Supersymmetric space forms

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$\Longrightarrow$ spatial universe is a 'space form'

Space forms

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- ‘maximally symmetric’

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where $\partial_i = \frac{\partial}{\partial x_i}$

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  i.e., $(\xi_p, \nabla\xi_p) \in T_pM \oplus \mathfrak{so}(T_pM)$

- $\dim\left(T_pM \oplus \mathfrak{so}(T_pM)\right)$

- $\dim\left(T_pM \oplus \mathfrak{so}(T_pM)\right) = n$

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- quadric is not simply-connected; its universal cover is AdS_n

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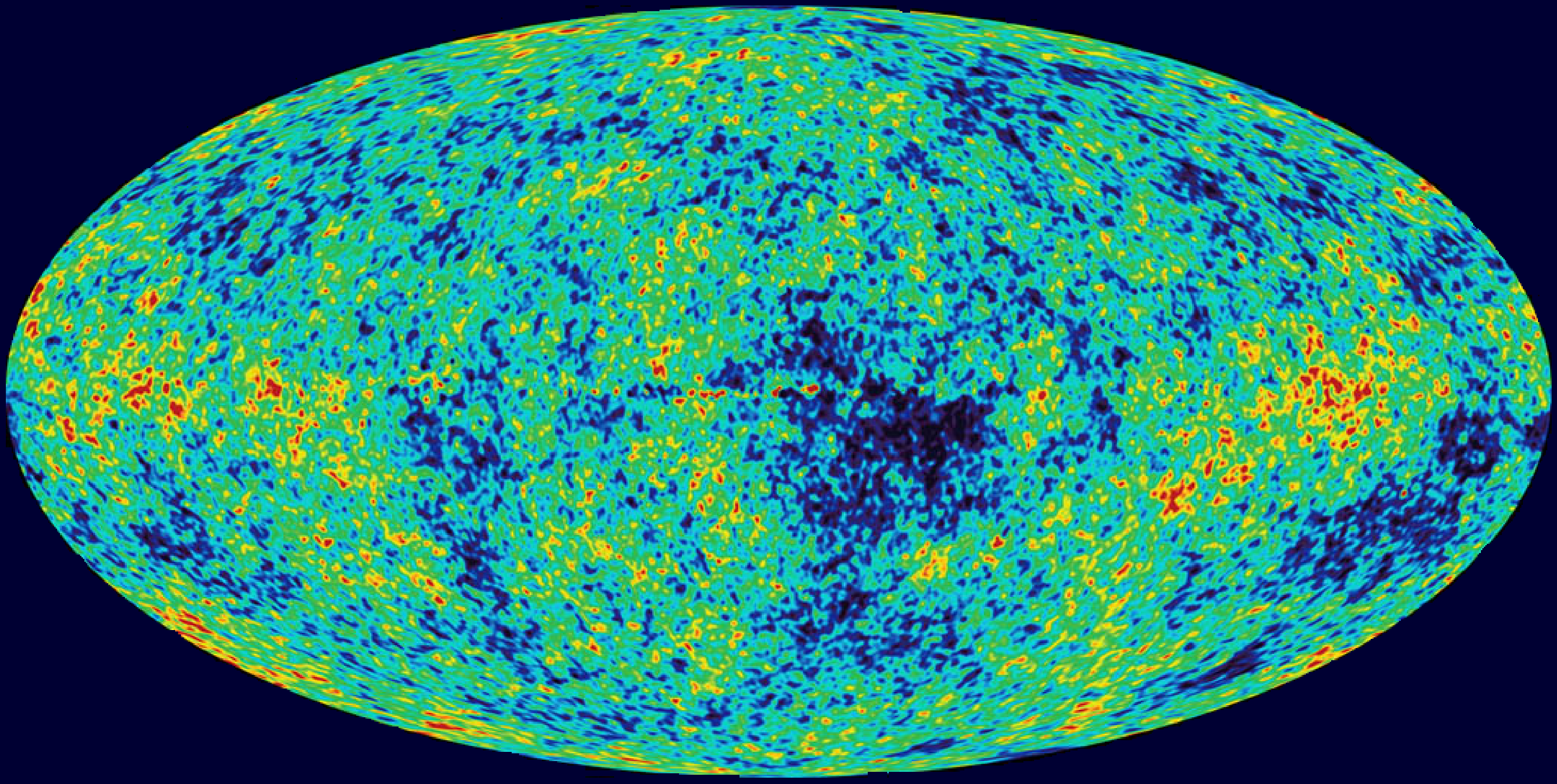

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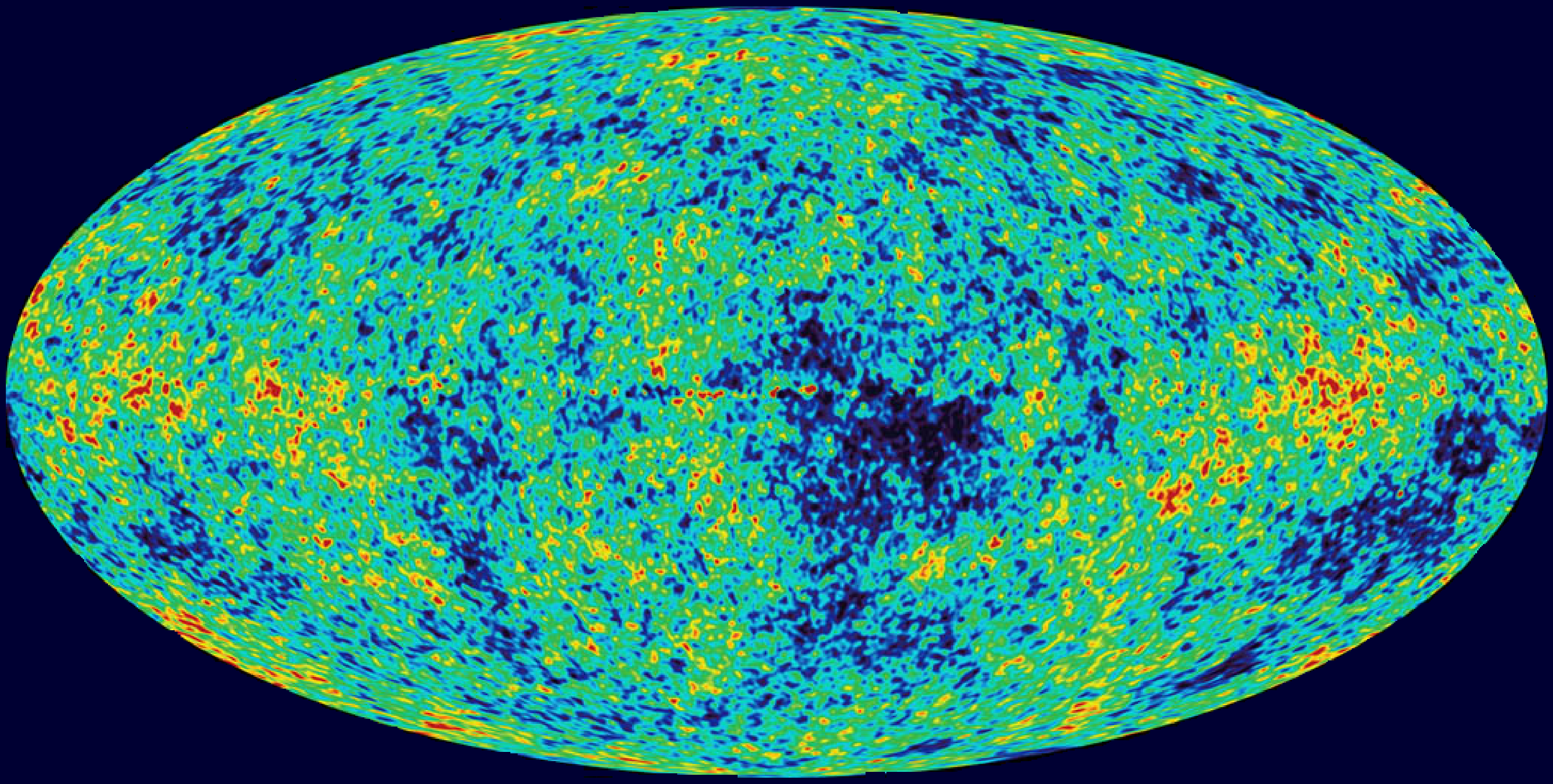

suggest that $g^{(3)}$ is the Poincaré dodecahedral space S^3/E_8

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Supergravities

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	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N = 2$			$N = 1$			
8	$N = 2$			$N = 1$			
7	$N = 4$			$N = 2$			
6	$(2, 2)$, $(3, 1)$, $(4, 0)$	$(2, 1)$, $(3, 0)$		$(1, 1)$, $(2, 0)$		$(1, 0)$	
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$\mu \neq 0 \implies$ same plane wave (Kowalski-Glikman, 1984)

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(Blau–FO–Hull–Papadopoulos, 2001)

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Watch this space.