SOME RESULTS IN THE BRST COHOMOLOGY OF THE OPEN BOSONIC STRING

José M. Figueroa-O'Farrill

and

Takashi Kimura

Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840, U. S. A.

ABSTRACT

We apply the techniques developed in a previous paper to the case of the open bosonic string proving some interesting results. We also comment on the relation with Kähler algebra. In particular we show that in the case of the open bosonic string — and hence in the NSR string — there is no positive definite inner product compatible with the natural Kähler structure.

§1 INTRODUCTION

This paper is in some sense a continuation of a previous paper[1] where we presented a general method of analyzing the cohomology of the BRST operator based on the existence of a particular positive definite inner product in the Fock space. This allowed us to prove a decomposition theorem for the BRST complex and in particular allowed us to identify the BRST cohomology space — which by definition is a subquotient — as an honest subspace of the Fock space, thus identifying a privileged representative of each BRST cohomology class. We called these representatives harmonic states since they are the zero modes of the BRST laplacian. In this paper we use this method to obtain some results on the BRST cohomology of the open bosonic string.

The results are not new in the sense that the BRST cohomology of the open bosonic string has already been explicitly computed so that these results can be checked. However, the proofs themselves are new. And moreover the techniques involved are quite general, transcending the particular theory under study. In fact some of the proofs carry over *mutatis mutandis* to the more general case where the constraints form a Lie algebra with a toral decomposition[2].

In this paper we make repeated use of the vanishing theorem for the BRST cohomology which was proven in [**3**] as a corollary of a general theorem valid for a large class of Lie algebras and representations. In [**1**] we gave a proof that the vanishing theorem is necessary for a consistent BRST quantization but so far we have not been able to prove it directly using the methods advocated there. This, we think, is the fundamental aim of these methods and such a proof would fill a large gap in this field.

Our initial goal at the beginning of this investigation was to find such a proof. In [**3**] a line of attack was suggested based on the formal analogy — discovered in [**4**] — between Kähler algebra and the BRST quantization of the open bosonic string. A explicit calculation shows, however, that there is no positive definite inner product compatible with the natural Kähler structure. We present this result as well.

This paper is organized as follows. In §2 we review the basic facts of the BRST quantization of the open bosonic string. We introduce the reduced BRST operator and prove that its cohomology is intimately related to the cohomology of the full BRST operator. We also introduce some operators which will be seen to be the BRST analogues of operators which arise very naturally in Kähler algebra. In particular we prove — using the vanishing theorem — that the physical states can be taken to be singlets of the Kähler $sl_2\mathbb{C}$ algebra. In §3 we make the correspondence between BRST and Kähler algebra after briefly discussing the relevant background material. We then present the result that there is no positive definite inner product compatible with the natural Kähler structure. In §4 we prove the "no-ghost" theorems for the open bosonic and NSR strings. The case of the open bosonic string was done originally in [**5**] but we include the relevant calculation for completeness since it is needed for the NSR string. Finally §5 contains some concluding remarks.

Throughout this paper, any section or equation number of the form §I.n or (I.n.m) respectively will refer to the corresponding section or equation number of [**1**].

§2 THE OPEN BOSONIC STRING

We follow for the most part the conventions used in [**6**]. We work in the conformal gauge and both the spacetime dimension and the value of the intercept are fixed to their critical values ($D = 26$ and $a = 1$) in order to insure nilpotency of the BRST operator.

The full Fock space of the BRST-quantized open bosonic string is constructed from three types of oscillators: a_n^μ, c_n, and b_n where $n \in \mathbb{Z}$ is the mode number and $\mu = 0 \ldots 25$ is the spacetime index. Since a_0^μ is identified with the momentum of the center of mass of the string we shall call it p^μ whenever it is convenient. The oscillators satisfy the following canonical (anti)commutation relations:

$$[a^\mu_m\,,\, a^\nu_n] = \mathrm{sign}(m)\,\delta_{m,-n}\eta^{\mu\nu} \tag{2.1}$$

$$\{b_n\,,\, c_m\} = \delta_{m,-n}\ . \tag{2.2}$$

All of the other (anti)commutators are zero. Notice that we are using normalized oscillators for the string coordinates. This is purely for computational convenience. It must also be mentioned that our ghost oscillators are not the natural ones but are unitarily related to them. In our conventions the mode expansion of the ghost and antighost fields at $\tau = 0$ are the following:

$$\begin{aligned} b(\sigma) &= b_0 + \sum_{m>0} \sqrt{m}\left(b_m\, e^{im\sigma} + b_{-m}\, e^{-im\sigma}\right) \\ c(\sigma) &= c_0 + \sum_{m>0} \frac{1}{\sqrt{m}}\left(c_m\, e^{im\sigma} + c_{-m}\, e^{-im\sigma}\right)\ . \end{aligned}$$

This seemingly unnatural choice of mode expansion turns out to be the natural one in our context. It will allow us to identify the involution $\mathcal{C}$ of §I.3 with ghost conjugation when acting on ghosts and antighosts. It also makes some of the calculations easier although it introduces some square roots in our expressions.

We now need to specify the Fock vacuum. Since the ghost zero modes b_0 and c_0 (anti)commute with all the other oscillators, we can factor the full Fock space into $\mathcal{F} \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the space upon which the zero modes act and $\mathcal{F}$ is the space upon which all of the other oscillators act. Because the anticommutation relations satisfied by the ghost zero modes define a Clifford algebra, $\mathcal{Z}$ —being an irreducible complex representation — is a two dimensional complex vector space. We may exhibit a basis comprised by the states $|\uparrow\rangle$ and $|\downarrow\rangle$ which satisfy

$$c_0\,|\downarrow\rangle = |\uparrow\rangle \qquad b_0\,|\uparrow\rangle = |\downarrow\rangle \qquad c_0\,|\uparrow\rangle = b_0\,|\downarrow\rangle = 0\ . \tag{2.3}$$

We introduce an inner product in $\mathcal{Z}$ via

$$\langle\uparrow|\downarrow\rangle = 1 \qquad \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0\ , \tag{2.4}$$

such that the ghost zero modes are hermitian:

$$b_0^\dagger = b_0 \qquad c_0^\dagger = c_0 \ . \tag{2.5}$$

This inner product is not positive definite as can be seen by choosing the orthogonal basis

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle) \ , \tag{2.6}$$

which diagonalizes the metric

$$\langle+|+\rangle = 1 \qquad \langle-|-\rangle = -1 \qquad \langle+|-\rangle = 0 \ . \tag{2.7}$$

A vacuum in $\mathcal{F}$, $|k\rangle$, is defined by

$$c_n\,|k\rangle = b_n\,|k\rangle = a_n^\mu\,|k\rangle = 0 \ \forall n > 0 \tag{2.8}$$

$$p^\mu\,|k\rangle = k^\mu\,|k\rangle \ . \tag{2.9}$$

It satisfies the normalization condition

$$\langle k|k\rangle = 1 \ . \tag{2.10}$$

Call the Fock space with $|k\rangle$ for the vacuum, $\mathcal{F}(k)$. The reality conditions on the classical fields induce the following hermiticity properties on the oscillators

$$a_n^{\mu\dagger} = a_{-n}^\mu \qquad b_n^\dagger = b_{-n} \qquad c_n^\dagger = c_{-n} \qquad \forall n \neq 0 \tag{2.11}$$

and

$$p^{\mu\dagger} = p^\mu \ . \tag{2.12}$$

Equations (2.1)–(2.12) define an inner product on $\mathcal{F}(k) \otimes \mathcal{Z}$.

The generators of the Virasoro algebra are given by

$$L_m^{(a)} = p \cdot a_m \sqrt{|m|} + \frac{1}{2} \sum_{n \neq 0} \sqrt{|m-n|\,|n|}\, {:}a_{m-n} \cdot a_n{:} \qquad m \neq 0$$

$$L_0^{(a)} = \frac{1}{2} p^2 + \sum_{n>0} n\, a_n^\dagger \cdot a_n \tag{2.13}$$

and

$$L_m^{(c)} = m \sqrt{|m|} b_m\, c_0 + \sum_{n \neq 0} (m-n) \sqrt{\frac{|m+n|}{|n|}} {:}\, b_{m+n} c_{-n}{:}\ , \tag{2.14}$$

where : : denotes normal ordering with respect to the above vacua. The BRST operator, Q, is given by the following expression:

$$\begin{aligned} Q = & \sum_{n>0} \left(p \cdot a_n\, c_n^\dagger + \text{h. c.} \right) + \sum_{\substack{m>0 \\ n>0}} \sqrt{\frac{(m+n)\, n}{m}} \left(a_{m+n}^\dagger \cdot a_n c_m + \text{h. c.} \right) \\ & + \frac{1}{2} \sum_{\substack{m>0 \\ n>0}} \sqrt{\frac{mn}{m+n}} \left(c_{m+n}^\dagger a_m \cdot a_n + \text{h. c.} \right) \\ & + \frac{1}{2} \sum_{\substack{m>0 \\ n>0}} (m-n) \sqrt{\frac{m+n}{mn}} \left(b_{m+n}^\dagger c_m c_n + \text{h. c.} \right) \\ & - \sum_{\substack{m>0 \\ n>0}} (2m+n) \sqrt{\frac{n}{(m+n)\, m}} \left(c_{m+n}^\dagger c_m b_n + \text{h. c.} \right) \ . \end{aligned} \tag{2.15}$$

Q is manifestly self-adjoint with respect to the above inner product. The dependence of Q on the ghost zero modes can be made manifest in the following way:

$$Q = \mathcal{Q} + c_0 H - 2 b_0 T\ , \tag{2.16}$$

where $\mathcal{Q}$, H, and T do not contain c_0 or b_0. The operator $\mathcal{Q}$ is the reduced BRST operator. The oscillator expressions for H and T are

$$H = L_0^{(a)} + L_0^{(c)} - 1 = \frac{1}{2} p^2 + \mathcal{L} - 1\ , \tag{2.17}$$

where

$$\mathcal{L} = \sum_{n>0} n \left(a_n^\dagger \cdot a_n + c_n^\dagger b_n + b_n^\dagger c_n \right) \ , \tag{2.18}$$

and

$$T = \sum_{n>0} c_n^\dagger c_n \ . \tag{2.19}$$

H and T can be projected out from Q using the relations

$$\{Q\,,\,b_0\} = H \qquad -\frac{1}{2}\{Q\,,\,c_0\} = T \ . \tag{2.20}$$

The nilpotency of Q gives us the relation

$$\mathcal{Q}^2 = 2HT \tag{2.21}$$

and the vanishing of all of the other (anti)commutators between H, T, and $\mathcal{Q}$.

There exists several operators which induce a grading on $\mathcal{F}$. An important one is the reduced ghost number operator

$$\mathcal{G} = \sum_{n>0} (c_n^\dagger b_n - b_n^\dagger c_n) \ , \tag{2.22}$$

which is related to the total ghost number operator $\mathcal{G}_{\text{tot}}$ via the relation

$$\mathcal{G}_{\text{tot}} = \mathcal{G} + \frac{1}{2}[c_0\,,\,b_0] \ . \tag{2.23}$$

The ghost number operator induces a grading of the reduced Fock space as follows

$$\mathcal{F} = \bigoplus_{g\in\mathbb{Z}} \mathcal{F}_g \ , \tag{2.24}$$

where $\psi \in \mathcal{F}_g \Leftrightarrow \mathcal{G}\,\psi = g\,\psi$. A finer structure reveals itself if we keep track of the ghost and antighost numbers separately. This allows us to decompose $\mathcal{F}_g$ further according to the following bigrading

$$\mathcal{F}_g = \bigoplus_{c-b=g} \mathcal{F}_{(b,c)} \ , \tag{2.25}$$

where $\mathcal{F}_{(b,c)}$ consists of states created from the vacuum by operators consisting of c ghost and b antighost creation operators.

Relative to this bigrading the reduced BRST operator splits as a sum of two operators

$$Q = Q' + Q'' \ , \tag{2.26}$$

defined uniquely by

$$Q' \ : \mathcal{F}_{(b,c)} \to \mathcal{F}_{(b,c+1)} \qquad\qquad Q'' \ : \mathcal{F}_{(b,c)} \to \mathcal{F}_{(b-1,c)} \ . \tag{2.27}$$

For reasons that will become obvious when we discuss Kähler algebra we shall refer to this decomposition of Q as a "holomorphic" split; and Q' (resp. Q'') will be referred to as the "holomorphic" (resp. "antiholomorphic") piece of the reduced BRST operator.

Finally, p^μ and $\mathcal{L}$ are mutually commuting operators which in turn commute with Q, $\mathcal{G}$, and $\mathcal{G}_{\rm tot}$ allowing us to decompose the full Fock space (and, in particular, $\mathcal{F}$) as direct sums of finite dimensional subspaces. Unless otherwise stated we shall always assume that we are in a particular eigenspace of p^μ and $\mathcal{L}$.

We now define the self-adjoint involution of §I.3, $\mathcal{C}$, as follows[**5**]

$$\mathcal{C}\,|k\rangle \otimes |\pm\rangle = \pm\,|k\rangle \otimes |\pm\rangle \tag{2.28}$$

$$\mathcal{C}a_n^0\mathcal{C} = -a_n^0 \qquad \mathcal{C}p^\mu\mathcal{C} = p^\mu \qquad \mathcal{C}a_n^i\mathcal{C} = a_n^i \qquad (\forall\ i = 1\ldots 25, n \neq 0) \tag{2.29}$$

$$\mathcal{C}c_n\mathcal{C} = b_n \qquad \mathcal{C}b_n\mathcal{C} = c_n \qquad (\forall\ n \neq 0) \tag{2.30}$$

$$\mathcal{C}c_0\mathcal{C} = b_0 \qquad \mathcal{C}b_0\mathcal{C} = c_0 \tag{2.31}$$

This allows us to define a positive definite inner product, $\langle , \rangle_{\mathcal{C}}$, in the full Fock space via equation (I.3.1).

Let $\hat{Q}$ denote the adjoint of the BRST operator with respect to this inner product. We can make its ghost zero mode dependence manifest, using the fact that H is still self adjoint, as follows:

$$\hat{Q} = \hat{\mathcal{Q}} + b_0 H - 2c_0 \hat{T} \ , \tag{2.32}$$

where $\hat{\mathcal{Q}}$, H, and $\hat{T}$ all commute amongst each other except for

$$\hat{\mathcal{Q}}^2 = 2H\hat{T} \ . \tag{2.33}$$

Furthermore, H, $\hat{T}$, and $\hat{\mathcal{Q}}$ can be recovered from Q by the relations

$$\hat{T} = -\frac{1}{2}\{\hat{Q}\,,\, b_0\} \qquad H = \{\hat{Q}\,,\, c_0\} \ . \tag{2.34}$$

The adjoint $\hat{\mathcal{Q}}$ of the reduced BRST operator also has a holomorphic split given by

$$\hat{\mathcal{Q}} = \hat{\mathcal{Q}}' + \hat{\mathcal{Q}}'' \ , \tag{2.35}$$

where

$$\hat{\mathcal{Q}}' \ : \mathcal{F}_{(b,c)} \to \mathcal{F}_{(b,c-1)} \qquad\qquad \hat{\mathcal{Q}}'' \ : \mathcal{F}_{(b,c)} \to \mathcal{F}_{(b+1,c)} \ . \tag{2.36}$$

As observed in [**7**], the operators T, $\hat{T}$, and $\mathcal{G}$ satisfy an $sl_2\mathbb{C}$ algebra

$$[T\,,\,\hat{T}] = \mathcal{G} \qquad [\mathcal{G}\,,\,T] = 2T \qquad [\mathcal{G}\,,\,\hat{T}] = -2\hat{T} \ . \tag{2.37}$$

This is to be compared with the similar algebraic structure surfacing in Kähler algebra about which we will have more to say in the sequel.

The reduced BRST operator $\mathcal{Q}$ and its adjoint $\hat{\mathcal{Q}}$ are nilpotent when restricted to the subspace $\ker H$. We can restrict ourselves to this subspace with no loss of generality as we now show. The key point is that b_0 is a chain homotopy connecting H with the zero map, *i.e.* $H = Q\, b_0 + b_0\, Q$. Using the fact that H is diagonalizable we see that if $H\,\psi = E\,\psi$ for $E \neq 0$ and $Q\,\psi = 0$ then $\psi = E^{-1}\, Q\, b_0\, \psi$ and hence cohomologous to zero.

On $\ker H$ we can see from equations (2.21) and (2.33) that

$$\mathcal{Q}^2 = \hat{\mathcal{Q}}^2 = 0 \; . \tag{2.38}$$

This allows us to define differential complexes with respect to $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ which we shall refer to as the reduced BRST complexes.

If we take (2.38) and decompose it according to the bigrading in (2.25) we find the following two sets of equations

$$(\mathcal{Q}')^2 = 0 \qquad (\mathcal{Q}'')^2 = 0 \qquad \{\mathcal{Q}'\,,\, \mathcal{Q}''\} = 0 \; , \tag{2.39}$$

and

$$(\hat{\mathcal{Q}}')^2 = 0 \qquad (\hat{\mathcal{Q}}'')^2 = 0 \qquad \{\hat{\mathcal{Q}}'\,,\, \hat{\mathcal{Q}}''\} = 0 \; , \tag{2.40}$$

This allows us in particular to define a family of complexes for the $\mathcal{Q}''$ operator analogous to the Dolbeault complexes in complex geometry. A priori there is no reason to expect any relation between the cohomology of the reduced BRST operator $\mathcal{Q}$ and that of its antiholomorphic piece $\mathcal{Q}''$, just like in general there is little relation between the de Rham and Dolbeault cohomologies in a complex manifold. However if the manifold is compact Kähler there is a theorem due to Hodge that relates them. The analogous result for the BRST complex was conjectured in [**3**]. We show in the next section that unfortunately such a theorem fails in this case.

Before we can justify restricting ourselves to the study of the reduced BRST operator we must make sure we don't lose any information. In order to see the relationship between $H(\mathcal{Q})$ and $H(Q)$, let us analyze the action of Q and $\hat{Q}$ on a general state $\Psi = \psi^{\uparrow} \otimes |\uparrow\rangle + \psi^{\downarrow} \otimes |\downarrow\rangle$ with definite $\mathcal{G}_{\text{tot}}$ number. Now, $Q\Psi = 0$ holds if and only if

$$\mathcal{Q}\psi^{\uparrow} = 0 \qquad \mathcal{Q}\psi^{\downarrow} + (-)^{\mathcal{G}} T\psi^{\uparrow} = 0 \ . \tag{2.41}$$

Similarly, $\hat{Q}\Psi = 0$ holds if and only if

$$\hat{\mathcal{Q}}\psi^{\uparrow} + (-)^{\mathcal{G}} \hat{T}\psi^{\downarrow} = 0 \qquad \hat{\mathcal{Q}}\psi^{\downarrow} = 0 \ . \tag{2.42}$$

Suppose that Ψ has $\mathcal{G}_{\text{tot}}$ number $g + \frac{1}{2}$ so that $\psi^{\uparrow}$ has $\mathcal{G}$ number g and $\psi^{\downarrow}$ has $\mathcal{G}$ number $g+1$. The previous equation shows that if Ψ is Q-harmonic then $\psi^{\uparrow}$ (resp. $\psi^{\downarrow}$) is a $\mathcal{Q}$-cocycle (resp. $\hat{\mathcal{Q}}$-cocycle). Therefore, there exists a map

$$\Phi : H^{g+\frac{1}{2}}(Q) \longrightarrow H^{g}(\mathcal{Q}) \oplus H^{g+1}(\hat{\mathcal{Q}}) \ , \tag{2.43}$$

defined via the relation

$$\Phi([\Psi]) \equiv (\left[\psi^{\uparrow}\right] \ , \ \left[\psi^{\downarrow}\right]) \tag{2.44}$$

where Ψ is the unique Q-harmonic representative of $[\Psi]$. Notice that since we have the relation $H^{g}(\hat{\mathcal{Q}}) \simeq H^{g}(\mathcal{Q})$, Φ induces a map $H^{g+\frac{1}{2}}(Q) \longrightarrow H^{g}(\mathcal{Q}) \bigoplus H^{g+1}(\mathcal{Q})$. We now show that Φ is injective.

Let Φ denote the map in (2.43) thought of as a map from the Q-harmonic states. We first show that Φ is injective. Let Ψ be the unique harmonic representative of some class in $H^{g+\frac{1}{2}}(Q)$. Then Ψ may be written as

$$\Psi = \psi^{\uparrow} \otimes |\uparrow\rangle + \psi^{\downarrow} \otimes |\downarrow\rangle \ , \tag{2.45}$$

where $\psi^{\uparrow}$ and $\psi^{\downarrow}$ obey equations (2.41) and (2.42). Let Ψ belong to the kernel of Φ. Injectivity of Φ is equivalent to Ψ being identically zero.

By the definition of Φ, Ψ lies in the kernel of Φ if and only if $\psi^{\uparrow}$ lies in $\mathrm{im}\,\mathcal{Q}$ and $\psi^{\downarrow}$ lies in $\mathrm{im}\,\hat{\mathcal{Q}}$. The non-trivial part of equations (2.41) and (2.42) can then be written in the following way

$$\mathcal{D}\ \Psi = \begin{pmatrix} (-1)^{\mathcal{G}} T & \mathcal{Q} \\ & \\ \hat{\mathcal{Q}} & (-1)^{\mathcal{G}} \hat{T} \end{pmatrix} \begin{pmatrix} \psi^{\uparrow} \\ \\ \psi^{\downarrow} \end{pmatrix} = 0\ . \tag{2.46}$$

Here $\mathcal{D}$ is to be thought of as an endomorphism of $\mathrm{im}\,\mathcal{Q} \oplus \mathrm{im}\,\hat{\mathcal{Q}}$. In fact, since $\mathcal{D}$ commutes with the level operator $\mathcal{L}$ and with the momentum operator p we can restrict ourselves to the finite dimensional eigenspaces of these operators. In order to prove that Ψ must be identically zero it is sufficient to prove that $\mathcal{D}$ has zero kernel.

The idea of the proof is very simple. We let $\hat{\mathcal{D}}$ denote the adjoint map of $\mathcal{D}$. Because the decomposition in (I.3.7) for the $\mathcal{Q}$ complex is orthogonal, $\hat{\mathcal{D}}$ is once again an endomorphism of $\mathrm{im}\,\mathcal{Q} \oplus \mathrm{im}\,\hat{\mathcal{Q}}$ and we may (and will) compose it with $\mathcal{D}$ to obtain the endomorphism $\mathcal{D} \circ \hat{\mathcal{D}}$. Notice that the kernel of $\mathcal{D} \circ \hat{\mathcal{D}}$ is precisely the kernel of $\hat{\mathcal{D}}$, *i.e.*

$$\langle \Psi,\, (\mathcal{D} \circ \hat{\mathcal{D}})\Psi \rangle_{\mathcal{C}} = \|\hat{\mathcal{D}}\Psi\|^2\ ; \tag{2.47}$$

and that the dimension of the kernel of $\mathcal{D}$ is equal to the dimension of the kernel of $\hat{\mathcal{D}}$, since the index of an endomorphism in a finite dimensional vector space is zero. Therefore we are done if we prove that $\mathcal{D} \circ \hat{\mathcal{D}}$ has zero kernel. However this is remarkably simple since it is a positive operator as we now show.

$\hat{\mathcal{D}}$ is explicitly given by

$$\begin{pmatrix} (-1)^{\mathcal{G}} \hat{T} & \mathcal{Q} \\ & \\ \hat{\mathcal{Q}} & (-1)^{\mathcal{G}} T \end{pmatrix}\ . \tag{2.48}$$

And therefore

$$
\begin{aligned}
\mathcal{D}\circ\hat{\mathcal{D}} &= \begin{pmatrix} (-1)^{\mathcal{G}}T & \mathcal{Q} \\ \\ \hat{\mathcal{Q}} & (-1)^{\mathcal{G}}\hat{T} \end{pmatrix} \begin{pmatrix} (-1)^{\mathcal{G}}\hat{T} & \mathcal{Q} \\ \\ \hat{\mathcal{Q}} & (-1)^{\mathcal{G}}T \end{pmatrix} \\
&= \begin{pmatrix} T\hat{T}+\mathcal{Q}\hat{\mathcal{Q}} & (-1)^{\mathcal{G}}[T\,,\,\mathcal{Q}] \\ \\ (-1)^{\mathcal{G}}[\hat{T}\,,\,\hat{\mathcal{Q}}] & \hat{\mathcal{Q}}\mathcal{Q}+\hat{T}T \end{pmatrix} \\
&= \begin{pmatrix} T\hat{T}+\mathcal{Q}\hat{\mathcal{Q}} & 0 \\ \\ 0 & \hat{\mathcal{Q}}\mathcal{Q}+\hat{T}T \end{pmatrix} \ , \qquad (2.49)
\end{aligned}
$$

where we have used the fact that $(-1)^{\mathcal{G}}\mathcal{Q} = -\mathcal{Q}(-1)^{\mathcal{G}}$ (similarly for $\hat{\mathcal{Q}}$) and that $[T\,,\,\mathcal{Q}] = [\hat{T}\,,\,\hat{\mathcal{Q}}] = 0$, which followed from nilpotency of Q.

It is evident from the explicit form of $\mathcal{D}\circ\hat{\mathcal{D}}$ that it is a non-negative operator. To prove that it is, in fact, positive we calculate its expectation value on some state Ψ:

$$
\begin{aligned}
\langle\Psi,\,(\mathcal{D}\circ\hat{\mathcal{D}})\Psi\rangle_{\mathcal{C}} &= \langle\psi^{\uparrow},\,(T\hat{T}+\mathcal{Q}\hat{\mathcal{Q}})\psi^{\uparrow}\rangle_{\mathcal{C}} + \langle\psi^{\downarrow},\,(\hat{\mathcal{Q}}\mathcal{Q}+\hat{T}T)\psi^{\downarrow}\rangle_{\mathcal{C}} \\
&= \|\hat{T}\psi^{\uparrow}\|^2 + \|\hat{\mathcal{Q}}\psi^{\uparrow}\|^2 + \|\mathcal{Q}\psi^{\downarrow}\|^2 + \|T\psi^{\downarrow}\|^2\ . \qquad (2.50)
\end{aligned}
$$

For this to vanish all terms must separately vanish. In particular, this implies that $\psi^{\uparrow}$ must belong to the kernel of $\hat{\mathcal{Q}}$. But it already belongs to the image of $\mathcal{Q}$. Since these two spaces are orthogonal complements of each other we conclude that $\psi^{\uparrow}$ must be zero. A similar argument holds for $\psi^{\downarrow}$. This completes the proof.

Now suppose we are given the vanishing theorem for the cohomology of the reduced BRST operator[1]

$$
H^g(\mathcal{Q}) = 0 \qquad \forall g \neq 0\ . \qquad (2.51)
$$

Then we can conclude that $H^g(Q) = 0$ unless $g = \pm\frac{1}{2}$ using the injectivity of Φ.

---

[1] In **[3]** the vanishing theorems for both the full and the reduced BRST complexes are proven (except for the exceptional case of $k^{\mu} = 0$, where the cohomology

Furthermore, if Ψ is a Q-cocycle with $\mathcal{G}_{\rm tot}$ number $\frac{1}{2}$ then $\psi^{\downarrow} = 0$. Similarly, if Ψ is a Q-cocycle with $\mathcal{G}_{\rm tot}$ number $-\frac{1}{2}$ then $\psi^{\uparrow} = 0$. In the exceptional case that $k^{\mu} = 0$ an explicit calculation shows that there are 4 extra states, two of which have ghost number different from zero: at -1 and $+1$. These two states induce (via Φ^{-1}) two states in the cohomology of the full BRST operator with total ghost numbers $-\frac{3}{2}$ and $+\frac{3}{2}$ respectively.

As a corollary of the vanishing theorem we can show that Φ is not just injective but indeed an isomorphism. To prove surjectivity of Φ let $([\psi^{\uparrow}], [\psi^{\downarrow}]) \in H(\mathcal{Q}) \oplus H(\hat{\mathcal{Q}})$. Without loss in generality we may choose $\psi^{\uparrow}$ and $\psi^{\downarrow}$ harmonic. And because of the vanishing theorem one of them is identically zero. For definiteness let us assume that it is $\psi^{\downarrow}$. The proof for the other case goes through in exactly the same way, the necessary changes having been made. Surjectivity of Φ is equivalent to the existence of a state $\widetilde{\psi}^{\uparrow}$ $\mathcal{Q}$-cohomologous to $\psi^{\uparrow}$ and of a state $\widetilde{\psi}^{\downarrow}$ $\hat{\mathcal{Q}}$-cohomologous to zero such that $\widetilde{\Psi} = \widetilde{\psi}^{\uparrow} \otimes |\uparrow\rangle + \widetilde{\psi}^{\downarrow} \otimes |\downarrow\rangle$ is Q-harmonic. That is, surjectivity is equivalent to the existence of states $\xi^{\uparrow} \in \mathrm{im}\, \hat{\mathcal{Q}}$ and $\xi^{\downarrow} \in \mathrm{im}\, \mathcal{Q}$ such that

$$\widetilde{\Psi} = (\psi^{\uparrow} + \mathcal{Q}\xi^{\uparrow}) \otimes |\uparrow\rangle + \hat{\mathcal{Q}}\xi^{\downarrow} \otimes |\downarrow\rangle$$

is Q-harmonic. We now proceed to prove their existence by explicitly constructing

---

can be computed explicitly). However it is the one for the reduced complex that was conjectured to be provable using Kähler techniques. We have chosen to present this proof of the vanishing theorem for the full complex given the same result for the reduced one because our proof is more in line with the techniques advocated in [**1**]. In fact the proof only uses the existence of a positive definite inner product and hence does not depend on the existence of a Kähler structure compatible with this inner product, which we later show to be impossible.

them. For $\widetilde{\Psi}$ to be harmonic the following equations must be satisfied

$$Q\widetilde{\Psi} = \begin{pmatrix} \mathcal{Q} & 0 \\ \\ (-1)^{\mathcal{G}}\,T & \mathcal{Q} \end{pmatrix} \begin{pmatrix} \widetilde{\psi}^{\uparrow} \\ \\ \widetilde{\psi}^{\downarrow} \end{pmatrix} = 0 \tag{2.52}$$

$$\hat{Q}\widetilde{\Psi} = \begin{pmatrix} \hat{\mathcal{Q}} & (-1)^{\mathcal{G}}\,\hat{T} \\ \\ 0 & \hat{\mathcal{Q}} \end{pmatrix} \begin{pmatrix} \widetilde{\psi}^{\uparrow} \\ \\ \widetilde{\psi}^{\downarrow} \end{pmatrix} = 0\ , \tag{2.53}$$

where $\widetilde{\psi}^{\uparrow} = \psi^{\uparrow} + \mathcal{Q}\,\xi^{\uparrow}$ and $\widetilde{\psi}^{\downarrow} = \hat{\mathcal{Q}}\,\xi^{\downarrow}$. From (2.52) and (2.53) respectively we get the following system of equations or $\xi^{\uparrow}$ and $\xi^{\downarrow}$:

$$(-1)^{\mathcal{G}}\,T\,(\psi^{\uparrow} + \mathcal{Q}\,\xi^{\uparrow}) + \mathcal{Q}\,\hat{\mathcal{Q}}\,\xi^{\downarrow} = 0 \tag{2.54}$$

$$\mathcal{Q}\,\hat{\mathcal{Q}}\,\xi^{\uparrow} + (-1)^{\mathcal{G}}\,\hat{T}\hat{\mathcal{Q}}\,\xi^{\downarrow} = 0\ . \tag{2.55}$$

Since $\mathcal{Q}\,\hat{\mathcal{Q}}$ coincides with the BRST laplacian in $\mathrm{im}\,\mathcal{Q}$ we may solve for $\xi^{\downarrow}$ from (2.54) by means of the Green's operator G (*cf.* §I.3)

$$\xi^{\downarrow} = -(-1)^{\mathcal{G}}\,G\,T\,(\psi^{\uparrow} + \mathcal{Q}\,\xi^{\uparrow})\ . \tag{2.56}$$

Notice that this only makes sense because $T\,\psi^{\uparrow} \in \mathrm{im}\,\mathcal{Q}$ and this is a direct consequence of the vanishing theorem. If $T\,\psi^{\uparrow}$ had a harmonic piece, we would never be able to solve for $\xi^{\downarrow}$.

Substituting (2.56) into (2.55) and after some straight-forward algebra we find the following equation for $\xi^{\uparrow}$

$$\hat{\mathcal{Q}}\,\mathcal{Q}\,\xi^{\uparrow} + \hat{T}\,\hat{\mathcal{Q}}\,G\,T\,Q\,\xi^{\uparrow} + \hat{T}\,\hat{\mathcal{Q}}\,G\,T\,\psi^{\uparrow} = 0\ . \tag{2.57}$$

Since T and $\mathcal{Q}$ commute and using the definition of the Green's operator we see that (2.57) becomes

$$(\hat{\mathcal{Q}}\,\mathcal{Q} + \hat{T}\,\pi_{\mathrm{im}\,\hat{\mathcal{Q}}}\,T)\,\xi^{\uparrow} = -\hat{T}\,\hat{\mathcal{Q}}\,G\,T\,\psi^{\uparrow}\ , \tag{2.58}$$

where $\pi_{\mathrm{im}\,\hat{\mathcal{Q}}}$ is the projector onto the image of $\hat{\mathcal{Q}}$. But notice that the operator $\hat{\mathcal{Q}}\,\mathcal{Q} + \hat{T}\,\pi_{\mathrm{im}\,\hat{\mathcal{Q}}}\,T$ is actually positive in $\mathrm{im}\,\hat{\mathcal{Q}}$ and hence it has an inverse. Therefore we can solve for $\xi^\uparrow$ and substituting this into (2.56) allows us to solve for both $\xi^\uparrow$ and $\xi^\downarrow$ purely in terms of $\psi^\uparrow$. This concludes the proof that Φ is surjective and thus an isomorphism.

Another intersting corollary of the vanishing theorem is that we can choose representatives for each cohomology class in $H^0(\mathcal{Q})$ such that they are singlets of the $sl_2\mathbb{C}$ algebra of (2.37).[2] The finite dimensional representations of $sl_2\mathbb{C}$ are extremely well known. They are fully reducible and the irreducible ones are generated by T acting on a vector of lowest weight annihilated by $\hat{T}$. Because T and $\hat{T}$ commute with $\mathcal{L}$ and p^μ, they stabilize their finite dimensional eigenspaces fully decomposing them into irreducible subspaces.

The idea of the proof is the following. Let $[\psi] \in H^0(\mathcal{Q})$. Then we will prove that we can find a state $\widetilde{\psi}$ cohomologous to ψ but which is annihilated by T. Then since it is also annihilated by $\mathcal{G}$ it is a singlet. We shall without loss of generality assume that ψ is in a particular eigenspace of p^μ and $\mathcal{L}$.

Let $\widetilde{\psi} = \psi + \mathcal{Q}\,\xi$ for some state ξ of ghost number -1. Imposing $T\,\widetilde{\psi} = 0$ we get $T\,\mathcal{Q}\,\xi = -T\,\psi$. From the fact that T and $\mathcal{Q}$ commute and the vanishing theorem we conclude that $T\,\psi = \mathcal{Q}\,\rho$ for a unique $\rho \in \operatorname{im}\hat{\mathcal{Q}}$ and with ghost number 1. Therefore the equation for ξ becomes $\mathcal{Q}\,(T\,\xi + \rho) = 0$. Hence all we need to do is solve the equation $T\,\xi = -\rho \pmod{\ker \mathcal{Q}}$. In fact we can do better and we can solve the equation exactly.

First of all let us break up ρ into its irreducible components. It is clear that we can restrict ourselves to each irreducible subspace at a time since T respects this. Therefore let us assume that ρ consists of exactly one such component. Then because ρ has ghost number 1 it cannot be annihilated by $\hat{T}$, since the kernel of $\hat{T}$ consists of lowest weight vectors and these have all non-positive ghost numbers. By similar reasoning ξ cannot be annihilated by T, and hence by $\hat{T}\,T$. Therefore

---

[2] This was used in [**7**] as a criterion to gauge away auxiliary fields.

we can solve for ξ as follows

$$\xi = -(\hat{T}\, T)^{-1}\, \hat{T}\, \rho \ , \tag{2.59}$$

where the inverse of $\hat{T}\, T$ exists in $\operatorname{im} \hat{T} = (\ker T)^{\perp} = (\ker \hat{T}\, T)^{\perp}$.

Noticing that $\rho = G\, \hat{\mathcal{Q}}\, T\, \psi$, we can write $\widetilde{\psi}$ as

$$\widetilde{\psi} = \Big(\mathbf{1} - \mathcal{Q}\, (\hat{T}\, T)^{-1}\, \hat{T}\, G\, \hat{\mathcal{Q}}\, T \Big)\ \psi \ . \tag{2.60}$$

The above operator turns out, after some straight forward algebra, to be a projection.

It is worth remarking that in general there is no unique singlet representative from each cohomology class. We find a counter example in level 1 already. Consider the state $\xi = b_1^\dagger\ |k\rangle$. It is clearly in $\ker \hat{\mathcal{Q}}$ since there are no states of ghost number -2 at level 1. By the vanishing theorem $\ker \hat{\mathcal{Q}} = \operatorname{im} \hat{\mathcal{Q}}$ and so $\mathcal{Q}\, \xi \neq 0$. Now $T\, \xi = c_1^\dagger\ |k\rangle$. which is in the kernel of $\mathcal{Q}$, since at level 1 there are no states of ghost number 2. Therefore since T and $\mathcal{Q}$ commute, $T\, \mathcal{Q}\, \xi = 0$. Hence given a singlet state ψ at level 1, we can always add to it $\mathcal{Q}\, \xi$ and still have a singlet.

A natural question to ask is whether the harmonic states are in fact singlets. We have verified this up to level 4; although we have been so far unable to prove that this is true in general. Furthermore the explicit construction suggests that they are not just singlets but in fact they do not contain any ghost and antighost oscillators.

We can, however, say something in general about the harmonic states; and that is that they fall into irreducible representations of the stability algebra of the center of mass momentum. This is easy to see as follows. In the expression for $\mathcal{Q}$ there is term which is linear in the momentum. If we were to consider the commutator of $\mathcal{Q}$ with the "spin" part of the Lorentz generators we find that it is precisely that linear term but with the transformed momentum. Hence the subalgebra of the Lorentz group which commutes with the reduced (and hence the full) BRST operator is the stability algebra of the momentum. With $\hat{\mathcal{Q}}$ the situation is similar except that we now have the time reversed momentum. However a momentum and its time reversed image share the same stability algebra. We therefore conclude that the stability algebra of the center of mass momentum commutes with both $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ and thus stabilizes the harmonic states. Finally, since they also commute with the level operator, the harmonic spectrum at each level breaks up into irreducible representations of the stability algebra.

We can illustrate these remarks by giving a table of the harmonic states at the first five levels. We choose convenient center of mass momenta. To obtain the states corresponding to different momenta one must merely Lorentz rotate the oscillators as well.

Level	Harmonic States	Representations
1	$U_i a^i_{-1} \|k_1\rangle \ , \ U_{25} = 0$	$\square$
2	$D_{ij} a^i_{-1} a^j_{-1} \|k_2\rangle$	
	$D_{ij} = D_{(ij)} \ , \ \delta^{ij} D_{ij} = 0$	$\square\!\square$
3	$T_{ijk} a^i_{-1} a^j_{-1} a^k_{-1} \|k_3\rangle$	
	$T_{ijk} = T_{(ijk)} \ , \ \delta^{ij} T_{ijk} = 0$	$\square\!\square\!\square$
	$A_{ij} a^i_{-1} a^j_{-2} \|k_3\rangle \ , \ A_{ij} = A_{[ij]}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
4	$C_{ijkl} \Big(a^i_{-1} a^j_{-1} a^k_{-1} a^l_{-1} - 2\sqrt{3}\, \delta^{kl} a^i_{-1} a^j_{-3}$	
	$-3\delta^{kl} a^i_{-2} a^j_{-2} \Big) \|k_4\rangle$	
	$C_{ijkl} = C_{(ijkl)}$	$\square\!\square\!\square\!\square \oplus \square\!\square \oplus \bullet$
	$S_{ijk} a^i_{-1} a^j_{-1} a^k_{-2} \|k_4\rangle$	
	$S_{ijk} = S_{(ij)k} \quad S_{(ijk)} = 0 \quad \delta^{ij} S_{ijk} = 0$	$\begin{smallmatrix}\square\square\\\square\ \ \end{smallmatrix}$

Several remarks are in order. First the indices i, j, k, l all run from 1 to 25. The momentum k_1 corresponds to the vector $(E, 0, \ldots, 0, E)$ and the momenta k_n for $n > 1$ correspond to the vectors $\left(\sqrt{2(n-1)}, 0, \ldots, 0\right)$. Finally, the representations described correspond to SO_{24} for level 1 and SO_{25} for all the other levels. Moreover due to our choice of momenta these groups are embedded naturally in the 26–dimensional Lorentz group.

§3 THE KÄHLER STRUCTURE

Having discussed in detail the BRST structure of the open bosonic string we are now ready to make contact with Kähler algebra. Although the relationship exhibited here is purely algebraic one can show[2] that there is some geometry behind it. Kähler algebra consists of algebraic relations between objects which made their mathematical debut as very natural differential operators in the study of Kähler manifolds. It is convenient therefore to introduce them in their original guise and later abstract those algebraic properties that are relevant to our case. We follow the notation of [**8**], where the reader may find the proofs of all the results asserted in this section concerning complex geometry.

Let X be a complex manifold. The complexified cotangent bundle $T^*_{\mathbb{C}}(X)$ splits into two sub-bundles

$$T^*_{\mathbb{C}}(X) = T^*(X)^{1,0} \oplus T^*(X)^{0,1} \ . \tag{3.1}$$

This split induces a decomposition of the exterior forms

$$\bigwedge^n T^*_{\mathbb{C}}(X) = \bigoplus_{p+q=n} \bigwedge^{p,q} T^*_{\mathbb{C}}(X) \ , \tag{3.2}$$

where

$$\bigwedge^{p,q} T^*_{\mathbb{C}}(X) = \bigwedge^p T^*(X)^{1,0} \otimes \bigwedge^q T^*(X)^{0,1} \ . \tag{3.3}$$

The smooth sections of these bundles are the complex-valued differential forms of type (p,q) which we denote by $\mathcal{E}^{p,q}(X)$. The decomposition in (3.2) induces a decomposition of the smooth n-forms, $\mathcal{E}^n(X)$ into (p,q)-forms as follows

$$\mathcal{E}^n(X) = \bigoplus_{p+q=n} \mathcal{E}^{p,q}(X) \ . \tag{3.4}$$

The (p,q)-forms are roughly the analogues of the states in $\mathcal{F}_{(b,c)}$ which consist *only* of ghost and antighost oscillators. In order to include the string coordinates oscillators in this formalism we must talk about (p,q)-forms with values in a complex vector space, or more generally in a complex vector bundle. To this end let $E \longrightarrow X$ denote a holomorphic vector bundle and let $\mathcal{E}^n(E)$ and $\mathcal{E}^{p,q}(E)$ denote respectively the smooth sections of the vector bundles $\bigwedge^n T^*_{\mathbb{C}}(X) \otimes E$ and $\bigwedge^{p,q} T^*_{\mathbb{C}}(X) \otimes E$.

Let d denote the exterior derivative. We can extend it trivially to act on the E-valued forms as $d \otimes 1$. It is still a nilpotent operator and it gives rise to the following differential complex known as the E-valued de Rham complex

$$\cdots \longrightarrow \mathcal{E}^{n-1}(E) \xrightarrow{d} \mathcal{E}^n(E) \xrightarrow{d} \mathcal{E}^{n+1}(E) \longrightarrow \cdots \tag{3.5}$$

Using the decomposition of n-forms into (p,q)-forms we can split the exterior derivative into the Cauchy-Riemann operators

$$d = \partial + \bar{\partial} \ , \tag{3.6}$$

where

$$\partial : \mathcal{E}^{p,q}(E) \longrightarrow \mathcal{E}^{p+1,q}(E) \ , \qquad \bar{\partial} : \mathcal{E}^{p,q}(E) \longrightarrow \mathcal{E}^{p,q+1}(E) \ . \tag{3.7}$$

Nilpotency of the exterior deriative implies, in a manner completely analogous to the argument leading to equations (2.39) or (2.40), that both ∂ and $\bar{\partial}$ are nilpotent and that they anticommute. In particular their nilpotency allows us to consider a family of complexes for, say, the Cauchy-Riemann operator $\bar{\partial}$ known as the Dolbeault complexes

$$\cdots \longrightarrow \mathcal{E}^{p,q-1}(E) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q}(E) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1}(E) \longrightarrow \cdots \tag{3.8}$$

Now let X be a compact Kähler manifold. The Kähler form is closed and the Hermitian metric associated to it is positive definite. Therefore there is an induced positive definite Hermitian inner product on the (p,q) forms — the Hodge metric — which allows us to define (formal) adjoints for the Cauchy-Riemann operators. The Hodge metric is constructed using the Hodge-Serre duality operator. We can define an operator $\bar{\star}$ mapping (p,q)-forms to $(m-p, m-q)$-forms (where m is the complex dimension of X) which is nothing but the usual Hodge star-operator associated with a Riemannian metric pre-composed with complex conjugation. Then the Hodge inner product of two complex-valued forms ϕ and ψ is

$$\langle \phi,\, \psi \rangle_H = \int_X \phi \wedge \bar{\star}(\psi) \ , \tag{3.9}$$

where $\bar{\star}(\psi) = \star\overline{\psi}$. Furthermore if $E \longrightarrow X$ is a Hermitian holomorphic vector bundle we can extend the inner product to the E-valued forms by essentially tensoring the two metrics. Let $\tau : E \longrightarrow E^*$ denote the conjugate-linear bundle isomorphism between E and its dual bundle E^* induced by the Hermitian structure. Then we can define $\bar{\star}_E$ on an E-valued form $\psi \otimes e$ by $\bar{\star}_E(\psi \otimes e) = \bar{\star}(\psi) \otimes \tau(e)$. Then the Hodge inner product for E-valued forms can be constructed just as in (3.9) but with $\bar{\star}_E$ instead of $\bar{\star}$ and using — before integration — the pointwise evaluation map $E \otimes E^* \longrightarrow \mathbb{C}$.

Let * denote hermitian conjugation with respect to this inner product so that d^*, ∂^* and $\bar{\partial}^*$ respectively denote the adjoints of the exterior derivative and the Cauchy-Riemann operators. The $\bar{\star}_E$ operator is the analogue of the self-adjoint involution $\mathcal{C}$ we introduced in §I.3. The $\bar{\star}$ operator corresponds to $\mathcal{C}$ restricted to the states that only consist of ghost and antighost oscillators.

Let us now introduce the operator L defined as exterior product with the Kähler form. Because the Kähler form is a $(1,1)$-form it maps

$$L\,:\mathcal{E}^{p,q}(E)\longrightarrow\mathcal{E}^{p+1,q+1}(E)\ , \tag{3.10}$$

and because the Kähler form is real L is a real operator. Its adjoint L^* is also a real operator and corresponds roughly (up to some factors depending on the bidegree of the form) to interior multiplication with the Kähler form. If we denote by $\Pi_{p,q}$ the projection operator onto the (p,q)-forms and Π_n the projection operator onto the n-forms then

$$[L\,,\,L^*]=\sum_{l=0}^{2m}(l-m)\,\Pi_l\ . \tag{3.11}$$

The operators L and L^* correspond to our operators T and $\hat{T}$.

There are three kinds of laplacian operators acting on forms on a compact complex manifold

$$\begin{aligned}\Box&=\partial\,\partial^*+\partial^*\,\partial\\ \bar{\Box}&=\bar\partial\,\bar\partial^*+\bar\partial^*\,\bar\partial\\ \triangle&=d\,d^*+d^*\,d\ .\end{aligned} \tag{3.12}$$

In general they are not related but if the manifold is Kähler then

$$\frac{1}{2}\triangle=\Box=\bar{\Box}\ . \tag{3.13}$$

In the process of proving this result a number of auxiliary relations are obtained

$$\begin{aligned}&[L\,,\,\partial]=[L\,,\,\bar\partial]=[L^*\,,\,\partial^*]=[L^*\,,\,\bar\partial^*]=0\\ &[L\,,\,\partial^*]=i\bar\partial\qquad\qquad [L\,,\,\bar\partial^*]=-i\partial\\ &[L^*\,,\,\partial]=i\bar\partial^*\qquad\qquad [L^*\,,\,\bar\partial]=-i\partial^*\ .\end{aligned} \tag{3.14}$$

Letting $d_c = -i(\partial - \bar{\partial})$ we see that

$$[L\,,\,d^*] = d_c \qquad\qquad [L^*\,,\,d] = -d_c^* \ . \tag{3.15}$$

Furthermore as a corollary we have that all laplacians commute with L, ∂, $\bar{\partial}$ and their adjoints.

The equality of the laplacians together with the relation between cohomology and harmonic forms allows us to relate the cohomology of the de Rham and Dolbeault complexes yielding the famous decomposition

$$H^n_{DR}(M) \cong \bigoplus_{p+q=n} H^{p,q}_{\bar{\partial}}(M)\ , \tag{3.16}$$

Finally consider the bundle $\overline{K} = \bigwedge^{\mathrm{top}} T^*(X)^{0,1}$, the conjugate of the canonical bundle. The smooth sections of $\overline{K}$ are just the $(0, m)$-forms where, again, m is the complex dimension of X and similarly the smooth sections of the bundle $\overline{K} \otimes E$ are nothing but the E-valued $(0, m)$-forms. If we define

$$F^{(b,c)} \equiv \mathcal{E}^{c,m-b}(E)\ , \tag{3.17}$$

then we see that

$$\partial\ :F^{(b,c)} \longrightarrow F^{(b,c+1)} \qquad\qquad \bar{\partial}\ :F^{(b,c)} \longrightarrow F^{(b-1,c)}\ . \tag{3.18}$$

This is to be compared with equation (2.27).

Having reviewed the basic facts of Kähler algebra we are finally ready to make the correspondence between the objects we have just introduced and those appearing in the BRST complex. The correspondence is obtained by making the following identifications:

BRST	Kähler Algebra
$\mathcal{F}_{(b,c)}$	$F^{(b,c)}$
$\mathcal{Q}'$	∂
$\mathcal{Q}''$	$\bar{\partial}$
T	L
$\dagger$	$\bar{\ }$
$\mathcal{C}$	$\bar{\star}_E$
$\hat{\ }$	$*$

First of all, a comment is in order. The definition in (3.17) only makes sense if the dimension of the complex manifold is finite; however, under the above dictionary this would correspond to a Fock space generated by a finite number of ghost and antighost oscillators which is not our case. The definition of $F^{(b,c)}$ can be given for the case of infinite dimensional manifolds and corresponds to the semi-infinite forms of [**9**]. It is in the spirit of clarity that we decided to present the finite dimensional theory since we are only using it as a heuristic guide. Moreover in the present work we are only drawing an analogy with the algebraic relations that will appear between the various operators and not with the spaces on which they act, although it should be kept in mind that ultimately there is. The fact that we have presented the finite dimensional theory also accounts for the factors of powers of $\sqrt{-1}$ that have been omitted in the above table and that would make the correspondence exact. As it stands the correspondence is to be understood modulo these factors. For instance, whereas the BRST operator $\mathcal{C}$ is an involution, the Hodge-Serre duality operator $\bar{\star}_E$ is not. This is remedied by redefining it by a phase which depends on the bidegree of the form. Since the Kähler adjoints are defined via this operator they too need some phase factors in order to exactly correspond under the above dictionary. We don't feel that the exact corespondence is crucial for our purposes and for simplicity we have decided to relax it. The interested reader can find the exact correspondence in [**3**]. Also notice that under the above dictionary the operator in the right hand side of equation (3.11) does in fact correspond to the ghost number operator $\mathcal{G}$.

Let us now investigate to what extent the Kähler identities are satisfied. As can be seen by equation (2.29) the involution $\mathcal{C}$ time reverses the string coordinates except for the center of mass coordinate which is left inert. Since the generators $\{L_n^{(a)}\}$ of the Virasoro algebra are bilinear in the string oscillators and invariant under the full Lorentz group they will be invariant under conjugation by $\mathcal{C}$ except for those terms which are linear in the center of mass momentum. It is therefore convenient to make the momentum dependence in $\mathcal{Q}$ manifest. To this end let us split $\mathcal{Q}(p)$ into

$$\mathcal{Q}(p) = p \cdot R + \mathbb{Q} \ , \tag{3.19}$$

which induces a holomorphic split

$$\begin{aligned} \mathcal{Q}'(p) &= p \cdot R' + \mathbb{Q}' \\ \mathcal{Q}''(p) &= p \cdot R'' + \mathbb{Q}'' \ . \end{aligned} \tag{3.20}$$

These obey the following algebra

$$(p \cdot R')^2 = (\mathbb{Q}')^2 = \{p \cdot R' \, , \, \mathbb{Q}'\} = 0 \tag{3.21}$$

$$(p \cdot R'')^2 = (\mathbb{Q}'')^2 = \{p \cdot R'' \, , \, \mathbb{Q}''\} = 0 \tag{3.22}$$

$$\{p \cdot R' \, , \, p \cdot R''\} = p^2 \, T \tag{3.23}$$

$$\{\mathbb{Q}' \, , \, \mathbb{Q}''\} = 2\,(\mathcal{L} - 1)\, T \tag{3.24}$$

$$\{p \cdot R' \, , \, \mathbb{Q}''\} = -\{\mathbb{Q}' \, , \, p \cdot R''\} \ , \tag{3.25}$$

where $\mathcal{L}$ is the level operator defined in (2.18). Of course in $\ker H$ there is a relation between p^2 and $\mathcal{L}$, namely $p^2 + 2(\mathcal{L} - 1) = 0$. After another straightforward oscillator calculation one finds that for the momentum independent piece

$\mathbb{Q}$ we obtain the expected Kähler relations analogoue to those in equation (3.14)

$$[\hat{T}\,,\,\mathbb{Q}'] = \hat{\mathbb{Q}}'' \qquad\qquad [\hat{T}\,,\,\mathbb{Q}''] = -\hat{\mathbb{Q}}'\ . \tag{3.26}$$

but for the momentum dependent piece we are not so lucky. In fact, one obtains

$$[\hat{T}\,,\,\mathcal{Q}'(p)] = \hat{\mathcal{Q}}''(\mathcal{T}p) \qquad\qquad [\hat{T}\,,\,\mathcal{Q}''(p)] = -\hat{\mathcal{Q}}'(\mathcal{T}p)\ , \tag{3.27}$$

where $\mathcal{T}p$ denotes the time reversed momentum $(\mathcal{T}p)^\mu = (-1)^{\delta_{\mu,0}} p^\mu$.

As a result the holomorphic and antiholomorphic laplacians do not agree. Had the Kähler relations been true the equality of the laplacians

$$\begin{aligned} \triangle' &\equiv \mathcal{Q}'\,\hat{\mathcal{Q}}' + \hat{\mathcal{Q}}'\,\mathcal{Q}' \\ \triangle'' &\equiv \mathcal{Q}''\,\hat{\mathcal{Q}}'' + \hat{\mathcal{Q}}''\,\mathcal{Q}'' \end{aligned} \tag{3.28}$$

would follow immediately. Indeed,

$$\begin{aligned} \triangle' &= \{\mathcal{Q}'\,,\,\hat{\mathcal{Q}}'\} \\ &= \{\mathcal{Q}'\,,\,[\mathcal{Q}''\,,\,\hat{T}]\} \qquad \text{by the K"ahler relations} \\ &= \{[\hat{T}\,,\,\mathcal{Q}']\,,\,\mathcal{Q}''\}\,\text{by Jacobi identity and (2.39)} \\ &= \{\hat{\mathcal{Q}}''\,,\,\mathcal{Q}''\} \qquad\qquad \text{by the Kähler relations} \\ \Rightarrow \triangle' &= \triangle''\ . \end{aligned} \tag{3.29}$$

This result would allow us to prove a decomposition theorem similar to that in equation (3.16).

Therefore we see how this positive definite inner product is not Kähler as exemplified by the equality of the holomorphic and antiholomorphic laplacians. Had we not time reversed the oscillators corresponding to the vibrational modes the inner product would have been Kähler but it would not have been positive definite and hence our decomposition theorem would not have gone through since we use positive-definiteness strongly. If on the other hand we would have treated the center of mass momentum on an equal footing with the other modes of the string we would have again obtained a Kähler inner product but again it would not have been positive definite.

These arguments by themselves do not constitute a proof of the non-existence of a positive-definite Kähler inner product. However a simple proof can be given that this is indeed the case. This uses the concept[10] of "disjointness," which is shown in the appendix to be a sufficient condition to guarantee a decomposition theorem as in [1]. Two operators A and B are said to be **disjoint** if and only if for all vectors ψ the following two conditions hold

$$\begin{aligned} A\,B\,\psi = 0 &\Rightarrow B\,\psi = 0 \\ B\,A\,\psi = 0 &\Rightarrow A\,\psi = 0\ . \end{aligned} \tag{3.30}$$

In particular, if $B = A^*$, where * denotes adjoint under a positive definite inner product $\langle , \rangle$, disjointness follows since

$$\begin{aligned} A\,A^*\,\psi = 0 &\Rightarrow \langle \psi,\, A\,A^*\,\psi\rangle = 0 \\ &\Rightarrow \|A^*\,\psi\| = 0 \\ &\Rightarrow A^*\,\psi = 0 \qquad\qquad \text{by positive-definiteness,} \end{aligned}$$

and similarly for the other identity.

Therefore, if there exists a positive-definite Kähler inner product then $\mathcal{Q}''$ and $\widetilde{\mathcal{Q}}' \equiv [\hat{T}\,,\, \mathcal{Q}']$ must be disjoint.[3] However an explicit calculation shows that this is

---

[3] We should really have the adjoint of T under the new inner product in the previous commutator. However T only contains ghost and antighost oscillators and the inner product is Kähler with ghosts and antighosts alone. Moreover T,

not the case. In fact all we must show is that there is something in $\ker \mathcal{Q}'' \cap \operatorname{im} \widetilde{\mathcal{Q}}'$. A completely straight-forward calculation shows that

$$\mathcal{Q}'' \, \widetilde{\mathcal{Q}}' \, c_2^\dagger \, |k\rangle = 0 \qquad \text{but} \qquad \widetilde{\mathcal{Q}}' \, c_2^\dagger \, |k\rangle \neq 0 \; . \tag{3.31}$$

Similarly one can verify that the decomposition analogous to equation (3.16) does not hold already at level 2.

One may also ask whether the operators $\mathcal{Q}$ and $\widetilde{\mathcal{Q}} \equiv [\hat{T}\, , \, \mathcal{Q}]$ could be disjoint even though their holomorphic and antiholomorphic parts are not separately disjoint. In this case an explicit calculation at level 2 shows that

$$\mathcal{Q} \, \widetilde{\mathcal{Q}} \, \left(k \cdot a_1^\dagger\right) \, c_1^\dagger \, |k\rangle = 0 \qquad \text{whereas} \qquad \widetilde{\mathcal{Q}} \, \left(k \cdot a_1^\dagger\right) \, c_1^\dagger \, |k\rangle \neq 0 \; . \tag{3.32}$$

§4 "NO-GHOST" THEOREMS

In this section we give proofs of the "no-ghost" theorems for the bosonic and NSR strings along the lines of §I.5. We assume in all cases that a vanishing theorem has been proven for the reduced BRST complex. In the case of the bosonic string this was done in **[3]** in great generality and in a manner independent of the "no-ghost" theorem, although several other proofs have appeared, as well as for the NSR string.[**11**] However these proofs follow always as byproducts of explicit computation of the cohomology in terms of DDF states. A direct proof of the vanishing theorem along the lines of **[1]** would be desirable, for then the "no-ghost" theorem follows as a more or less trivial consequence as we will show in this section. We only discuss open strings because this is sufficient to prove "no-ghost" theorems for the closed

$\hat{T}$ and $\mathcal{G}$ form *the* Kähler $sl_2\mathbb{C}$ algebra and therefore any Kähler inner product must have $\hat{T}$ as the adjoint of T. Any other Kähler structure would not be the natural one.

strings as well. The spectra of the closed strings can be obtained by truncating the tensor products of the spectra of the open strings. And hence if there are no negative norm states in the spectra of the open strings there cannot be any negative norm states in the spectra of the closed strings.

The Open Bosonic String

As we remarked in §I.5 to prove the "no-ghost" theorem all we need to prove — given the vanishing theorem — is the identity in (I.5.8). We now proceed to calculate this. As mentioned in §I.5 we compute weighted traces. The reduced Fock space $\mathcal{F}$ is easily seen to have the following structure

$$\mathcal{F} = \bigotimes_{\mu=0}^{25} \bigotimes_{n=1}^{\infty} S_n^\mu \bigotimes_{n=1}^{\infty} A_n \ , \tag{4.1}$$

where S_n^μ is the one particle Hilbert space corresponding to the oscillator $a_n^{\mu\dagger}$ and A_n is the Hilbert space corresponding to the oscillators $\{b_n^\dagger, c_n^\dagger\}$. The space S_n^μ is isomorphic to the polynomial algebra in one variable: $a_n^{\mu\dagger}$ whereas the space A_n is isomorphic to the exterior algebra on two generators: $b_n^\dagger$ and $c_n^\dagger$.

Therefore using the fact that the trace is multiplicative over tensor products the left hand side of (I.5.8) becomes

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}}\,(-1)^{\mathcal{G}}\, q^{\mathcal{L}} &= \prod_{\mu=0}^{25}\prod_{n=1}^{\infty} \mathrm{Tr}_{S_n^\mu}\, q^{n a_n^{\mu\dagger} a_n^\mu} \times \prod_{n=1}^{\infty} \mathrm{Tr}_{A_n} \left[(-1)^{c_n^\dagger b_n - b_n^\dagger c_n}\, q^{n(c_n^\dagger b_n + b_n^\dagger c_n)}\right] \\
&= \left[\prod_{n=1}^{\infty}\left(\sum_{m=0}^{\infty} q^{nm}\right)\right]^{26} \times \prod_{n=1}^{\infty}\left(1 - q^n - q^n + q^{2n}\right) \\
&= \prod_{n=1}^{\infty} (1-q^n)^{-26} \cdot (1-q^n)^2 \\
&= \prod_{n=1}^{\infty} (1-q^n)^{-24} \ .
\end{aligned} \tag{4.2}$$

As for the right hand side we have

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{F}}\, \mathcal{C}\, q^{\mathcal{L}} &= \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n^\mu}\, \mathcal{C}\, q^{n a_n^{\mu\,\dagger} a_n^\mu} \times \prod_{n=1}^{\infty} \mathrm{Tr}_{A_n}\, \mathcal{C}\, q^{n(c_n^\dagger b_n + b_n^\dagger c_n)} \\
&= \prod_{\mu=0}^{25} \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} \Big((-1)^{\delta_{\mu,0}}\, q^n \Big)^m \times \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \\
&= \prod_{n=1}^{\infty} (1+q^n)^{-1} \cdot (1-q^n)^{-25} \cdot (1-q^n) \cdot (1+q^n) \\
&= \prod_{n=1}^{\infty} (1-q^n)^{-24} \quad . \qquad (4.3)
\end{aligned}
$$

Therefore the identity in (I.5.8) is satisfied and the “no-ghost” theorem for the open bosonic string is proven. Notice how this partition function is precisely the one obtained from the light-cone quantization.

The Neveu-Schwarz Sector of the Open NSR String

We shall be very brief in this and the next subsections. We assume the reader is familiar with the standard BRST treatment of the NSR string as found for instance in **[6]**. We merely list some basic properties in order to clarify the notation.

In the Fock space of the Neveu-Schwarz sector we find the following oscillators: $\{a_n^\mu, b_r^\mu, b_n, c_n, \beta_r, \gamma_r\}$, where $n \in \mathbb{Z}$, $r \in \mathbb{Z} + \frac{1}{2}$, and $\mu = 0, \ldots, 9$, where again we use p^μ and a_0^μ interchangingly. These oscillators enjoy the following hermiticity properties:

$$
\begin{array}{llll}
a_n^{\mu\,\dagger} = a_{-n}^\mu & b_n^\dagger = b_{-n} & c_n^\dagger = c_{-n} & \forall n \in \mathbb{Z} \\
b_r^{\mu\,\dagger} = b_{-r}^\mu & \beta_r^\dagger = \beta_{-r} & \gamma_r^\dagger = -\gamma_{-r} & \forall r \in \mathbb{Z} + \dfrac{1}{2}\ .
\end{array} \qquad (4.4)
$$

They obey the following canonical (anti)commutation relations:

$$
[a_n^\mu\, ,\, a_m^{\nu\,\dagger}] = \eta^{\mu\nu} \delta_{mn} \qquad \{b_n\, ,\, c_m^\dagger\} = \delta_{mn} \qquad \forall\, m, n \in \mathbb{N}
$$

$$\begin{aligned} &\{b^\mu_r\,,\,b^{\nu\dagger}_s\} = \eta^{\mu\nu}\delta_{rs} \qquad [\beta_r\,,\,\gamma^\dagger_s] = \delta_{rs} \qquad \forall\, r,s \in \mathbb{N} - \frac{1}{2} \\ &\{b_0\,,\,c_0\} = 1\ , \end{aligned} \tag{4.5}$$

and all other (anti)commutators vanish.

A vacuum $|k\rangle$ in the reduced Fock space is defined by

$$\begin{aligned} &p^\mu\,|k\rangle = k^\mu\,|k\rangle \qquad a^\mu_n\,|k\rangle = 0 \qquad b_n\,|k\rangle = 0 \qquad c_n\,|k\rangle = 0 \\ &b^\mu_r\,|k\rangle = 0 \qquad \beta_r\,|k\rangle = 0 \qquad \gamma_r\,|k\rangle = 0 \qquad \forall\, n > 0\,, r \geq \frac{1}{2}\ . \end{aligned}$$

A vacuum in the full Fock space is obtained by tensoring $|k\rangle$ with any state in the representation space for the Clifford algebra defined by the ghost zero modes $\{b_0, c_0\}$.

The dependence of the BRST operator on the ghost zero modes can be made manifest by the following decomposition

$$Q = c_0\,H - 2b_0\,T + \mathcal{Q}\ , \tag{4.6}$$

where

$$T = \sum_{n>0} c^\dagger_n c_n - \sum_{r\geq\frac{1}{2}} \gamma^\dagger_r \gamma_r \tag{4.7}$$

$$H = \frac{1}{2}p^2 + \mathcal{L} - \frac{1}{2} \tag{4.8}$$

and $\mathcal{Q}$ is the rest. It is convenient to split the level operator $\mathcal{L}$ as follows

$$\mathcal{L} = \mathcal{L}^{(a)} + \mathcal{L}^{(b)} + \mathcal{L}^{(b,c)} + \mathcal{L}^{(\beta,\gamma)} \tag{4.9}$$

where

$$\mathcal{L}^{(a)} = \sum_{n>0} n\, a_n^\dagger \cdot a_n \tag{4.10}$$

$$\mathcal{L}^{(b)} = \sum_{r\geq\frac{1}{2}} r\, b_r^\dagger \cdot b_r \tag{4.11}$$

$$\mathcal{L}^{(b,c)} = \sum_{n>0} n \left(b_n^\dagger c_n + c_n^\dagger b_n \right) \tag{4.12}$$

$$\mathcal{L}^{(\beta,\gamma)} = \sum_{r\geq\frac{1}{2}} r \left(\beta_r^\dagger \gamma_r + \gamma_r^\dagger \beta_r \right) \ . \tag{4.13}$$

In the Neveu-Schwarz sector of the NSR string the relation between the reduced and the full Fock spaces is essentially the same as in the bosonic string due to the absence of zero modes for the fermionic and super-ghost fields. In particular we can restrict ourselves to $\ker H$ without loss of generality and study the cohomology of the reduced BRST operator $\mathcal{Q}$. Given the vanishing theorem for the cohomology of $\mathcal{Q}$ we can then prove the vanishing theorem for the cohomology of Q and moreover prove that the cohomology of Q is isomorphic to two copies of the cohomology of $\mathcal{Q}$.

The reduced Fock space has the following structure

$$\mathcal{F} = \mathcal{F}^{(a)} \otimes \mathcal{F}^{(b,c)} \otimes \mathcal{F}^{(b)} \otimes \mathcal{F}^{(\beta,\gamma)} \tag{4.14}$$

where

$$\mathcal{F}^{(a)} = \bigotimes_{\mu=0}^{9} \bigotimes_{n=1}^{\infty} S_n^\mu \tag{4.15}$$

$$\mathcal{F}^{(b,c)} = \bigotimes_{n=1}^{\infty} A_n \tag{4.16}$$

$$\mathcal{F}^{(b)} = \bigotimes_{\mu=0}^{9} \bigotimes_{r=\frac{1}{2}}^{\infty} A_r^\mu \tag{4.17}$$

$$\mathcal{F}^{(\beta,\gamma)} = \bigotimes_{r=\frac{1}{2}}^{\infty} S_r \ . \tag{4.18}$$

The first two terms are just like in the open bosonic string except that $D = 10$. Therefore we shall concentrate on the last two terms. Here A_r^μ is the Hilbert space of the $b_r^{\mu\dagger}$ oscillator and is isomorphic to the exterior algebra on one generator; and S_r is the Hilbert space of the $\{\beta_r^\dagger, \gamma_r^\dagger\}$ oscillators and is isomorphic to the polynomial algebra in two variables.

The contribution to the left hand side of equation (I.5.8) coming from the first two terms in the above decomposition of the reduced Fock space can be readily obtained from the similar computation done in the previous subsection. The result is $\prod_{n=1}^\infty (1-q^n)^{-8}$. The contribution coming from the Neveu-Schwarz oscillators can be computed as follows

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(b)}}\, q^{\mathcal{L}^{(b)}} &= \prod_{\mu=0}^{9} \prod_{r=\frac{1}{2}}^{\infty} Tr_{A_r^\mu}\, q^{r\, b_r^{\mu\dagger} b_r^\mu} \\
&= \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^{10} \quad ,
\end{aligned}$$

whereas the contribution from the super-ghosts is

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\, (-1)^{\mathcal{G}}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{S_r}\, (-1)^{N_\gamma - N_\beta}\, q^{r(N_\gamma + N_\beta)} \\
&= \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{S_r}\, (-q^r)^{N_\gamma + N_\beta} \\
&= \prod_{r=\frac{1}{2}}^{\infty} \sum_{n,m=0}^{\infty} (-q^r)^{n+m} \\
&= \prod_{r=\frac{1}{2}}^{\infty} \left(\sum_{n=0}^{\infty} (-q^r)^n \right)^2 \\
&= \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^{-2} \quad ,
\end{aligned}$$

where N_β (resp. N_γ) is the number operator corresponding to the $\{\beta_r\}$ (resp. $\{\gamma_r\}$) oscillators. Putting everything together we find that

$$\mathrm{Tr}_{\mathcal{F}}\, (-1)^{\mathcal{G}}\, q^{\mathcal{L}} = \prod_{n=1}^{\infty} (1-q^n)^{-8} \times \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^8 \quad . \tag{4.19}$$

In order to compute the right hand side of equation (I.5.8) we must first discuss the conjugation $\mathcal{C}$. The conjugation on the $\{a_n^\mu, b_n, c_n\}$ oscillators is the same as in (2.28)–(2.30) but with $D = 10$. For the other oscillators the conjugation with the desired properties turns out to be the following

$$\begin{aligned}
\mathcal{C}\, b_s^\mu\, \mathcal{C} &= (-1)^{\delta_{\mu,0}}\, b_s^\mu \qquad \forall\, s \in \mathbb{Z} + \frac{1}{2} \\
\mathcal{C}\, \gamma_r\, \mathcal{C} &= \beta_r \qquad \mathcal{C}\, \beta_r\, \mathcal{C} = \gamma_r \\
\mathcal{C}\, \gamma_{-r}\, \mathcal{C} &= -\beta_{-r} \qquad \mathcal{C}\, \beta_{-r}\, \mathcal{C} = -\gamma_{-r} \qquad \forall\, r \in \mathbb{N} - \frac{1}{2}\ .
\end{aligned} \tag{4.20}$$

Once again the contribution now to the right hand side of (I.5.8) coming from the $\{a_n^\mu, b_n, c_n\}$ oscillators can be read from the calculations in the previous subsection and yields $\prod_{n=1}^{\infty} (1-q^n)^{-8}$. The contribution from the Neveu-Schwarz oscillators is

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(b)}}\, \mathcal{C}\, q^{\mathcal{L}^{(b)}} &= \prod_{\mu=0}^{9} \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{A_r^\mu}\, \mathcal{C}\, q^{r\, b_r^{\mu\dagger} b_r^\mu} \\
&= \prod_{\mu=0}^{9} \prod_{r=\frac{1}{2}}^{\infty} \left(1 + (-1)^{\delta_{\mu,0}} q^r\right) \\
&= \prod_{r=\frac{1}{2}}^{\infty} (1-q^r)\cdot(1+q^r)^9\ .
\end{aligned}$$

Finally we compute the contribution coming from the super-ghosts. Notice that because of the nature of the conjugation $\mathcal{C}$ we only pick a contribution to the trace from states whose β and γ occupation numbers coincide. Therefore

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\, \mathcal{C}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{r=\frac{1}{2}}^{\infty} \mathrm{Tr}_{S_r}\, \mathcal{C}\, q^{r(N_\beta + N_\gamma)} \\
&= \prod_{r=\frac{1}{2}}^{\infty} \sum_{n=0}^{\infty} q^{2rn}
\end{aligned}$$

$$= \prod_{r=\frac{1}{2}}^{\infty} \left(1-q^{2r}\right)^{-1} \ .$$

Combining all results we find

$$\mathrm{Tr}_{\mathcal{F}}\, \mathcal{C}\, q^{\mathcal{L}} = \prod_{n=1}^{\infty} (1-q^n)^{-8} \times \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^8 \ , \tag{4.21}$$

which agrees with (4.19), hence proving the "no-ghost" theorem.

The Ramond Sector of the Open NSR String

Again we shall be brief. In the Ramond sector of the NSR string we find the following oscillators: $\{a_n^\mu, b_n, c_n, d_n^\mu, \beta_n, \gamma_n\}$ for all $n \in \mathbb{Z}$ and $\mu = 0, \ldots, 9$ and where, as usual, we shall confuse a_0^μ with p^μ. These oscillators enjoy the hermiticity properties

$$\begin{aligned} &a_n^{\mu\,\dagger} = a_{-n}^\mu \qquad b_n^\dagger = b_{-n} \qquad c_n^\dagger = c_{-n} \\ &d_n^{\mu\,\dagger} = d_{-n}^\mu \qquad \beta_n^\dagger = \beta_{-n} \qquad \gamma_n^\dagger = -\gamma_{-n} \qquad \forall n \in \mathbb{Z} \end{aligned} \tag{4.22}$$

and they obey the following canonical (anti)commutation relations:

$$\begin{aligned} &[a_n^\mu \,,\, a_m^{\nu\,\dagger}] = \eta^{\mu\nu}\delta_{mn} \qquad \{b_n \,,\, c_m^\dagger\} = \delta_{mn} \\ &\{d_m^\mu \,,\, d_n^{\nu\,\dagger}\} = \eta^{\mu\nu}\delta_{mn} \qquad [\beta_m \,,\, \gamma_n^\dagger] = \delta_{mn} \qquad \forall\, m,n \in \mathbb{N} \\ &\{b_0 \,,\, c_0\} = 1 \qquad [\gamma_0 \,,\, \beta_0] = 1 \ , \end{aligned} \tag{4.23}$$

and all other (anti)commutators vanish.

A vacuum in the reduced Fock space is defined by

$$\begin{aligned} &p^\mu \,|k\rangle = k^\mu \,|k\rangle \qquad a_n^\mu \,|k\rangle = 0 \qquad b_n \,|k\rangle = 0 \qquad c_n \,|k\rangle = 0 \\ &d_n^\mu \,|k\rangle = 0 \qquad \beta_n \,|k\rangle = 0 \qquad \gamma_n \,|k\rangle = 0 \qquad \forall\, n > 0 \ . \end{aligned}$$

To define a vacuum in the full Fock space we must tensor $|k\rangle$ with any state in the Hilbert space of zero modes: $\{d_0^\mu, \beta_0, \gamma_0, b_0, c_0\}$. The ghost zero modes $\{b_0, c_0\}$

define a Clifford algebra on two generators and hence its irreducible representations are two dimensional. Similarly the zero modes of the Ramond field d_0^μ define a Clifford algebra with 10 generators and hence its unique irreducible representation is 32-dimensional. Finally the super-ghost zero modes $\{\beta_0, \gamma_0\}$ define a Heisenberg algebra whose unique irreducible representation (the "Schrödinger representation") is infinite dimensional. Hence there is an infinite degeneracy in the space of vacua. This presents some difficulties especially when arguing that the study of the reduced BRST operator suffices. Some progress was made in **[12]** but in our opinion the results presented there are inconclusive and work is presently under way to put this on a more solid foundation. We will however for the purposes of this paper accept the results in **[12]** although what is given in that paper is not a proof but just a strong indication.

Because of the proliferation of zero modes the decomposition of the BRST operator analogous to equation (4.6) is more complicated:

$$Q = c_0\, H - 2b_0\, T - \gamma_0^2 b_0 + \mathbb{Q} \ , \tag{4.24}$$

where

$$\mathbb{Q} = \beta_0\, K + \gamma_0\, F + \mathcal{Q} \ , \tag{4.25}$$

where

$$H = \frac{1}{2}p^2 + \mathcal{L} \tag{4.26}$$

$$T = \sum_{n>0} \left(c_n^\dagger c_n - \gamma_n^\dagger \gamma_n \right) \tag{4.27}$$

$$F = F_0 + \frac{1}{2}\sum_{n>0} \sqrt{n} \left(c_n^\dagger \beta_n - \beta_n^\dagger c_n \right) + 2 \sum_{n>0} \sqrt{n} \left(\gamma_n^\dagger b_n - b_n^\dagger \gamma_n \right) \tag{4.28}$$

$$K = \frac{3}{2}\sum_{n>0} \sqrt{n} \left(c_n^\dagger \gamma_n + \gamma_n^\dagger c_n \right) \ , \tag{4.29}$$

and where

$$F_0 = p \cdot d_0 + \sum_{n>0} \sqrt{n} \left(a_n^\dagger \cdot d_n + d_n^\dagger \cdot a_n \right) \ . \tag{4.30}$$

The level operator $\mathcal{L}$ may be further decomposed as

$$\mathcal{L} = \mathcal{L}^{(a)} + \mathcal{L}^{(n)} + \mathcal{L}^{(b,c)} + \mathcal{L}^{(\beta,\gamma)} \tag{4.31}$$

where

$$\mathcal{L}^{(a)} = \sum_{n>0} n\, a_n^\dagger \cdot a_n \tag{4.32}$$

$$\mathcal{L}^{(d)} = \sum_{n>0} n\, d_n^\dagger \cdot d_n \tag{4.33}$$

$$\mathcal{L}^{(b,c)} = \sum_{n>0} n\, \left(b_n^\dagger c_n + c_n^\dagger b_n \right) \tag{4.34}$$

$$\mathcal{L}^{(\beta,\gamma)} = \sum_{n>0} n\, \left(\beta_n^\dagger \gamma_n + \gamma_n^\dagger \beta_n \right) \ . \tag{4.35}$$

The nilpotency of the BRST operator Q implies the following identities

$$\mathbb{Q}^2 = 0 \qquad F^2 = H \qquad [F\, ,\, T] = K \qquad \mathcal{Q}^2 = 2H\,T + F\,K \ , \tag{4.36}$$

and all other (anti)commutators vanish. Just as before we can restrict ourselves to $\ker H$. In this subspace F is nilpotent and hence its cohomology can be studied. It is shown in **[12]** that as long as $k^\mu \neq 0$ this cohomology is trivial and for a particular class of vacua we can restrict ourselves to the subspace $\ker F$ where $\mathcal{Q}$ is nilpotent. Therefore at least in some sense we can restrict ourselves to the reduced BRST operator. However it is not clear in this case what the relation between the two cohomology spaces $H(Q)$ and $H(\mathcal{Q})$ is; although it seems plausible that $H(Q)$ is just a infinite number of copies of $H(\mathcal{Q})$.

Allowing ourselves to study the cohomology of the reduced BRST operator and assuming the vanishing theorem for its cohomology one can again prove the "no-ghost" theorem. The reduced Fock space has the following structure

$$\mathcal{F} = \mathcal{F}^{(a)} \otimes \mathcal{F}^{(b,c)} \otimes \mathcal{F}^{(d)} \otimes \mathcal{F}^{(\beta,\gamma)} \ , \tag{4.37}$$

where $\mathcal{F}^{(a)}$ and $\mathcal{F}^{(b,c)}$ are given by (4.15) and (4.16) respectively. As for the rest

$$\mathcal{F}^{(d)} = \bigotimes_{\mu=0}^{9} \bigotimes_{n=1}^{\infty} A_n^\mu \tag{4.38}$$

$$\mathcal{F}^{(\beta,\gamma)} = \bigotimes_{n=1}^{\infty} S_n \ . \tag{4.39}$$

Here A_n^μ is the Hilbert space of the $d_n^{\mu\dagger}$ oscillator and is isomorphic to the exterior algebra on one generator; and S_n is the Hilbert space of the $\{\beta_n^\dagger, \gamma_n^\dagger\}$ oscillators and is isomorphic to the polynomial algebra in two variables.

Again the contribution to the Euler characteristic of the complex coming from the first two terms in the above decomposition of $\mathcal{F}$ is $\prod_{n=1}^{\infty}(1-q^n)^{-8}$. The Ramond oscillators contribute

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(d)}}\, q^{\mathcal{L}^{(d)}} &= \prod_{\mu=0}^{9} \prod_{n=1}^{\infty} Tr_{A_n^\mu}\, q^{n\, d_n^{\mu\dagger} d_n^\mu} \\
&= \prod_{n=1}^{\infty} (1+q^n)^{10} \ ,
\end{aligned}$$

and the contribution from the super-ghosts is

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\, (-1)^{\mathcal{G}}\, q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n}\, (-1)^{N_\gamma - N_\beta}\, q^{n(N_\gamma + N_\beta)} \\
&= \prod_{n=1}^{\infty} \mathrm{Tr}_{S_n}\, (-q^n)^{N_\gamma + N_\beta} \\
&= \prod_{n=1}^{\infty} \sum_{m,p=0}^{\infty} (-q^n)^{m+p} \\
&= \prod_{n=1}^{\infty} \left(\sum_{m=0}^{\infty} (-q^n)^m \right)^2 \\
&= \prod_{n=1}^{\infty} (1+q^n)^{-2} \ ,
\end{aligned}$$

where N_β (resp. N_γ) is the number operator corresponding to the $\{\beta_n\}$ (resp.

$\{\gamma_n\}$) oscillators. Putting everything together we find that

$$\mathrm{Tr}_{\mathcal{F}}\,(-1)^{\mathcal{G}}\,q^{\mathcal{L}} = \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^8 \quad . \tag{4.40}$$

In order to compute the signature of the complex we need to discuss the conjugation $\mathcal{C}$. This is similar to the one for the Neveu-Schwarz sector (*cf.* (4.20))

$$\begin{aligned}
\mathcal{C}\,d_m^\mu\,\mathcal{C} &= (-1)^{\delta_{\mu,0}}\,d_m^\mu \qquad \forall\, m\in\mathbb{Z}\\
\mathcal{C}\,\gamma_n\,\mathcal{C} &= \beta_n \qquad \mathcal{C}\,\beta_n\,\mathcal{C} = \gamma_n\\
\mathcal{C}\,\gamma_{-n}\,\mathcal{C} &= -\beta_{-n} \qquad \mathcal{C}\,\beta_{-n}\,\mathcal{C} = -\gamma_{-n} \qquad \forall\, n\in\mathbb{N}\ .
\end{aligned} \tag{4.41}$$

The contribution to the signature of the complex coming from the $\{a_n^\mu, b_n, c_n\}$ is once again $\prod_{n=1}^{\infty}(1-q^n)^{-8}$. The Ramond oscillators contribute

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(d)}}\,\mathcal{C}\,q^{\mathcal{L}^{(d)}} &= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty}\mathrm{Tr}_{A_n^\mu}\,\mathcal{C}\,q^{n\,d_n^{\mu\dagger}d_n^\mu}\\
&= \prod_{\mu=0}^{9}\prod_{n=1}^{\infty}\left(1+(-1)^{\delta_{\mu,0}}q^n\right)\\
&= \prod_{n=1}^{\infty}(1-q^n)\cdot(1+q^n)^9 \quad .
\end{aligned}$$

Finally we compute the contribution coming from the super-ghosts. Just as in the Neveu-Schwarz sector we only pick a contribution to the trace from states whose β and γ occupation numbers coincide. Indeed,

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{F}^{(\beta,\gamma)}}\,\mathcal{C}\,q^{\mathcal{L}^{(\beta,\gamma)}} &= \prod_{n=1}^{\infty}\mathrm{Tr}_{S_n}\,\mathcal{C}\,q^{n(N_\beta+N_\gamma)}\\
&= \prod_{n=1}^{\infty}\sum_{m=0}^{\infty}q^{2nm}
\end{aligned}$$

$$= \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)^{-1} \; .$$

Combining all results we find

$$\mathrm{Tr}_{\mathcal{F}}\, \mathcal{C}\, q^{\mathcal{L}} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^8 \; , \tag{4.42}$$

which agrees with (4.40), hence proving the "no-ghost" theorem.

Finally we remark that the GSO projected NSR string is also free of ghosts. This is true because modular invariance also forces the GSO projection on the super-ghost spectrum which goes hand in hand with the GSO projection in the spectrum of the Neveu-Schwarz and Ramond oscillators. We leave the trivial details of this calculation as an exercise.

§5 CONCLUSIONS

In this paper we have applied the techniques of [**1**] to the particular case of the open bosonic string. It is remarkable that the proofs are so simple. We think that the importance of these techniques has been established.

Moreover the importance of the vanishing theorem both for consistency of the BRST quantization, as was proven in [**1**], and as a powerful tool in proving interesting technical results can hardly be overemphasized. But as mentioned in the introduction the fundamental aim of this paper is still unfulfilled. We hope that a self-contained simple proof of the vanishing theorem using these techniques can be found, thus providing a sense of completion to this chapter of BRST quantization.

ACKNOWLEDGEMENTS

We would like to acknowledge conversations with Anders Karlhede, Ulf Lindström, Jim McCarthy, Martin Roček and Warren Siegel.

Appendix A DISJOINTNESS AND DECOMPOSITION

For convenience and since it is in this way that we use the result in §4 we assume that $\mathbb{V}$ is a finite dimensional vector space and d and δ are two nilpotent endomorphisms. We recall that we said that d and δ are **disjoint** if and only if

$$
\begin{aligned}
&d\,\delta\, v = 0 \Rightarrow \delta\, v = 0 \ , \\
&\delta\, d\, v = 0 \Rightarrow d\, v = 0 \qquad \forall\, v \in \mathbb{V}
\end{aligned}
\tag{A.1}
$$

This is the case, for instance, when d and δ are adjoints of each other with respect to a positive definite inner product in $\mathbb{V}$. We saw in §3 how in this case there was a decomposition theorem. In this appendix we show that in fact the weaker condition of disjointness suffices[10]. If d and δ are disjoint we define the endomorphism $\triangle \equiv d\,\delta + \delta\, d$. Then we have the following:

Theorem. *Let d and δ be disjoint and $\triangle$ be as above. Then the following hold:*

(1) $\ker \triangle = \ker d \cap \ker \delta$

(2) $\mathbb{V} = \mathrm{im}\, d \oplus \mathrm{im}\, \delta \oplus \ker \triangle$

(3) If H_d denotes the cohomology space $\ker d / \mathrm{im}\, d$, *then the canonical projection* $\pi : \ker d \to H_d$ *induces an isomorphism* $\pi : \ker \triangle \xrightarrow{\cong} H_d$.

Proof: (1) It is obvious that $\ker d \cap \ker \delta \subset \ker \triangle$. Now let $v \in \ker \triangle$ and let $w = -\delta\, d\, v = d\,\delta\, v$. Clearly $w \in \ker d \cap \ker \delta$ by nilpotency of d and δ. But by disjointness, $-d\,\delta\, d\, v = 0 \Rightarrow \delta\, d\, v = 0 \Rightarrow d\, v = 0$ and $\delta\, d\,\delta\, v = 0 \Rightarrow d\,\delta\, v = 0 \Rightarrow \delta\, v = 0$. Hence $v \in \ker d \cap \ker \delta$ as well.

(2) Let $x = d\,v = \delta\,w$ then by nilpotency of d, $d\,\delta\,w = 0$, which by disjointness implies that $\delta\,w = 0$. Thus $\mathrm{im}\,d \cap \mathrm{im}\,\delta = \mathbb{O}$ and hence $\mathrm{im}\,d + \mathrm{im}\,\delta$ is direct. We claim that $\ker\,\triangle \cap (\mathrm{im}\,d \oplus \mathrm{im}\,\delta) = \mathbb{O}$. In fact let $x \in \ker\triangle$. If $x \in \mathrm{im}\,d \oplus \mathrm{im}\,\delta$ it can be written (uniquely) as $x = d\,v + \delta\,w$. But by (1), $d\,x = 0$ and $\delta\,x = 0$. Hence, $d\,\delta\,w = 0$ and $\delta\,d\,v = 0$ which by disjointnes forces $\delta\,w = 0$ and $d\,v = 0$ and consequently $x = 0$. From its definition, $\mathrm{im}\,\triangle \in \mathrm{im}\,d \oplus \mathrm{im}\,\delta$. Hence $\ker\,\triangle + \mathrm{im}\,\triangle$ is direct and since $\mathbb{V}$ is finite dimensional $\mathbb{V} = \ker\,\triangle \oplus \mathrm{im}\,\triangle$. In particular, $\mathrm{im}\,\triangle = \mathrm{im}\,d \oplus \mathrm{im}\,\delta$.

(3) Let $v, w \in \ker\triangle$. If $\pi\,v = \pi\,w$ then $v = w + d\,z$, $\exists z$. But since $\delta\,v = \delta\,w = 0$ we conclude that $\delta\,d\,z = 0$, which by disjointness imlies that $d\,z = 0$ and hence that $v = w$. Therefore the map $\pi\,:\ker\triangle \to H_d$ is injective. To prove that it is a bijection all we need to do is count dimensions. By (1) and (2) and the nilpotency of d, $\ker d = \ker\,\triangle \oplus \mathrm{im}\,d$. Therefore $\dim\,\ker\triangle = \dim\ker d - \dim\,\mathrm{im}\,d$ which is precisely $\dim\,H_d$. ■

Corollary. *If H_δ is the cohomology space* $\ker\delta/\mathrm{im}\,\delta$ *then $H_d \cong H_\delta$.*

Proof: Just replace δ with d in the previous theorem and using (3) we find the required isomorphisms $H_d \cong \ker\triangle \cong H_\delta$. ■

Moreover if d and δ are disjoint and related by an involution $\star \in \mathrm{End}\,\mathbb{V}$ in such a way that $\delta = \star\,d\,\star$, then $\star$ gives the isomorphism of the previous corollary since $\star\,\triangle = \triangle\,\star$ as can be easily verified.

REFERENCES

[**1**] J. M. Figueroa-O'Farrill and T. Kimura, (ITP-SB-88-34 (revised))

[**2**] J. M. Figueroa-O'Farrill and T. Kimura, unpublished.

[**3**] I. B. Frenkel, H. Garland and G. J. Zuckerman, *Proc. Natl. Acad. Sci. USA* **83** (1986) 8442

[**4**] T. Banks and R. Peskin, *Nucl. Phys.* **B264** (1986) 513

[5] M. Spiegelglas, *Nucl. Phys.* **B283** (1987) 205

[6] M. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, (Cambridge 1987)

[7] W. Siegel and B. Zwiebach, *Nucl. Phys.* **B263** (1986) 105

[8] R. O. Wells, *Differential Analysis on Complex Manifolds*, (Springer-Verlag 1980)

[9] B. Feigin, *Usp. Mat. Nauk* **39** (1984) 195 (English Translation: *Russian Math Surveys* **39** (1984) 155)

[10] B. Kostant, *Ann. of Math.* **74** (1961) 329

[11] R. C. Brower and C. B. Thorn, *Nucl. Phys.* **B31** (1971) 136;
R. C. Brower, *Phys. Rev.* **D6** (1972) 1655;
P. Goddard and C. B. Thorn, *Phys. Lett.* **40B** (1972) 235;
M. Kato and K. Ogawa, *Nucl. Phys.* **B212** (1983) 443;
M. Freeman and D. Olive, *Phys. Lett.* **175B** (1986) 151;
M. Henneaux, *Phys. Lett.* **177B** (1986) 35;
N. Ohta, *Phys. Lett.* **179B** (1986) 347

[12] M. Henneaux, *Phys. Lett.* **183B** (1987) 59