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BIHAMILTONIAN STRUCTURE OF THE KP HIERARCHY AND THE W_{KP} ALGEBRA

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ABSTRACT

We construct the second hamiltonian structure of the KP hierarchy as a natural extension of the Gel'fand-Dickey brackets of the generalized KdV hierarchies. The first structure—which has been recently identified as $W_{1+\infty}$—is coordinated with the second structure and arises as a trivial (generalized) cocycle. The second structure gives rise to a non-linear algebra, denoted W_{KP}, with generators of weights $1, 2, \ldots$. The reduced algebra obtained by setting the weight 1 field to zero contains a centerless Virasoro subalgebra, and we argue that this is a universal W-algebra from which all W_n-algebras are obtained through reduction.

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Introduction

The KP hierarchy **[1]** has become an ubiquitous subject in string theory ever since the surprising discovery of the connection between the matrix model formulation of two-dimensional gravity and the generalized KdV hierarchies (see *e.g.* **[2]**). Nevertheless the KP hierarchy—as well as its reductions—has been around for much longer in the field of classical integrable systems. Its exact integrability properties and its relation with free fermion theories via infinite dimensional grassmannians are by now well understood; as well as its hamiltonian nature with respect to the natural Kirillov bracket on a coadjoint orbit of a formal Lie algebra of pseudo-differential operators **[3]** **[4]** , which has been recently identified **[5]** **[6]** with the $W_{1+\infty}$ algebra. These results notwithstanding, a bihamiltonian structure for the KP hierarchy has not, to the best of our knowledge, been constructed. It is the purpose of this letter to do so. As a byproduct we will see that the conserved charges obey (Lenard) recursion relations. As a further byproduct, which is interesting in its own right, we find a nonlinear algebra generated by fields of weights $1, 2, \ldots$ from which $W_{1+\infty}$ can be recovered as a trivial (generalized) cocycle. Furthermore, setting the field of weight 1 equal to zero yields, upon hamiltonian reduction, a nonlinear algebra containing a centerless Virasoro subalgebra—*i.e.* , containing a subalgebra isomorphic to the Lie algebra of Diff S^1. It is reasonable to expect and, in fact, we shall argue that this is the case, that this new nonlinear algebra is a universal W-algebra in the sense that all W_n-algebras **[7]** can be obtained from it through hamiltonian reduction.

The KP hierarchy is defined as the isospectral problem for the pseudo-differential operator (ΨDO)

$$\Lambda = \partial + \sum_{i=0}^{\infty} \partial^{-i} u_i \ . \tag{1}$$

The KP flows are given by

$$\frac{\partial \Lambda}{\partial t_i} = [\Lambda_+^i \, , \, \Lambda] \ . \tag{2}$$

The $n^{\rm th}$-generalized KdV hierarchy is obtained by imposing the constraint $\Lambda^n_- = 0$. It is well known that the flows in (2) preserve this condition and that they are bihamiltonian; that is, hamiltonian relative to two coordinated Poisson brackets: the first and second Gel'fand-Dickey brackets **[8]** . Furthermore, it is by now common knowledge that the second bracket gives rise to a classical realization of the W_n-algebra, and that the first bracket can be obtained from the second as a trivial (generalized) cocycle.

It is therefore a natural question to ask whether the bihamiltonian structure of the $n^{\rm th}$ KdV hierarchy is inherited under reduction from a similar structure in the KP hierarchy. We partially answer this question affirmatively by proving that the KP hierarchy is in fact bihamiltonian. Moreover we present some evidence that leads us to the conjecture that the reduction actually works. The two hamiltonian structures of the KP hierarchy are obtained by a straightforward modification of the hamiltonian formalism for generalized Lax operators.

Lack of space forbids all but a sketchy description of the formalism and hence we refer those readers not familiar with the formal calculus of pseudo-differential operators and/or the basics of integrable systems to Dickey's comprehensive treatment **[9]** .

Bihamiltonian Structure of the KP Hierarchy

We want to define Poisson brackets in the space M of KP operators Λ of the form (1) . In order to define Poisson brackets we need to define several geometric objects: the class of functions on which we wish to define the Poisson brackets, the vector fields and 1-forms, and a map sending (the gradient of) a function to its associated hamiltonian vector field.

We will define Poisson brackets on functions of the form

$$F[\Lambda] = \int f(u) \ , \tag{3}$$

where $f(u)$ is a differential polynomial in the u_i, and the integral sign stands for any reasonable linear map annihilating perfect derivatives so that we can "integrate by parts."

The tangent space $\mathcal{T}$ to M at Λ is isomorphic to the infinitesimal deformations of Λ, which are clearly given by ΨDO's of the form $\sum_{i=0}^{\infty} \partial^{-i} a_i$. Each $A \in \mathcal{T}$ gives rise to a vector field ∂_A whose action on a function $F = \int f$ is given by

$$\partial_A F \equiv \frac{d}{d\epsilon} F[\Lambda + \epsilon A]\Big|_{\epsilon=0} = \int \sum_{k=0}^{\infty} a_k \frac{\delta F}{\delta u_k} \ , \tag{4}$$

where the Euler variational derivative is given by

$$\frac{\delta F}{\delta u_k} = \sum_{i=0}^{\infty} (-\partial)^i \frac{\partial f}{\partial u_k^{(i)}} \ . \tag{5}$$

Notice that both sums are actually finite for f a differential polynomial and thus the action of ∂_A is well defined on our class of functions.

The space $\mathcal{S}$ of ΨDO's of the form $\sum_{i \geq 0} b_i \partial^{i-1}$ parametrizes the 1-forms. Notice that by definition the above sum is finite since ΨDO's are formal Laurent series in ∂^{-1}. Decomposing the ring $\mathcal{P}$ of ΨDO's as $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$, where $\mathcal{P}_\pm$ are respectively the subrings of differential and integral operators, we see that $\mathcal{S} \cong \mathcal{P}/\partial^{-1}\mathcal{P}_-$. The dual pairing between vectors and 1-forms is naturally given by the Adler trace Tr defined as follows: for any ΨDO $P = \sum_i p_i \partial^{-i}$, $\mathrm{Tr}\, P \equiv \int \mathrm{res}\, P \equiv \int p_1$. Thus, if $A \in \mathcal{T}$ and $X \in \mathcal{S}$ then $(\partial_A, X) = \mathrm{Tr}(AX)$. The reader can check that the pairing is nondegenerate.

If $F = \int f$ is any function on M, we define its gradient dF by $\partial_A F = (\partial_A, dF)$ whence it follows that

$$dF = \sum_{k \geq 0} \frac{\delta F}{\delta u_k} \partial^{k-1} \ . \tag{6}$$

It is obvious that for any function in the class defined above, its gradient is a 1-form.

The last ingredient required to define Poisson brackets is a linear map Ω from 1-forms to vector fields which takes the gradient of a function to its associated hamiltonian vector field. This map is naturally induced from a linear map $J : \mathcal{S} \to \mathcal{T}$ by $\Omega(X) = \partial_{J(X)}$. The Poisson bracket is then given by

$$\{F\,,\,G\} = (\Omega(dF), dG) = \mathrm{Tr}(J(dF)dG)\;. \tag{7}$$

Demanding that this be antisymmetric and obey the Jacobi identity, constrains the allowed maps J. We call the allowed maps hamiltonian.

We now proceed to construct a one-parameter family of such maps. Let λ be any constant parameter and define $\widetilde{\Lambda} = \Lambda + \lambda$. Then define J by

$$J(X) = (\widetilde{\Lambda}X)_+\widetilde{\Lambda} - \widetilde{\Lambda}(X\widetilde{\Lambda})_+ = \widetilde{\Lambda}(X\widetilde{\Lambda})_- - (\widetilde{\Lambda}X)_-\widetilde{\Lambda}\;, \tag{8}$$

for X any ΨDO. From the first equality it follows that J annihilates $\partial^{-1}\mathcal{P}_-$, hence only the image of X in $\mathcal{S}$ really matters, whereas from the second it follows that $J(X)$ lies in $\mathcal{T}$; in other words, J defines a map $\mathcal{S} \to \mathcal{T}$. The proof that the Poisson brackets defined by J are antisymmetric and obey the Jacobi identity follows closely the proof of Gel'fand and Dickey **[8]** for their analogous result (*cf.* also **[9]**) and thus we omit it here.

Making the dependence in λ manifest we can write $J(X) = J_2(X) + \lambda J_1(X)$ where

$$J_1(X) = [X_+\,,\,\Lambda]_-\;, \tag{9}$$

$$J_2(X) = (\Lambda X)_+\Lambda - \Lambda(X\Lambda)_+\;. \tag{10}$$

Both J_1 and J_2 define (coordinated) Poisson brackets which, as we will now show, are the first and second hamiltonian structures of the KP hierarchy of equations. Notice that J_1 is obtained from J_2 by shifting $u_0 \mapsto u_0 + \lambda$. The fundamental Poisson brackets computed form J_1 have recently been shown to be isomorphic **[5]** **[6]** to $W_{1+\infty}$.

Let us define functions $H_k = \frac{1}{k}\mathrm{Tr}\,\Lambda^k$. They are clearly conserved quantities for the Lax flows, since the Adler trace annihilates commutators. Moreover they are nontrivial for all k. Using the cyclic property of the trace one finds that $\partial_A H_k = \mathrm{Tr}\,(A\Lambda^{k-1})$ from where the gradients immediately follow

$$dH_k = \Lambda^{k-1} \mod \partial^{-1}\mathcal{P}_- \ . \tag{11}$$

Given a hamiltonian map J, there is a way to associate a flow to a function H by $\frac{\partial\Lambda}{\partial t} = J(dH)$. Choosing J_2 and H_k we find,

$$\begin{aligned} J_2(dH_k) &= J_2(\Lambda^{k-1}) \\ &= [\Lambda^k_+ \, , \, \Lambda] \\ &= \frac{\partial\Lambda}{\partial t_k} \ ; \qquad \text{from (2)} \end{aligned}$$

so that the KP flows (2) are hamiltonian relative to the second hamiltonian structure. Morevoer, since $[\Lambda^k_+ \, , \, \Lambda] = J_1(dH_{k+1})$ we see that they are also hamiltonian with respect to the first structure. Thus the KP flows are bihamiltonian, as summarized by the Lenard relations relating the conserved charges

$$\frac{\partial\Lambda}{\partial t_k} = J_1(dH_{k+1}) = J_2(dH_k) \ . \tag{12}$$

Hamiltonian reduction for $u_0 = 0$

The free term (u_0) in the KP operator does not evolve under any of the KP flows given by (2) and thus the submanifold $\widetilde{M}$ of KP opeartors with $u_0 = 0$ is preserved under the KP dynamics. These dynamics can be described intrinsically relative to a bihamiltonian structure on $\widetilde{M}$ induced by that on M and which we now proceed to describe.

As we will now show, the first structure restricts trivially to $\widetilde{M}$ since u_0 is central relative to it; whereas relative to the second structure, $\widetilde{M}$ is, at least formally, a symplectic submanifold of M and the induced brackets are the Dirac brackets. To see this we need to introduce a few calculational tools. If $F_A \equiv \mathrm{Tr} A\Lambda$, $F_B \equiv \mathrm{Tr} B\Lambda$ are linear functions on M—*i.e.* , $A, B \in \mathcal{S}$ independent of Λ— so that $dF_A = A$ and $dF_B = B$, then their Poisson bracket relative to a hamiltonian map J is given by

$$\{F_A\, ,\, F_B\} = \mathrm{Tr} J(A)\, B = -\mathrm{Tr} A\, J(B)\ . \tag{13}$$

If $A = \sum_{i\geq 0} a_i \partial^{i-1}$ and $B = \sum_{i\geq 0} b_i \partial^{i-1}$, then (13) can be rewritten as

$$\{F_A\, ,\, F_B\} = -\sum_{i,j} \int a_i (\Omega_{ij} \cdot b_j)\ , \tag{14}$$

where the Ω_{ij} are differential operators defined by $J(B) = \sum_{i,j} \partial^{-i} (\Omega_{ij} \cdot b_j)$. In terms of the coordinates u_i on M, the fundamental Poisson brackets are given by

$$\{u_i(x)\, ,\, u_j(y)\} = -\Omega_{ij} \cdot \delta(x-y)\ , \tag{15}$$

where Ω_{ij} is taken at the point x.

From the expression (9) for the first hamiltonian structure we see that since only A_+ constributes, $\Omega_{0i} = 0$ for all i. Hence u_0 is central as claimed.

From the expression (10) for the second hamiltonian structure we see that $\Omega_{00} = -\partial$ and hence it is formally invertible. However, the Dirac brackets, involving a Ω_{00}^{-1} could be potentially nonlocal. We will see, however that this is not the case. The fundamental brackets on $\widetilde{M}$

$$\{u_i(x)\, ,\, u_j(y)\} = -\widetilde{\Omega}_{ij} \cdot \delta(x-y) \tag{16}$$

are given from those on M via the celebrated Dirac formula

$$\widetilde{\Omega}_{ij} = \Omega_{ij} - \Omega_{i0}\, \Omega_{00}^{-1}\, \Omega_{0j}\ . \tag{17}$$

This formula induces brackets on functions F, G on $\widetilde{M}$ obtained from those of M as follows. We first extend the functions to M which we also denote F, G. Since the extension is not unique, the terms $\frac{\delta F}{\delta u_0}, \frac{\delta G}{\delta u_0}$ in the expression for the gradients are rendered undefined. This ambiguity is fixed by demanding that the associated hamiltonian vector field be tangent to $\widetilde{M}$. In other words, we fix $\frac{\delta F}{\delta u_0}$ by the requirement that $J_2(dF)$ should have no free term. Let the gradient of F be given by $dF = \sum_{i\geq 0} X_i \partial^{i-1}$. Then $X_i = \frac{\delta F}{\delta u_i}$ for $i > 0$ and X_0 is to be determined by demanding that $J_2(dF) = \sum_{i,j\geq 0} \partial^{-i}(\Omega_{ij} \cdot X_j)$ have no free term. In other words,

$$X_0 = -\sum_{j>0} \Omega_{00}^{-1} \Omega_{0j} \cdot X_j \ , \tag{18}$$

whence the expression for the Dirac brackets follows at once after rewriting $J_2(dF)$ in terms of the X_i for $i > 0$:

$$J_2(dF) = \sum_{i,j>0} \partial^{-i}(\widetilde{\Omega}_{ij} \cdot X_j) \ . \tag{19}$$

This condition on X_0 can be written more invariantly as follows. The free term of $J_2(dF)$ is given by $\operatorname{res} J_2(dF)\partial^{-1}$. Setting this to zero we find,

$$\begin{aligned} 0 =& \operatorname{res} J_2(dF)\partial^{-1} \\ =& \operatorname{res} \big(\Lambda(dF\Lambda)_-\partial^{-1} - (\Lambda dF)_-\Lambda\partial^{-1}\big) \\ =& \operatorname{res} \big(dF\Lambda(\Lambda\partial^{-1})_+ - \Lambda dF(\Lambda\partial^{-1})_+\big) \\ =& \operatorname{res} [dF \, , \, \Lambda] \ . \end{aligned} \tag{20}$$

As an equation on X_0, this simply says that X_0' is to be equal to the Adler residue of a commutator, which is always a perfect derivative. Therefore X_0 can always be solved as a differential polynomial in the X_i and hence the induced brackets on $\widetilde{M}$ are local.

Notice that if we assign weights $[\partial]=1$ and $[u_i]=i+1$, then the KP operator Λ in (1) is homogeneous of weight 1. It is easy to verify that if we define $[f']=[f]+1$ for any homogeneous differential polynomial f, the multiplication in the ring of ΨDO's preserves the weight and thus becomes a graded ring. Since the hamiltonian structures are defined using only the ring structure of the ΨDO's (up to some harmless pojections), it is clear that the differential operators Ω_{ij} (and also $\widetilde{\Omega}_{ij}$) are homogeneous of weight $i+j+1$. Thus the coefficients of Ω_{ij} (and also of $\widetilde{\Omega}_{ij}$) can be a priori differential polynomials in the $u_1, u_2, \ldots, u_{i+j}$; although symmetry considerations actually forbid the appearence of u_{i+j}.

A straight-forward computation yields the first few Ω_{ij}:

$$
\begin{aligned}
\Omega_{00} &= -\partial \\
\Omega_{11} &= \partial u_1 + u_1 \partial \\
\Omega_{20} &= u_1 \partial \\
\Omega_{12} &= 2\partial u_2 + u_2 \partial - \partial^2 u_1 \\
\Omega_{22} &= \left(u_1^2 + 4u_3 - 2u_2'\right)\partial + u_1 u_1' - u_2'' + 2u_3'
\end{aligned}
\tag{21}
$$

with $\Omega_{ji} = \Omega_{ij}^*$ and all other Ω_{ij} being zero. The $\widetilde{\Omega}_{ij}$ follow from (17) :

$$
\begin{aligned}
\widetilde{\Omega}_{11} &= \partial u_1 + u_1 \partial \\
\widetilde{\Omega}_{12} &= 2\partial u_2 + u_2 \partial - \partial^2 u_1 \\
\widetilde{\Omega}_{22} &= \left(2u_1^2 + 4u_3 - 2u_2'\right)\partial + 2u_1 u_1' - u_2'' + 2u_3' \ ,
\end{aligned}
\tag{22}
$$

where as before $\widetilde{\Omega}_{ji} = \widetilde{\Omega}_{ij}^*$. The first equation exhibits explicitly a subalgebra isomorphic to the Lie algebra of Diff S^1. The second equation says that $u_2 - \frac{1}{2}u_1'$ is a DiffS^1 tensor of weight 3. Presumably this remains true for the $u_{j>2}$ and they can be modified by adding differential polynomials of the lower u_j that make them into tensors. The Poisson bracket of u_2 with itself involves u_1 non-linearly while also the field u_3 shows up. Therefore the fundamental Poisson brackets on $\widetilde{M}$ relative to the second structure define a nonlinear extension of the Lie algebra of DiffS^1 by fields of weights $3, 4, \ldots$.

Conclusions

The generalization of the Gel'fand-Dickey brackets to the space of KP operators has allowed us to display the KP hierarchy as a bihamiltonian integrable hierarchy. The fundamental brackets coming from the first structure have been recently identified with the $W_{1+\infty}$ algebra **[5]** **[6]** , whereas the ones coming from the second structure yield a nonlinear algebra which we denote W_{KP}. The fundamental brackets induced on the submanifold of KP operators without free term by restricting the second hamiltonian structure, yields a nonlinear extension of the Lie algebra of Diff S^1 by tensors of weights $3, 4, \ldots$.

Imposing the constraint $\Lambda^n_- = 0$ reduces the KP hierarchy to the n^{th} order generalized KdV hierarchy which is bihamiltonian relative to the Gel'fand-Dickey brackets. These are brackets defined on the space $\mathcal{N}$ of Lax (differential) operators $L = \partial^n + \cdots$. Let $N \subset M$ denote the submanifold of KP operators obeying the constraint $\Lambda^n_- = 0$. Then N is isomorphic to $\mathcal{N}$, the isomorphism being explicitly given by $\Lambda \mapsto \Lambda^n$. The functions H_i generating the KP flows restrict to N and can be pulled back to $\mathcal{N}$. Moreover the Lax flows (2) preserve N and induce flows on $\mathcal{N}$ which are bihamiltonian relative to the Gel'fand-Dickey brackets and are generated by the functions induced on $\mathcal{N}$ by the H_i.

Now, N can be given a bihamiltonian structure for the KdV flows in two different ways. On the one hand we can pull back the Gel'fand-Dickey brackets on $\mathcal{N}$ to N. It is straightforward to derive reasonably explicit formulas for the induced brackets. On the other hand, one expects that N inherits a bihamiltonian structure from the one on M that we have constructed in this paper.

It is natural to conjecture that these two structures coincide. It is not enough that the Lax flows correspond, since the vector fields corresponding to the Lax flows commute and thus only span a (possibly maximal) isotropic subspace of the tangent space. However, this correspondence presents strong evidence for our conjecture. The difficulty in proving the conjecture arises in deriving an expression for the brackets inherited by N from M. The constraint $\Lambda^n_- = 0$ translates into an infinite number of constraints involving the coordinates u_i—namely setting each $u_{j\geq n}$ equal to a differential polynomial in the $u_{i<n}$—which makes the Dirac prescription difficult to implement. Work on this is in progress.

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