SOME RESULTS ON THE BRST COHOMOLOGY OF THE NSR STRING

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ABSTRACT

We summarize some results obtained on the BRST cohomology of the NSR string: among them vanishing theorems for the full and relative complexes, extending the work of Frenkel, Garland, and Zuckerman for the bosonic string. Using these results we give simple proofs of the "no-ghost" theorems for both sectors.

§1 Introduction

It is by now well known that the Dirac quantization of a Hamiltonian system with first class constraints can be translated into the problem of analyzing the cohomology of the BRST operator. Of course, many properties of this cohomology theory will depend on the particular physical theory; but, nevertheless, quite a lot can be said in general by the study of the structures arising in a BRST quantized theory.

The data defining a BRST cohomology theory consists of a differential complex $(\mathcal{H}, Q)$ where $\mathcal{H}$ is a Fock space with an indefinite metric $\langle , \rangle$ relative to which Q (the BRST operator) is self-adjoint, and a skew-adjoint operator $\mathcal{G}$ (the ghost number operator) which defines an integral (or half-integral in some cases) grading of $\mathcal{H}$ relative to which Q has degree 1. We can then define the space of physical quanta $\mathcal{H}_{\text{phys}}$ as the cohomology at zero ghost number of Q. Because of the hermiticity of the BRST operator, this space inherits a well-defined inner product: *i.e.* it is independent of the representatives chosen from the cohomology classes. Physical observables, *i.e.* self-adjoint endomorphisms of $\mathcal{H}_{\text{phys}}$, can quite generally be shown to arise from BRST invariant endomorphisms of $\mathcal{H}$. In fact, in [**1**] it is shown that in many cases $H_Q(\operatorname{End}\mathcal{H}) \cong \operatorname{End} H_Q(\mathcal{H})$.

However to have a sensible quantum theory in $\mathcal{H}_{\text{phys}}$ we require that the inherited inner product be positive definite. This absence of negative norm states — unfortunately misnamed the "no-ghost" theorem — must be checked in each particular physical theory. We showed in [**1**], generalizing ideas of [**2**], that if all the BRST cohomology is concentrated in the zero ghost number sector (the "vanishing" theorem) then the verification of the "no-ghost" theorem reduced itself to the calculation of certain partition-function-like traces.

But the vanishing theorem plays a more fundamental rôle than just as a calculational nicety. In [**1**] we argued that it is also necessary for the consistency of the BRST quantization. We showed that if the vanishing theorem does not hold then $\mathcal{H}_{\text{phys}}$ may contain states with ghost excitations, which would not be there had we quantized the theory in some other way which did not require the introduction of ghosts.

An important example of a BRST cohomology theory is the semi-infinite cohomology of Feigin[3] of a graded Lie algebra with coefficients in a module of the category $\mathcal{O}$. A vanishing theorem valid for a large class of such modules was proven by Frenkel, Garland, and Zuckerman in [**2**]. This included the BRST cohomology of the open bosonic string away from zero center-of-mass momentum. Also in [**2**], and later in [**4**], the "no-ghost" theorem for the open bosonic string was proven in the way we later generalized in [**1**].

In this letter we summarize some results obtained for the NSR string along these lines: the vanishing and "no-ghost" theorems for both the Neveu-Schwarz and Ramond sectors. The interested reader will find the details in a forthcoming paper[5]. We should emphasize that the results in this letter are not new, having appeared, for example, in [**9**]. The methods used, however, are new and are more suitable for generalization.

§2 The Neveu-Schwarz Sector

We follow for the most part the notation and conventions of [**6**]. Let $\mathcal{NS}$ denote the centrally extended complexified Neveu-Schwarz algebra. Let $\mathbb{M}$ denote the "matter" Fock space of the Neveu-Schwarz of the NSR string: that is, the Fock space for the $\{\alpha_n^\mu\}$ and $\{b_r^\mu\}$ oscillators. The full Fock space $\mathcal{H}$ (including the ghost and anti-ghost oscillators) can be thought of, by analogy with the case of Lie algebras, as the space of semi-infinite cochains of $\mathcal{NS}$ with coefficients in $\mathbb{M}$. The differential in this complex is the BRST operator Q. Making the dependence of the ghost and anti-ghost zero modes manifest, the BRST operator becomes

$$Q = \mathcal{Q} - 2b_0\, T + c_0\, L_0\ . \tag{2.1}$$

We denote the cohomology of Q in $\mathcal{H}$, by analogy, as $H_\infty(\mathcal{NS};\mathbb{M})$: the semi-infinite cohomology of the Lie super-algebra $\mathcal{NS}$ with coefficients in the module $\mathbb{M}$. This cohomology is graded by ghost number.

Brower and Friedman in [**7**] showed that $\mathbb{M}$ admits a decomposition $\mathbb{M} = (\bigoplus_\lambda \mathbb{V}_\lambda) \oplus \mathbb{V}_0$ where $\mathbb{V}_0$ is the Fock space at zero center-of-mass momentum and the $\mathbb{V}_\lambda$ are Verma modules whose highest weight vectors are obtained by repeated application of the creation operators in the full spectrum generating algebra. It turns out that $\mathbb{V}_0$ is not a Verma module. In this case, however, the BRST cohomology is very easy to calculate explicitly. According to the decomposition of $\mathbb{M}$, the semi-infinite cohomology also breaks up as $\bigoplus_\lambda H_\infty(\mathcal{NS}; \mathbb{V}_\lambda) \oplus H_\infty(\mathcal{NS}; \mathbb{V}_0)$. We ignore $\mathbb{V}_0$ from now on and we focus on $H_\infty(\mathcal{NS}; \mathbb{V})$ where $\mathbb{V}$ is a fixed $\mathbb{V}_\lambda$. Consider the relative subcomplex consisting of vectors $\omega \in \mathcal{H}$ satisfying $b_0\,\omega = L_0\,\omega = 0$. The induced differential in this subcomplex is easily seen to be the operator $\mathcal{Q}$ in (2.1) . The cohomology of this complex is nothing but the relative semi-infinite cohomology $H_\infty(\mathcal{NS}, \mathcal{NS}_0; \mathbb{V})$ where $\mathcal{NS}_0$ is the subalgebra generated by L_0.

Then there exists a filtration of this complex giving rise to a spectral sequence (see *e.g.* [**8**]) converging to $H_\infty(\mathcal{NS}, \mathcal{NS}_0; \mathbb{V})$, whose E_1 term obeys $E_1^m = 0$ for $m < 0$. Therefore $H_\infty^m(\mathcal{NS}, \mathcal{NS}_0; \mathbb{V}) = 0$ for $m < 0$. The "Poincaré duality" theorem proven in [**1**]:

$$H_\infty^m(\mathcal{NS}, \mathcal{NS}_0; \mathbb{V}) \cong H_\infty^{-m}(\mathcal{NS}, \mathcal{NS}_0; \mathbb{V}) \ , \tag{2.2}$$

allows us to conclude that

$$H_\infty^{m\neq 0}(\mathcal{NS}, \mathcal{NS}_0; \mathbb{V}) = 0 \ . \tag{2.3}$$

Putting all Verma modules $\mathbb{V}$ together we obtain the vanishing theorem for the full relative subcomplex away from zero center-of-mass momentum.

From this and the fact that there is another spectral sequence relating the relative cohomology $H_\infty(\mathcal{NS}, \mathcal{NS}_0; \mathbb{M})$ to the full cohomology $H_\infty(\mathcal{NS}; \mathbb{M})$, we can determine the full cohomology from a knowledge of the relative one. In fact, this spectral sequence collapses at the E_1 term, which is given by

$$E_1^{m+\frac{1}{2}} = H_\infty^m(\mathcal{NS}, \mathcal{NS}_0; \mathbb{M}) \oplus H_\infty^{m+1}(\mathcal{NS}, \mathcal{NS}_0; \mathbb{M}) \ . \tag{2.4}$$

Therefore from (2.3) we conclude that

$$H_\infty^m(\mathcal{NS}; \mathbb{M}) \cong \begin{cases} H_\infty^0(\mathcal{NS}, \mathcal{NS}_0; \mathbb{M}) & \text{for } m = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \ . \tag{2.5}$$

The reason that the full cohomology is half-integrally graded is that the ghost number operator acting on the ghost zero modes has half-integral eigenvalues.

§3 The Ramond Sector

If, in the Ramond sector, we try to repeat the steps leading to the vanishing theorem for the Neveu-Schwarz sector we are immediately faced with a dilemma: the choice of representation for the superconformal ghost zero modes: β_0 and γ_0. Canonical quantization induces hermiticity properties on these modes such that β_0 is antihermitian and γ_0 is hermitian. If we wished to preserve these hermiticity properties the natural representation for these modes is the Schrödinger representation, in which the representation space is the closure of the polynomial algebra $\mathbb{C}[a^\dagger]$, where $\beta_0 = \frac{1}{\sqrt{2}}(a^\dagger - a)$ and $\gamma_0 = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and $[a\,,\,a^\dagger] = 1$. The BRST charge then becomes

$$Q = c_0\, L_0 - 2b_0\, T - \gamma_0^2 b_0 + \mathbb{Q} \ , \tag{3.1}$$

where

$$\mathbb{Q} = \mathcal{Q} + \frac{1}{\sqrt{2}}(F_0 + K)\, a^\dagger + \frac{1}{\sqrt{2}}(F_0 - K)\, a \ . \tag{3.2}$$

In this representation the ghost number operator has a piece $(a^\dagger)^2 - a^2$ which is not diagonalizable. Therefore the cohomology of Q (or $\mathbb{Q}$) is not graded. Therefore the vanishing theorem does not make sense.

On the other hand, following Henneaux[9], we may work with a different representation for the superconformal ghost zero modes whose carrier space is $\mathbb{C}[\gamma_0]$. It must be remarked[9] that there is no positive definite inner product on $\mathbb{C}[\gamma_0]$ compatible with the hermiticity conditions on β_0 and γ_0 induced via canonical quantization. If we demand a positive definite inner product we would have to alter their hermiticity properties in such a way that β_0 and γ_0 are mutually adjoint. Inner product aside we call this the Henneaux representation. In this representation the vanishing theorem does make sense and moreover, as we shall see, it holds.

Since cohomology is a purely algebraic object it is independent of the particular inner product we choose for our representation space. But it clearly does depend on the representation chosen: the BRST operator is different (see below) in each representation and it is therefore not *a priori* obvious that their cohomologies are isomorphic. In fact we already saw that in the Schrödinger representation the cohomology is not graded whereas in the Henneaux representation it is; although, as we will later show, they are isomorphic as (ungraded) vector spaces.

Canonical quantization consists of finding a representation of an operator algebra with involution as operators in a Hilbert space where the involution corresponds to taking adjoints. Therefore we are not free to alter the hermiticity conditions (*i.e.* the involution). This is particularly important in BRST quantization where the hermiticity properties of the BRST operator Q are instrumental in guaranteeing that the inner product in cohomology is independent of the representative.

In working with the Henneaux representation we are not changing the quantization procedure. For us the Henneaux representation is an auxiliary device with nice formal properties which will allow us to prove, for example, the "no-ghost" theorem for the Schrödinger representation.

In the Henneaux representation the BRST operator is

$$Q = c_0\, L_0 - 2b_0\, T - \gamma_0^2 b_0 + \beta_0\, K + \gamma_0\, F_0 + \mathcal{Q}\ . \tag{3.3}$$

Consider the subcomplex defined by those $\omega \in \mathcal{H}$ such that $F_0\,\omega = b_0\,\omega = \beta_0\,\omega = 0$. The differential in this complex is $\mathcal{Q}$ and its cohomology is the relative cohomology $H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{M})$; where $\mathcal{R}_0$ is the subalgebra spanned by F_0 and L_0, and $\mathbb{M}$ is the "matter" Fock space of the Ramond sector. Again results of Brower and Friedman[7] tell us that $\mathbb{M}$ breaks up as $(\bigoplus_\lambda \mathbb{V}_\lambda)\oplus\mathbb{V}_0$ where $\mathbb{V}_\lambda$ are Verma modules and $\mathbb{V}_0$ is the submodule at zero center-of-mass momentum, which, as in the Neveu-Schwarz case, is not a Verma module. Again the cohomology breaks up as $\bigoplus_\lambda H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{V}_\lambda)\oplus H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{V}_0)$. The cohomology $H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{V}_0)$ is easy to calculate explicitly and, in fact, it is all concentrated in the zero ghost number sector.

Fix $\mathbb{V}$ to be one of the $\mathbb{V}_\lambda$. One can show that there exists a spectral sequence converging to $H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{V})$ such that its E_1 term obeys $E_1^m = 0$ for $m<0$, which together with "Poincaré duality"[1] implies the vanishing theorem for $H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{V})$.

As a result[9] of the triviality of the cohomology of F_0 (which is nilpotent when restricted to states $\omega\in\mathcal{H}$ such that $b_0\,\omega = L_0\,\omega = 0$) we can prove the following isomorphism between relative cohomologies

$$H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{V}) \cong H_\infty(\mathcal{R},\mathcal{L}_0;\mathbb{V}) \ , \tag{3.4}$$

where $\mathcal{L}_0$ is the subalgebra spanned by L_0. Putting all the $\mathbb{V}_\lambda$ together (and even $\mathbb{V}_0$ in this sector) we obtain a vanishing theorem for $H_\infty(\mathcal{R},\mathcal{L}_0;\mathbb{M})$. A spectral sequence argument as in the Neveu-Schwarz case further implies that

$$H^m_\infty(\mathcal{R};\mathbb{M}) \cong \begin{cases} H^0_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{M}) & \text{for } m=\pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \ . \tag{3.5}$$

Now let's go back to the Schrödinger representation. In this case one can show the following isomorphism of cohomologies

$$[H_\infty(\mathcal{R},\mathcal{L}_0;\mathbb{M})]_S \cong [H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{M})]_H \ , \tag{3.6}$$

where the subscripts H and S refer to the Henneaux and the Schrödinger representation respectively. The idea of the proof is first to show that each class in $[H_\infty(\mathcal{R},\mathcal{L}_0;\mathbb{M})]_S$ contains a representative ψ such that it is annihilated by a. Then $e^T\,\psi$ is a representative of a class in $[H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{M})]_H$. Choosing a suitable normalization the above isomorphism is actually an isometry, which makes sense since $[H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{M})]_H$ and $[H_\infty(\mathcal{R},\mathcal{L}_0;\mathbb{M})]_S$ both inherit well-defined inner products because their respective differentials ($\mathcal{Q}$ and $\mathbb{Q}$) are hermitian. Therefore proving the "no-ghost" theorem for $[H_\infty(\mathcal{R},\mathcal{R}_0;\mathbb{M})]_H$ proves it for $[H_\infty(\mathcal{R},\mathcal{L}_0;\mathbb{M})]_S$. This we do in the next section.

§4 The "No-Ghost" Theorems

With these results in mind we can now prove the "no-ghost" theorem for the NSR string along the lines suggested in [**1**]. This method was used to prove the similar result for the bosonic string in [**2**] and [**4**]. We briefly recall the method.

Let K denote the appropriate relative subcomplex of the string. We use the relative subcomplexes since, as we have seen before, the full complex is just two copies of the relative one; hence proving positive-definiteness of the inner product in the relative subcomplex suffices. Let $\mathcal{C}$ denote the conjugation[**1**] used to redefine the inner product in order to make it positive definite. The existence of this positive definite inner product allows us to define a BRST laplacian whose kernel — denoted by $\mathbb{H}$ and referred to as the space of harmonic states — is isomorphic to the BRST cohomology. Because $\mathcal{C}$ commutes with the laplacian it stabilizes its kernel. Moreover since $\mathcal{C}$ reverses ghost number it stabilizes also $\mathbb{H}_0$, the space of harmonic states at zero ghost number, which is isomorphic to the physical space. From its definition (see [**1**] for the details and [**5**] for its explicit construction in this case) $\mathcal{C}$ is the identity on states of positive norm and minus the identity on states of negative norm. Therefore we see that[1]

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[1] As it stands this next equation is ill-defined since $\mathbb{H}_0$ is infinite dimensional. These quantities are to be understood as weighted traces; the dimension being understood as the trace of the identity.

$$\mathrm{Tr}_{\mathbb{H}_0}\, \mathcal{C} \leq \dim\, \mathbb{H}_0 \ , \tag{4.1}$$

where the bound is saturated if and only if $\mathbb{H}_0$ is positive definite. Since the inner product on the cohomology does not depend on the particular representative, the saturation of the above bound is equivalent to the "no-ghost" theorem.

As shown in [**1**], we can extend the trace of $\mathcal{C}$ over $\mathbb{H}_0$ over all of K without picking any further contributions. This makes the left hand side of (4.1) easy to compute. As for the right hand side we notice, using the vanishing theorem, that $\dim\, \mathbb{H}_0$ is nothing but the Euler characteristic $\chi(K)$ of the relative subcomplex: the alternating sum of the dimensions of the cohomology spaces. Using the Euler-Poincaré principle we can write the Euler characteristic as $\mathrm{Tr}_K\, (-1)^{\mathcal{G}}$ where $\mathcal{G}$ is the ghost number operator in the relative subcomplex. This again is easy to compute. Moreover, since the relative subcomplex is graded by the level operator $\mathcal{L}$ (the momentum independent piece of L_0) and each level eigenspace is finite dimensional both $\mathrm{Tr}_K\, q^{\mathcal{L}}\mathcal{C}$ and $\mathrm{Tr}_K\, q^{\mathcal{L}}(-1)^{\mathcal{G}}$ converge for sufficiently small q. Therefore the "no-ghost" theorem is equivalent to the saturation of the inequality

$$\mathrm{Tr}_K\, q^{\mathcal{L}}\mathcal{C} \leq \mathrm{Tr}_K\, q^{\mathcal{L}}(-1)^{\mathcal{G}} \ . \tag{4.2}$$

These traces are straight-forward to calculate so we will only give the results. For the Neveu-Schwarz sector one finds that

$$\mathrm{Tr}_K\, (-1)^{\mathcal{G}}\, q^{\mathcal{L}} = \prod_{n=1}^{\infty} (1-q^n)^{-8} \times \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^8 \ ; \tag{4.3}$$

and exactly the same expression for the left hand side of (4.2) . For the Ramond sector (in the Henneaux representation) one finds that

$$\mathrm{Tr}_K\, (-1)^{\mathcal{G}}\, q^{\mathcal{L}} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^8 \ , \tag{4.4}$$

which agrees with the left hand side of (4.2) . By the remarks at the end of the previous section, this proves the "no-ghost" theorem for the Schrödinger representation.

Finally we remark that the GSO projected NSR string is also free of ghosts. This is true because modular invariance also forces the GSO projection on the superghost spectrum which goes hand in hand with the GSO projection in the spectrum of the Neveu-Schwarz and Ramond oscillators.

ACKNOWLEDGEMENTS

We would like to thank Martin Roček for suggesting the problem. One of us (JMF) would like to thank Claude LeBrun for some valuable insight into spectral sequences. We would also like to acknowledge conversations and/or seminars with Francisco Figueirido, Ulf Lindström, Jim McCarthy, Eddie Ramos, Martin Roček, and Warren Siegel.

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