\delta_2\} \oplus \delta_2^2\ , \tag{3.10}$$

where we have separated it in terms of different bidegree and arranged them in increasing c-degree. The first three terms are zero but, in general, the other two will not vanish. The idea behind the BRST construction is to keep defining higher $\delta_i : K^{c,b} \to K^{c+i,b+i-1}$ such that their partial sums $D_i = \delta_0 + \cdots + \delta_i$ are nilpotent up to terms of higher and higher c-degree until eventually $D_k^2 = 0$. The proof of this statement will follow by induction from the acyclicity of the Koszul complex, but first we need to introduce some notation that will help us organize the information.

Let us define $F^p K = \bigoplus_{c\geq p} \bigoplus_b K^{c,b}$. Then $K = F^0 K \supseteq F^1 K \supseteq \cdots$ is a filtration of K. Let $\operatorname{Der} K$ denote the derivations (with respect to the $\wedge$ product) of K. We say that a derivation has bidegree (i,j) if it maps $K^{c,b} \to K^{c+i,b+j}$. $\operatorname{Der} K$ is naturally bigraded

$$\operatorname{Der} K = \bigoplus_{i,j} \operatorname{Der}^{i,j} K\ , \tag{3.11}$$

where $\operatorname{Der}^{i,j} K$ consists of derivations of bidegree (i,j). This decomposition makes $\operatorname{Der} K$ into a bigraded Lie superalgebra under the graded commutator:

$$[\cdot\,,\,\cdot] : \operatorname{Der}^{i,j} K \times \operatorname{Der}^{k,l} K \to \operatorname{Der}^{i+k,j+l} K\ . \tag{3.12}$$

We define $F^p\operatorname{Der} K = \bigoplus_{i\geq p} \bigoplus_j \operatorname{Der}^{i,j} K$. Then $F\operatorname{Der} K$ gives a filtration of $\operatorname{Der} K$ associated to the filtration $F\,K$ of K.

The remarks immediately following (3.10) imply that $D_2^2 \in F^3\operatorname{Der} K$. Moreover, it is trivial to check that $\left[\delta_0\,,\,D_2^2\right] \in F^4\operatorname{Der} K$. In fact,

$$\left[\delta_0\,,\,D_2^2\right] = \left[D_2\,,\,D_2^2\right] - \left[\delta_1\,,\,D_2^2\right] - \left[\delta_2\,,\,D_2^2\right] \tag{3.13}$$

where the first term vanishes because of the Jacobi identity and the last two terms are clearly in $F^4\operatorname{Der} K$. Therefore the part of D_2^2 in $F^3\operatorname{Der} K / F^4\operatorname{Der} K$ is a δ_0-chain map: that is, $[\delta_0\,,\,\{\delta_1\,,\,\delta_2\}] = 0$. Since it has non-zero b-degree, the quasi-acyclicity of the Koszul complex implies that it induces the zero map in Koszul cohomology. By the following algebraic lemma (see below), there exists a derivation δ_3 of bidegree $(3,2)$ such that $\{\delta_0\,,\,\delta_3\} + \{\delta_1\,,\,\delta_2\} = 0$. If we define $D_3 = \sum_{i=0}^3 \delta_i$, this is equivalent to $D_3^2 \in F^4\operatorname{Der} K$. But by arguments identical to the ones above we deduce that $\left[\delta_0\,,\,D_3^2\right] \in F^5 \operatorname{Der} K$, and so on. It is not difficult to formalize these arguments into an induction proof of the following theorem:

THEOREM 3.14. *We can define a derivation $D = \sum_{i=0}^{k} \delta_i$ on K, where δ_i are derivations of bidegree $(i, i-1)$, such that $D^2 = 0$.*

Finally we come to the proof of the algebraic lemma used above.

LEMMA 3.15. *Let*

$$\cdots \longrightarrow K_2 \xrightarrow{\delta_0} K_1 \xrightarrow{\delta_0} K_0 \to 0 \tag{3.16}$$

denote the Koszul complex where $K_b = \bigoplus_c K^{c,b}$. Let $d : K_b \to K_{b+i}$, $(i \geq 0)$ be a derivation which commutes with δ_0 and which induces the zero map on cohomology. Then there exists a derivation $K : K_b \to K_{b+i+1}$ such that $d = \{\delta_0 \,, K\}$.

PROOF: Since $C^\infty(M)$ is an $\mathbb{R}$-algebra it is, in particular, a vector space. Let $\{f_\alpha\}$ be a basis for it. Then, since $\delta_0 \, f_\alpha = 0$, $\delta_0 \, d \, f_\alpha = 0$. Since d induces the zero map in cohomology, there exists λ_α such that $d \, f_\alpha = \delta_0 \, \lambda_\alpha$. Define $K \, f_\alpha = \lambda_\alpha$. Similarly, since $\delta_0 \, d \, c^i = 0$, there exists μ^i such that $d \, c^i = \delta_0 \, \mu^i$. Define $K \, c^i = \mu^i$. Since $C^\infty(M)$ and the $\{c^i\}$ generate K_0, we can extend K to all of K_0 as a derivation and, by construction, in such a way that on K_0, $d = \{\delta_0 \,, K\}$. Now, $\delta_0 \, d \, b_i = d \, \delta_0 \, b_i$. But since $\delta_0 \, b_i \in K_0$, $\delta_0 \, d \, b_i = \delta_0 \, K \, \delta_0 \, b_i$. Therefore $\delta_0 \, (d \, b_i - K \, \delta_0 \, b_i) = 0$. Since $d \, b_i \in K^{i+1}$ for some $i \geq 0$, the quasi-acyclicity of the Koszul complex implies that there exists ξ_i such that $d \, b_i - K \, \delta_0 \, b_i = \delta_0 \, \xi_i$. Define $K \, b_i = \xi_i$. Therefore, $d \, b_i = \{\delta_0 \,, K\} \, b_i$. We can now extend K as a derivation to all of K. Since d and $\{\delta_0 \,, K\}$ are both derivations and they agree on generators, they are equal. $\square$

Defining the total complex $K = \bigoplus_n K^n$, where $K^n = \bigoplus_{c-b=n} K^{c,b}$, we see that $D : K^n \to K^{n+1}$. Its cohomology is therefore graded, that is, $H_D = \bigoplus_n H_D^n$. Notice that since all terms in D have non-negative filtration degree with respect to $F\,K$, there exists a spectral sequence associated to this filtration which converges to the cohomology of D. The E_1 term is the cohomology of the associated graded object $\mathrm{Gr}^p K \equiv F^p K / F^{p+1} K$, with respect to the induced differential. The induced differential is the part of D of c-degree 0, that is, δ_0. Therefore the E_1 term is given by

$$E_1^{c,b} \cong \bigwedge{}^c \mathbb{V}^* \otimes H^b(K(\Phi)) \; . \tag{3.17}$$

That is, $E_1^{c,0} \cong \Omega_V^c(M_o)$ and $E_1^{c,b>0} = 0$.

The E_2 term is the cohomology of E_1 with respect to the induced differential d_1. Tracking down the definitions we see that d_1 is induced by δ_1 and hence it is just the vertical derivative d_V. Therefore, $E_2^{c,0} \cong H_V^c(M_0)$ and $E_2^{c,b>0} = 0$. Notice, however, that the spectral sequence is degenerate at this term, since the higher differentials $d_2, d_3, \ldots$ all have b-degree different from zero. Therefore we have proven the following theorem:

THEOREM 3.18. *The BRST cohomology is given by*

$$H_D^n \cong \begin{cases} 0 & \text{for } n < 0 \\ H_V^n(M_o) & \text{for } n \geq 0 \end{cases} . \tag{3.19}$$

In particular, $H_D^0 \cong C^\infty(\widetilde{M})$.

There is one important detail that we have left for the end. The construction of the BRST complex depends on the choice of constraints functions $\{\phi_i\}$. However what determines the dynamics is the constraint submanifold M_o itself. One should therefore expect that choosing other constraints $\{\phi'_i\}$ defining the same M_o we would obtain the same cohomology. One could go ahead and show this directly by proving as in [**4**] that there is a chain homotopy relating the BRST complexes constructed from different choices of constraints. However we have avoided this by first characterizing the BRST cohomology purely in terms of M_o[2]. Hence the answer justifies the construction and shows its uniqueness.

§4 POISSON STRUCTURE OF BRST COHOMOLOGY

So far in the construction of the BRST complex no use has been made of the Poisson structure of the smooth functions on M. In this section we remedy the situation. It turns out that the complex K introduced in the last section is a Poisson superalgebra and its cohomology can be computed by a differential which is also an inner Poisson derivation. It will then follow that in cohomology all constructions based on the Poisson structures will be preserved. So let us first review the concepts associated to Poisson algebras and then define the relevant Poisson structures in K and explore its consequences.

Poisson Superalgebras and Poisson Derivations

Recall that a Poisson superalgebra is a $\mathbb{Z}_2$-graded vector space $P = P_0 \oplus P_1$ together with two bilinear operations preserving the grading:

$$\begin{aligned} P \times P &\to P \\ (a,b) &\mapsto ab \end{aligned} \qquad \text{(multiplication)}$$

and

$$\begin{aligned} P \times P &\to P \\ (a,b) &\mapsto [a\,,\,b] \end{aligned} \qquad \text{(Poisson bracket)}$$

obeying the following properties

[2] The BRST cohomology depends manifestly on M_o and on the foliation $\mathcal{F}$; but the foliation is determined by M_o. In fact it is the characteristic (null) foliation of $i^*\Omega$, the pull-back of the symplectic form via the inclusion $i : M_o \to M$.

(P1) P is an associative supercommutative superalgebra under multiplication:

$$\begin{aligned} a(bc) &= (ab)c \\ ab &= (-1)^{|a||b|}\, ba \ ; \end{aligned}$$

(P2) P is a Lie superalgebra under Poisson bracket:

$$\begin{aligned} [a\,,\, b] &= (-1)^{|a||b|}\ [b\,,\, a] \\ [a\,,\, [b\,,\, c]] &= [[a\,,\, b]\ ,\, c] + (-1)^{|a||b|}\ [b\,,\, [a\,,\, c]] \ ; \end{aligned}$$

(P3) Poisson bracket is a derivation over multiplication:

$$[a\,,\, bc] = [a\,,\, b]\, c + (-1)^{|a||b|}\, b\, [a\,,\, c] \ ;$$

for all $a, b, c \in P$ and where $|a|$ equals 0 or 1 according to whether a is even or odd, respectively.

The algebra $C^\infty(M)$ of smooth functions of a symplectic manifold (M, Ω) is clearly an example of a Poisson superalgebra where $C^\infty(M)_1 = 0$. On the other hand, if $\mathbb{V}$ is a finite dimensional vector space and $\mathbb{V}^*$ its dual, then the exterior algebra $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ posseses a Poisson superalgebra structure. The associative multiplication is given by exterior multiplication $(\wedge)$ and the Poisson bracket is defined for $u, v \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{V}^*$ by

$$[\alpha\,,\, v] = \langle \alpha, v\rangle \qquad [v\,,\, w] = 0 = [\alpha\,,\, \beta] \ , \tag{4.1}$$

where $\langle , \rangle$ is the dual pairing between $\mathbb{V}$ and $\mathbb{V}^*$. We then extend it to all of $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ as an odd derivation. In **[3]** it is shown that this Poisson bracket is induced from the supercommutator in the Clifford algebra $\mathrm{Cl}\,(\mathbb{V} \oplus \mathbb{V}^*)$ with respect to the non-degenerate inner product on $\mathbb{V} \oplus \mathbb{V}^*$ induced by the dual pairing.

To show that K is a Poisson superalgebra we need to discuss tensor products. Given two Poisson superalgebras P and Q, their tensor product $P \otimes Q$ can be given the structure of a Poisson superalgebra as follows. For $a, b \in P$ and $u, v \in Q$ we define

$$(a \otimes u)(b \otimes v) = (-1)^{|u||b|}\, ab \otimes uv \tag{4.2}$$

$$[a \otimes u\,,\, b \otimes v] = (-1)^{|u||b|}\ ([a\,,\, b] \otimes uv + ab \otimes [u\,,\, v]) \ . \tag{4.3}$$

The reader is invited to verify that with these definitions (P1)-(P3) are satisfied. From this it follows that $K = C^\infty(M) \otimes \bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ becomes a Poisson superalgebra which can, in fact, be interpreted as the algebra of functions of a smooth symplectic supermanifold. We will not, however, pursue this interpretation in this article.

Now let P be a Poisson superalgebra which, in addition, is $\mathbb{Z}$-graded, that is, $P = \bigoplus_n P^n$ and $P^n P^m \subseteq P^{m+n}$ and $[P^n \, , \, P^m] \subseteq P^{m+n}$; and such that the $\mathbb{Z}_2$-grading is the reduction modulo 2 of the $\mathbb{Z}$-grading, that is, $P_0 = \bigoplus_n P^{2n}$ and $P_1 = \bigoplus_n P^{2n+1}$. We call such an algebra a graded Poisson superalgebra. Notice that P^0 is an even Poisson subalgebra of P.

For example, letting $K = C^\infty(M) \otimes \bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ we can define $K^n = \bigoplus_{c-b=n} K^{c,b}$. This way K becomes a $\mathbb{Z}$-graded Poisson superalgebra. Although the bigrading is preserved by the exterior product, the Poisson bracket does not preserve it. In fact, the Poisson bracket obeys

$$[\, , \,] : K^{i,j} \times K^{k,l} \to K^{i+k,j+l} \oplus K^{i+k-1,j+l-1} \; . \tag{4.4}$$

By a Poisson derivation of degree k we will mean a linear map $D : P^n \to P^{n+k}$ such that

$$D(ab) = (Da)b + (-1)^{k|a|}\, a(Db) \tag{4.5}$$

$$D\,[a\, , \, b] = [Da\, , \, b] + (-1)^{k|a|}\; [a\, , \, Db] \; . \tag{4.6}$$

The map $a \mapsto [Q\, , \, a]$ for some $Q \in P^k$ automatically obeys (4.5) and (4.6). Such Poisson derivations are called inner.

<u>The BRST Differential as a Poisson Derivation</u>

The BRST differential D constructed in the previous section is a derivation over the exterior product. Nothing in the way it was defined guarantees that it is a Poisson derivation and, in fact, it need not be so. However one can show that the δ_i's — which were, by far, not unique — can be defined in such a way that the resulting differential is an inner Poisson derivation. In fact, what we will show, is the existence of an element $Q \in K^1$ such that $[Q\, , \cdot]$ computes the cohomology of the BRST complex. We will show that there exists $Q = \sum_{i \geq 0} Q_i$, where $Q_i \in K^{i+1,i}$, such that $[Q\, , \, Q] = 0$ and that the cohomology of the operator $[Q\, , \cdot]$ is isomorphic to that of D. This was first proven by Henneaux in **[5]** and later in a completely algebraic way by Stasheff in **[7]**. Our proof is a simplified version of this latter proof.

From the discussion previous to Theorem 3.18 we know that the only parts of D which affect its cohomology are δ_0, which is the Koszul differential, and δ_1 acting on the Koszul cohomology. Hence we need only make sure that the Q_i we construct realize these differentials. Notice that if $Q_i \in K^{i+1,i}$, $[Q_i\, , \cdot]$ has terms of two different bidegrees $(i+1, i)$ and $(i, i-1)$. Hence the only term which can contribute to the Koszul differential is Q_0. There is a unique element $Q_0 \in K^{1,0}$ such that $[Q_0\,,\,b_i] = \delta_0\, b_i = \phi_i$. This is given by

$$Q_0 = \sum_i c^i\, \phi_i\ . \tag{4.7}$$

Notice that

$$\begin{aligned} [Q_0\,,\,b_i] &= \delta_0\, b_i = \phi_i && (4.8)\\ \left[Q_0\,,\,c^i\right] &= \delta_0\, c^i = 0 && (4.9)\\ [Q_0\,,\,f] &= (\delta_0 + \delta_1)\, f = \sum_i\, [\phi_i\,,\,f]\; c^i\ . && (4.10) \end{aligned}$$

There is now a unique $Q_1 \in K^{2,1}$ such that $\left[Q_1\,,\,c^i\right] = \delta_1\, c^i$, namely,

$$Q_1 = -\tfrac{1}{2} \sum_{i,j,k} f_{ij}{}^k\, c^i \wedge c^j \wedge b_k\ . \tag{4.11}$$

If we define $R_1 = Q_0 + Q_1$ we then have that

$$\begin{aligned} [R_1\,,\,b_i] &= (\delta_0 + \delta_1)\, b_i && (4.12)\\ \left[R_1\,,\,c^i\right] &= (\delta_0 + \delta_1)\, c^i && (4.13)\\ [R_1\,,\,f] &= (\delta_0 + \delta_1 + \delta_2)\, f\ . && (4.14) \end{aligned}$$

In particular, two things are imposed upon us: $\delta_2\, f$ and $\delta_1\, b_i$; the latter imposition agrees with the choice made in (3.9).

Letting FK denote the filtration of K defined in the previous section, and using the notation in which, if $O \in K$ is an odd element, O^2 stands for $\frac{1}{2}\,[O\,,\,O]$, the following are satisfied:

$$R_1^2 \in F^3K \qquad \text{and} \qquad \left[Q_0\,,\,R_1^2\right] \in F^4K\ . \tag{4.15}$$

That means that the part of R_1^2 which lives in F^3K/F^4K is a δ_0-cocycle, since the $(0,-1)$ part of Q_0 is precisely δ_0. By the acyclicity of the Koszul complex it is a coboundary, say, $-\delta_0\, Q_2$ for some $Q_2 \in K^{3,2}$. In other words, there exists $Q_2 \in K^{3,2}$ such that if $R_2 = Q_0 + Q_1 + Q_2$, then $R_2^2 \in F^4K$. If this is the case then

$$\left[Q_0\,,\,R_2^2\right] = \left[R_2\,,\,R_2^2\right] - \left[Q_1\,,\,R_2^2\right] - \left[Q_2\,,\,R_2^2\right]\ . \tag{4.16}$$

But the first term is zero because of the Jacobi identity and the last two terms are clearly in F^5K due to the fact that, from (4.4),

$$[F^pK\,,\,F^qK] \subseteq F^{p+q-1}K\ . \tag{4.17}$$

Hence, $\left[Q_0\,,\,R_2^2\right] \in F^5K$, from where we can deduce the existence of $Q_3 \in K^{4,3}$ such that $R_3 = Q_0 + Q_1 + Q_2 + Q_3$ obeys $R_3^2 \in F^5K$, and so on. It is easy to formalize this into an induction proof of the following theorem:

THEOREM 4.18. *There exists* $Q = \sum_i Q_i$, *where* $Q_i \in K^{i+1,i}$ *such that* $[Q\,,\,Q] = 0$.

Now let $D = [Q\,,\,\cdot]$. Then $D^2 = 0$ and repeating the proof of Theorem 3.18 we obtain the following.

THEOREM 4.19. *The cohomology of* D *is given by*

$$H^n_D \cong \begin{cases} 0 & \text{for } n < 0 \\ H^n_V(M_o) & \text{for } n \geq 0 \end{cases} . \tag{4.20}$$

In particular, $H^0_D \cong C^\infty(\widetilde{M})$.

With this choice of D it is easy to verify that $\ker D$ becomes a Poisson subalgebra of K and $\operatorname{im} D$ is a Poisson ideal of $\ker D$. Therefore the cohomology space $H_D = \ker D / \operatorname{im} D$ naturally inherits the structure of a Poisson superalgebra. Moreover since K is a graded Poisson superalgebra and D is homogeneous with respect to this grading, the cohomology naturally becomes a graded Poisson superalgebra. In particular, H^0_D is a Poisson subalgebra and, since H^0_D is isomorphic to $C^\infty(\widetilde{M})$, we see that the Poisson brackets get induced. Therefore if we wished to compute the Poisson brackets of two smooth functions on $\widetilde{M}$ we merely need to find suitable BRST cocycles representing them and compute their Poisson bracket in K.

Explicit Isomorphism with Vertical Cohomology

To illustrate this, we now give the explicit isomorphism as Poisson algebras between BRST cohomology and the cohomology of vertical differential forms. Throughout this section, we shall identify the space of vertical differential forms $\Omega^c_V(M_o)$ with $H^{c,0}_{\delta_0}$ where H_{δ_0} is defined to be the the cohomology of the differential complex

$$\cdots \longrightarrow K^{c,b+1} \xrightarrow{\delta_0} K^{c,b} \xrightarrow{\delta_0} \cdots \; . \tag{4.21}$$

Furthermore, the vertical derivative d_V is the derivative induced by δ_1 on H_{δ_0}.

PROPOSITION 4.22. *Let* $\chi : H_D(K) \to H_V(M_o)$ *be defined via*

$$[x] \mapsto \begin{cases} \mathbf{0} & \text{for } g < 0 \\ [x'_0] & \text{for } g \geq 0 \end{cases}$$

where $[x] \in H^g_D(K)$, x_i *is the component of* x *in* $K^{g+i,i}$ *for* $g, i \geq 0$, *and* x'_0, *when thought of as an element in* H_{δ_0}, *is identified with the equivalence class* $\langle x_0 \rangle \in H_{\delta_0}$. *In this case,* $H_V(M_o)$ *can be endowed with the structure of a Poisson superalgebra such that* χ *is an isomorphism of Poisson superalgebras.*

PROOF: We first check the map is well defined. Consider any $y \in K^g$ for $g \geq 0$ then we have the decomposition $y = y_0 + y_1 + \cdots$ where $y_i \in K^{g+i,i}$ for all $i \geq 0$. In this case, Dy belongs to K^{g+1} and the component of Dy in $K^{g+1,0}$ is given by $\delta_1 \, y_0 + \delta_0 \, y_1$. Therefore, we see that

$$\begin{aligned}\chi([Dy]) = [\langle \delta_1 \, y_0 + \delta_0 \, y_1 \rangle] \\ &= [\langle \delta_1 \, y_0 \rangle] \\ &= [d_V \, \langle y_0 \rangle] \\ &= 0\end{aligned}$$

and therefore, the map is well-defined.

Before we prove that this map is an isomorphism we shall need the following result. Let's decompose the BRST differential as $D = \sum_{i=1}^{r} \delta_i$ where $\delta_i : K^{c,b} \to K^{c+i,b+i-1}$. Then the fact that $D^2 = 0$ is equivalent to a string of identities

$$\sum_{i=1}^{p} \delta_i \delta_{p-i} = 0 \ , \tag{4.23}$$

for each p and where δ_i is conventionally set to zero for $i > r+1$.

We now proceed to prove that the map is injective. Consider $x \in K^g$ for $g \geq 0$ which decomposes into $x = x_0 + x_1 + x_2 + \ldots$ where $x_i \in K^{g+i,i}$ such that $\chi([x]) = [\langle x_0 \rangle] = 0$. We need to show that in this case, there exists $y \in K^{g-1}$ such that $x = Dy$. Decomposing the previous equation, this is equivalent to asserting that there exists $y_i \in K^{g-1+i,i}$ such that for all $p \geq 0$,

$$x_p = \sum_{i=0}^{p+1} \delta_i \, y_{p+1-i}. \tag{4.24}$$

Now, $[\langle x_0 \rangle] = 0$ means that $\langle x_0 \rangle = d_V \, \langle y_0 \rangle$ for some $y_0 \in K^{g-1,0}$ or $\langle x_0 - \delta_1 \, y_0 \rangle = 0$. Therefore, $x_0 - \delta_1 \, y_0 - \delta_0 \, y_1 = 0$ for some $y_1 \in K^{g,1}$. In other words,

$$x_0 = \delta_0 \, y_1 + \delta_1 \, y_0 \tag{4.25}$$

for some y_0 and y_1. This is just (4.24) for $p = 0$. We now need to show that y_p exists for all $p > 1$ which satisfies (4.24). We now proceed to show this by induction. Suppose that there exist $y_i \in K^{g-1+i,i}$ for all $i = 0, \ldots, r$ which satisfy (4.24) for all $p = 0, \ldots, r-1$. In this case, observe that

$$\delta_0 \, (x_r - \sum_{j=1}^{r+1} \delta_j \, y_{r+1-j}) = \delta_0 \, x_r - \sum_{j=1}^{r+1} \delta_0 \, \delta_j \, y_{r+1-j}$$

$$
\begin{aligned}
&= \delta_0\, x_r + \sum_{j=1}^{r+1} \sum_{l=1}^{j} \delta_l\, \delta_{j-l}\, y_{r+1-j} \quad \text{using (4.23)} \\
&= \delta_0\, x_r + \sum_{l=1}^{r+1} \sum_{j=l}^{r+1} \delta_l\, \delta_{j-l}\, y_{r+1-j} \\
&= \delta_0\, x_r + \sum_{l=1}^{r+1} \delta_l \sum_{s=0}^{r+1-l} \delta_s\, y_{r+1-s-l} \\
&= \delta_0 x_r + \sum_{l=1}^{r+1} \delta_l x_{r-l} \text{ by induction hypothesis} \\
&= \sum_{l=0}^{r+1} \delta_l\, x_{r-l} = 0
\end{aligned}
$$

where we have used the fact that x is a D-cocycle in the last step. Using the quasiacyclicity of the Koszul complex, this implies that there exists an element $y_{r+1} \in K^{g+r,r+1}$ which satisfies

$$
x_r = \sum_{j=0}^{r+1} \delta_j\, y_{r+1-j}\, .
$$

This completes our induction and, therefore, χ is injective.

Finally we prove that the map is surjective. Consider any $[\langle y \rangle] \in H^g_V(M_o)$ where $g \geq 0$. Decomposing y into the sum of $y_i \in K^{g+i,i}$ for $i \geq 0$, we see that since y is a Koszul cycle, this implies that each y_i is a Koszul cycle. However, by the quasiacyclicity of the Koszul complex, this implies that y_i is a Koszul boundary for $i \geq 1$. Therefore, we see that $[\langle y \rangle] = [\langle y_0 \rangle]$. We shall now construct an $x \in K^g$ which decomposes into the sum of $x_i \in K^{g+i,i}$ such that $x_0 = y_0$ and x is a D-cocycle. The latter condition is equivalent to imposing that

$$
\sum_{i=0}^{p} \delta_i\, x_{p-i} = 0
$$

for all $p \geq 0$. First of all, it is clear that $\delta_0\, x_0 = 0$. We now proceed by induction. Suppose that there exists $x_0, \ldots, x_r$ such that

$$
\sum_{i=0}^{p} \delta_i\, x_{p-i} = 0
$$

for all $p = 0, \ldots, r$. In this case, observe that

$$
\delta_0 \sum_{i=1}^{r+1} \delta_i\, x_{r+1-i} = \sum_{i=1}^{r+1} \delta_0\, \delta_i\, x_{r+1-i}
$$

$$
\begin{aligned}
&= -\sum_{i=1}^{r+1}\sum_{j=1}^{i} \delta_j\, \delta_{i-j}\, x_{r+1-i} && \text{using (4.23)}\\
&= -\sum_{j=1}^{r+1}\sum_{i=j}^{r+1} \delta_j\, \delta_{i-j}\, x_{r+1-i}\\
&= -\sum_{j=1}^{r+1} \delta_j \sum_{i=j}^{r+1} \delta_{i-j}\, x_{r+1-i}\\
&= -\sum_{j=1}^{r+1} \delta_j \sum_{s=0}^{r+1-j} \delta_s\, x_{r+1-i-j}\\
&= 0
\end{aligned}
$$

where we have used the induction hypothesis in the last step. Once again, by the quasiacyclicity of the Koszul complex, this means that there exists $x_{r+1} \in K^{g+r+1,r+1}$ such that

$$
\sum_{i=0}^{r+1} \delta_i\, x_{r+1-i} = 0.
$$

This complete our induction and, therefore, χ is surjective.

Furthermore, it is clear that χ is an isomorphism of associative algebras since if $x \in K^g$ and $y \in K^h$ which decompose into sums of $x_i \in K^{g+i,i}$ and $y_i \in K^{h+i,i}$, respectively, then

$$
\begin{aligned}
\chi([x]\,[y]) &= \chi([xy])\\
&= [\langle (xy)_0 \rangle]\\
&= [\langle x_0 y_0 \rangle]\\
&= [\langle x_0 \rangle]\,[\langle y_0 \rangle]\\
&= \chi([x])\chi([y])\,.
\end{aligned}
$$

Finally, we can give $H_V(M_o)$ the structure of a Poisson superalgebra by defining the Poisson bracket on elements $x, y \in H_V(M_o)$ by

$$
[x\,,\,y] = \chi(\big[\chi^{-1}(x)\,,\,\chi^{-1}(y)\big]).
$$

$\square$

The Case of a Group Action

Finally, since the case when the constraints arise from a moment map is of special interest, it is worth looking at it in some more detail. We will be able to relate the BRST cohomology with a Lie algebra cohomology group with coefficients in an infinite dimensional (differential) representation.

So let G be a Lie group and $\mathfrak{g}$ its Lie algebra and let there be a Poisson action of G on M giving rise to an equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Let $\{b_i\}$ be a basis for $\mathfrak{g}$ and $\{c^i\}$ be the canonical dual basis for $\mathfrak{g}^*$. Notice that the dual of the moment map gives rise to a map $\mathfrak{g} \to C^\infty(M)$ sending $b_i \mapsto \phi_i$, where ϕ_i are the coefficients of the moment map relative to the $\{c^i\}$:

$$\langle \Phi(m), b_i \rangle = \phi_i(m) \ , \tag{4.26}$$

which is precisely the map δ_K in the Koszul complex. In particular, we can identify $\mathbb{V}$ with $\mathfrak{g}$. Since the action is Poisson, the functions $\{\phi_i\}$ represent the algebra under the Poisson bracket: $[\phi_i\, , \phi_j] = \sum_k {f_{ij}}^k \phi_k$, where the ${f_{ij}}^k$ are the structure constants of $\mathfrak{g}$ in the chosen basis. Let $Q = Q_0 + Q_1$ where Q_0 and Q_1 are given by (4.7) and (4.11), respectively. Since the ${f_{ij}}^k$ are constant and satisfy the Jacobi identity, $[Q\, , Q] = 0$, and hence the extra $Q_{i>1}$ are not necessary. Hence

$$Q = \sum_i c^i\, \phi_i - \tfrac{1}{2} \sum_{i,j,k} {f_{ij}}^k\, c^j \wedge c^k \wedge b_i \ . \tag{4.27}$$

Notice that this is precisely the operator found in [**1**].

We can now make contact with Lie algebra cohomology. The BRST cohomology is exactly the cohomology of the vertical derivative which is computed by the complex C defined by

$$C^\infty(M_o) \xrightarrow{D} \mathfrak{g}^* \otimes C^\infty(M_o) \xrightarrow{D} \bigwedge^2 \mathfrak{g}^* \otimes C^\infty(M_o) \xrightarrow{D} \cdots \ , \tag{4.28}$$

where D is defined on the generators by

$$\begin{aligned} Df &= \sum_i c^i \otimes [\phi_i\, , f] \\ Dc^i &= -\tfrac{1}{2} \sum_{j,k} {f_{ij}}^k\, c^j \wedge c^k \ . \end{aligned}$$

Hence, for the case of a Poisson group action, the classical Lie algebra cohomology is just the Lie algebra cohomology $H(\mathfrak{g}; C^\infty(M_o))$. Furthermore, as shown in [**13**], if the action of G is free and proper, this is isomorphic to $C^\infty(\widetilde{M}) \otimes H_{\mathrm{dR}}(G)$, where $H_{\mathrm{dR}}(G)$ is the real de Rham cohomology of the Lie group G.

§5 OUTLOOK AND SOME FURTHER RESULTS

In this paper we have described symplectic reduction from a homological point of view. Although the motivation stemmed from perhaps the simplest case of symplectic reduction—namely when $\widetilde{M}$ is indeed a symplectic manifold—the construction goes through even in the singular case. In fact, perhaps the most encouraging feature of this approach is that it allows one to treat the singular case at the same level as the non-singular one. In that sense it accomplishes the same thing as the formalism of **[8]** with the important difference that the cohomological description is easier to work with than the normalizer $N(J)$. This is particularly evident when it comes to the quantization problem. In fact, the quantization of the reduced system is no harder than the quantization of the original system $C^\infty(M)$, since $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ is straightforward to quantize: one merely passes to the Clifford algebra $\mathrm{Cl}\,(\mathbb{V} \oplus \mathbb{V}^*)$ as described, for example, in **[3]**.

The quantization of constrained systems is, at least from the point of view of physics, the area where BRST cohomology plays its most relevant role. But, although it has been used successfully in many different systems, it is still on a rather shaky logical basis. In particular, there are no definite results investigating when the BRST quantization of a symplectic manifold agrees with the quantization of its symplectic reduction. The formalism described in this paper opens the possibility of answering this question by exploiting the natural Poisson structure of BRST cohomology. It is natural to expect that any quantization scheme that is based purely on the Poisson structure of $C^\infty(M)$ can be extended to quantize the BRST cohomology theory. One such method is geometric quantization. In **[14]** and **[15]** we have started to formalize BRST quantization within this framework. In particular we have shown that, at least for finite dimensional systems, prequantization and reduction commute. Another quantization method that exploits the Poisson structure is the method of Poisson deformations. It would be interesting to define a BRST quantization in this way. A most desirable consequence of this approach would be to state the conditions under which the deformed Poisson brackets before and after reduction correspond. Another intriguing aspect of quantum BRST cohomology lies not so much in quantization but in the intrinsic quantum description of BRST cohomology in the context of non-commutative differential geometry. The non-commutative versions of all the ingredients for the construction of the BRST cohomology exist. Thus the difficult aspects of such a theory would only lie in its physical interpretation.

It is perhaps worth noticing that the construction in section 2 of a projective resolution for $C^\infty(M_o)$ holds provided M_o is $\Phi^{-1}(\mathbf{0})$ for $\mathbf{0}$ a regular value of a smooth map $\Phi : M \to \mathbb{R}^k$. In particular, in the case where M is symplectic and the components of Φ correspond to second class constraints, this gives a resolution for the functions on the symplectic manifold M_o. It would be very interesting to investigate the possibility of quantizing this resolution, since it may provide a clue as to the covariant quantization of systems of second class constraints without the need to solve for the constraints explicitly as it is usually the case.

On the mathematical side, there are interesting extensions to the case which can be investigated such as the case of infinite-dimensional symplectic manifolds modeled, for instance, on Banach spaces; and to supermanifolds in the sense of Kostant.

Finally, we comment briefly on the extension to the case where the constraints are not regular. In this case the role of the Koszul resolution is played by the Tate resolution [**16**]. Roughly speaking one adds chains to the Koszul complex in order to kill whatever cohomology there is in positive dimension. A BRST cohomology theory can be constructed in this case along similar lines to the regular case. This has recently been done in [**17**]. Choosing constraints appropriately, it contains the case where M_o has a non-trivial normal bundle and therefore cannot be described globally as the zero locus of a collection of smooth functions such that zero is a regular value of this collection.

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