

Generalised Spencer Cohomology and 5-Dimensional Supersymmetry

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Abstract

We introduce graded and filtered algebras and superalgebras, filtered deformations and the generalised Spencer cohomology. We calculate some of the cohomologies of the Poincaré algebra $\mathfrak{p}$ and superalgebra $\mathfrak{s}$, focusing in the latter case on $\mathcal{N} = 1$ and $D = 5$, and find the general form of their filtered deformations. The second cohomology of the superalgebra is used to define the notion of Killing spinors on a 5-dimensional Lorentzian spin manifold. Necessary and sufficient conditions for maximal supersymmetry and sufficient conditions for Killing spinors to generate a Killing superalgebra are derived. We make an attempt at classification of the maximally supersymmetric geometries up to local isometry. Our results are compared with their analogues in 5-dimensional supergravity.

Declaration

I declare that this dissertation was composed entirely by myself.

Chapters 2 and 3 provide an introduction to the subject area, some conventions and useful elementary results, and a description of some relevant background theory on the topic. They do not contain original research but do contain some routine but independently derived results.

Chapters 4 and 5 contain my original work done in collaboration with Professor José Figueroa-O'Farrill, based on previous work [1–4] done by Professor Figueroa-O'Farrill in collaboration with Andrea Santi and Paul de Medeiros. Chapters 6 and 7 consist of my own analysis and discussion of the results and conclusions based on them.

Personal Statement

My first two weeks of work on the project consisted of building up a working knowledge of the algebraic structures and techniques which would be used in the main calculations which were to be carried out, in particular carefully reading up on the 11-dimensional case [1, 2], the general structure of the Spencer complex of the Poincaré superalgebra, the Clifford and exterior algebras over a Lorentzian vector space in general finite dimension and the relationship between them, the spin group, the Lorentz algebra $\mathfrak{so}(V)$ and their vector and spinor representations. As a toy calculation, I computed the first two Spencer cohomology groups of the Poincaré algebra and its relevant subalgebras in order to find its filtered (sub)deformations, finding that these are given by a space of algebraic curvature operators. This calculation is vastly simplified compared to the superalgebra calculation owing to the lack of spinor structure.

In the following two weeks, I turned to the particular case of 5 dimensions, initially studying the quaternionic structure of the relevant Clifford algebra, its spinor representation and symplectic Majorana spinors. I read up on complex and quaternionic structures and representations, proved various identities for products of higher-rank gamma matrices found in [5] and derived the Fierz identity and various other useful identities involving quantities defined in terms of the Majorana spinors. I made an initial attempt to solve the relevant cocycle conditions, initially finding that the space of solutions was given by 3-forms (or Hodge-dually 2-forms), but from the known spinor connection in $D = 5$ supergravity we knew that there was a problem with the solution. Working with my supervisor to fix the errors in this calculation, finding the most efficient way of setting it out and writing up the work done so far and some of the background material took up the following two to three weeks.

Next, I moved on to figuring out the fundamentals of the geometric part of the project, understanding spin structures on manifolds, the spin lift of the Levi-Civita connection, the spinorial Lie derivative and the definition of the superconnection. I derived the flatness conditions for the curvature of the superconnection and showed that these conditions are sufficient to cause the Killing superalgebra to close and form a Lie superalgebra. This work, along with further writeup, took another two weeks.

The final weeks of the project were spent learning about Lorentzian and Riemannian symmetric spaces, finding all of the the maximally supersymmetric backgrounds, on the way learning a little about the supersymmetric solutions of 5-dimensional supergravity, and finally finishing the dissertation.

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Finally, thanks to the second marker, and apologies for the length of this document.

Foreword for the revised version

This version of this work has undergone a number of revisions since submission. For the most part, these are corrections of errors in typography, spelling, grammar, and terminology. Some minor mathematical corrections have also been made, however, some more significant errors in Chapter 5 have been left in place, as they would significantly change the latter parts of this document, including its main results. These errors are commented on in footnotes and in Chapter 6. The corrected versions of material found in those sections will appear in an upcoming research article.

Foreword

This dissertation was prepared as part of the requirements for the award of Mathematical Physics MSc at the University of Edinburgh. It is intended to be understandable by readers with a reasonable familiarity with differential geometry, general relativity and Lie theory, such as might be gained by attending the relevant lecture courses available on the aforementioned degree programme. We provide what is hopefully a sufficiently comprehensive presentation of essential background material beyond this level in the early chapters. That being said, this work is primarily a presentation of original research, rather than a literature review or pedagogical exposition, and its presentation style reflects this. The pace is fast and much of Chapters 2 and 3 are simply collations of definitions and essential results; we endeavour to present reasonable motivation and justification for these but full proofs and discussion are in some cases deferred to the literature. Chapter 5 uses some more advanced geometric concepts that we do not have time to discuss in detail, but nonetheless we strive to make our arguments as simply as possible and use some powerful theorems from the literature to help us through the pain.

Chapter 1 provides an introduction to higher-dimensional supersymmetric theories, in particular supergravity theories, explains the reasoning behind the approach taken in this project and puts the project in the context of recent literature. The essential algebraic structures of graded and filtered algebras, superalgebras and Lie (super)algebra cohomology are presented in Chapter 2, where we also discuss filtered deformations and the Spencer cohomology, before applying these ideas to the Poincaré algebra. Chapter 3 discusses the Clifford algebra of a Lorentzian vector space and its relationship to the exterior and Lorentz algebras, before discussing the particular case of a 5-dimensional Lorentzian vector space and introducing a number of additional structures we put on the spinor module. In Chapter 4, the relevant parts of the Spencer cohomology of the Poincaré superalgebra in 5 dimensions are calculated. Finally, in Chapter 5 we move on from the algebraic part of the project to the geometric part, using the homological results of Chapter 4 to define a superconnection, Killing spinors and the Killing superalgebra. We find necessary and sufficient conditions for a background to be maximally supersymmetric and sufficient conditions for well-definition of the Killing superalgebra, and attempt to classify the maximally supersymmetric backgrounds and their Killing superalgebras. The remaining chapters summarise and discuss the results in the context of the existing literature and consider possible future work.

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Chapter 1

Introduction

1.1 Higher-dimensional supersymmetry

Supergravity is a geometric theory of physics which combines the principles of general relativity with those of supersymmetry. It can be seen as a generalisation of general relativity via the inclusion of local supersymmetry [5]. It is considered to represent a potential step towards a *Theory of Everything (ToE)*, a unified theory of the small-scale physics of fundamental particles and the large-scale physics of gravity, for a number of reasons, some of which are fairly elementary, while others include some of the most tantalising results in theoretical physics of the last half-century.

Supersymmetry is a unification of fermionic and bosonic physics, a resolution of a fundamental dichotomy in particle and statistical physics. It is an internal symmetry, such as the $U(1)$ phase invariance of quantum theories or the $SU(3)$ flavour symmetry of hadronic physics, except while those involve mapping fermions to fermions or bosons to bosons, supersymmetry is a property of physical theories which are invariant under a mapping of fermions to bosons and vice-versa.

Supergravity is a further unification of supersymmetry with the geometric description of gravity, and therefore large-scale fundamental physics, which is the mathematical basis of general relativity. A ToE would certainly have to encompass all of these, and supergravity is an attempt at a first-principles theory which does so. Of course, a true ToE would have to at least include a quantum-mechanical description of physics, so it is also a necessity to study the quantisation of supergravity. Indeed, the quantum aspects of supergravity are a subject of intense study, and part of the attractiveness of supergravity (and supersymmetric theories in general) is that the mathematical divergences encountered in quantum interactions are softer than in non-supersymmetric theories due to cancellations between bosonic and fermionic loops in Feynman diagrams; in fact, some of these theories may even be finite [6].

Supergravity is also interesting as the low-energy limit of *superstring theory*, a candidate ToE which replaces particles with strings as the dynamical objects, includes supersymmetry, gravity and quantum mechanics and can include gauge symmetries and chiral interactions. Superstring theory (like other gauge theories) suffers from *gauge anomalies*,

quantum effects which cause mathematical inconsistencies that can be fatal for such theories. These anomalies cancel out, making quantum superstring theory viable, only when spacetime has dimension 10, known as the *critical dimension* [7]. In this dimension, taking a low-energy limit of superstring theory recovers supergravity. More precisely, there are a number of different superstring and supergravity theories: in the superstring case these go by the inspiring names of Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$; 10-dimensional supergravity comes in flavours of Type I (which can be augmented by gauge symmetries) and Types IIA and IIB. The low-energy limit of each superstring theory recovers a corresponding 10-dimensional supergravity theory in each case.

Yet a further highly speculative unification, known as M-theory, which is expected to appear as a “strong coupling limit” of Type IIA superstring theory is thought to exist in 11 dimensions [8–10]. Upon taking appropriate limits it reproduces Type IIA superstring theory in 10 dimensions (while the other types can be obtained via a number of “dualities” relating superstring theories) and supergravity in 11 dimensions, which in turn can reproduce Type IIA supergravity in 10 dimensions via dimensional reduction, as described below. However, supergravity remains sufficiently interesting on its own and it may represent the low-energy limit of some more fundamental dynamics which is not superstring or M-theory.

Given the apparent preference of these theories for higher numbers of spacetime dimensions, it is necessary to consider how they are supposed to describe the universe in which we live, which is apparently 4-dimensional. The basis for one way of achieving this was first provided long before string theory or supergravity by Kaluza, who suggested that it may be possible to obtain a description of a 4-dimensional universe with gravity and electromagnetism from a description of a 5-dimensional universe with only gravity [11] and Klein, who helped to realise Kaluza’s ideas [12–14]. The 5-dimensional universe is modelled as the manifold $\mathbb{R}^{1,3} \times S^1$, a spacetime where one of the spatial dimensions is rolled up into a circle or *compactified*, on which the physics is governed by General Relativity. The behaviour of the various gravitational degrees of freedom (components of the metric) is studied in the limit where the radius of the circle goes to zero; the degrees of freedom associated with the compactified direction can then be reinterpreted as fields on $\mathbb{R}^{1,3}$. This process is known as *dimensional reduction*.

The hope of Kaluza and Klein to recover electromagnetism plus gravity in 4 dimensions was somewhat successful but couldn’t reproduce the Maxwell and Einstein theories precisely¹. Nonetheless, this process of Kaluza-Klein dimensional reduction provides a solution to our problem of obtaining a description of a 4-dimensional universe from a higher-dimensional theory: suppose we have a theory which is consistent in some spacetime dimension $D > 4$ and the macroscopic spacetime we directly observe is a 4-dimensional manifold M . Our D -dimensional theory describes physics on the D -dimensional manifold $M \times N$, where N is a $(D - 4)$ -dimensional compact manifold which is an “internal space” not apparent to us because it is spatially microscopic². Whatever the degrees of freedom

¹While 4-dimensional gravity and the electromagnetic field can be recovered, there is also a nonphysical scalar “dilaton” field which can’t be consistently removed from the theory.

²More generally, a compactification may consist of a nontrivial fibration with M as the base space and N the typical fibre.

of the fields D -dimensional theory are, they must then be reinterpreted as fields on M .

This however opens up an even greater problem: even if the higher-dimensional theory (be it supergravity, superstring theory, M-theory or something else) is unique, the resulting 4-dimensional theory obtained after compactification is certainly not; there are many choices of manifolds M and N^3 . Even though some geometric constraints can be put on N , the number of possible choices is still enormous. The problem might be made more tractable by looking only for those compactifications which produce the physics we observe in 4 dimensions, such as the fields and symmetries of the Standard Model or the geometry and energy content of the Lambda-CDM cosmological model, but this is epistemologically problematic as it is essentially a fine-tuning argument: it has been argued [15, 16] that the goal of a physical theory should be to produce a unique set of falsifiable predictions, not provide such a wide range of possible predictions that a tiny subset of them end up agreeing with observation by accident! The challenge for proponents of Kaluza-Klein theories is thus to demonstrate that their theory is prejudiced towards a reduction to a 4-dimensional physics which agrees with observations, not just amenable to such a reduction. Of course, one could take the opposite point of view, as some have, that the non-uniqueness of these theories is theoretical evidence of multiple universes [17]!

An alternative idea to that of Kaluza and Klein for finding our universe in one with higher dimensions is the *brane-world*, where the extra dimensions remain macroscopic but our universe exists on the surface of a submanifold or *brane*. For more on this, see [18].

1.2 Supergravity backgrounds

Related to the problem of getting a unified description of our 4-dimensional universe from a higher dimensional theory is more generally the understanding of the possible *bosonic backgrounds* of supergravity theories, the geometries on which supergravity theories live, in various dimensions. Such a background is called *supersymmetric* if it admits *Killing spinors* (defined in Chapter 5). The “amount” of supersymmetry possessed by a background - given by the dimension of its space of Killing spinors - is taken to be the primary way of organising them. The (super)symmetries of a supersymmetric background are encoded by a Lie superalgebra known as a Killing superalgebra.

A lot of work has been done on the problem of finding possible supergravity backgrounds. A full classification in all dimensions is not currently within reach, but much progress has been made in the past decade and a half on finding the supersymmetric backgrounds [19–23]; the case of maximal supersymmetry (backgrounds with a space of Killing spinors with the maximum possible dimension) is particularly amenable to full classification. In the especially interesting case of 11-dimensional supergravity, many supersymmetric backgrounds have been found and a number of other results have been achieved, but a full classification has so far not been arrived at.

³In general there are also more choices of fibration.

1.3 Killing superalgebras

Recent work by the supervisor of this project, (Professor Figueroa-O'Farrill) and collaborators (Andrea Santi, Paul de Medeiros) has sought to attack the problem of finding supersymmetric backgrounds from a new direction by putting primary focus on the Killing superalgebras [1–4]. The key insight is that the Killing superalgebra of a supergravity background is a filtered subdeformation⁴ of a Poincaré superalgebra in the same spacetime dimension. The angle of attack is therefore to seek out the possible such deformations and to see if they can be geometrically realised.

Cheng and Kac have laid out the general theory of filtered deformations of Lie superalgebras [24, 25]. The deformation theory is governed by generalised Spencer cohomology, a refinement of the Chevalley-Eilenberg cohomology of a particular subalgebra of a $\mathbb{Z}$ -graded Lie superalgebra with coefficients in the adjoint representation (See Chapter 2). In our case, the cohomological calculation for (subalgebras of) the Poincaré superalgebra also provides geometric information in the form of a definition for a superconnection on the spinors of an arbitrary Lorentzian spin manifold. Killing spinor fields are then defined as those spinor fields which are parallel with respect to this superconnection. Conditions for these Killing spinors to generate a Lie superalgebra must then be found, as well as corresponding backgrounds. In this way we can claim that the Spencer cohomology provides a geometric notion of supersymmetry independent of physical theories. In some cases, for example in 11 spacetime dimensions, this notion coincides precisely with that of supergravity [1, 2], while in others it is more general [4].

It is possible through this process, therefore, to rediscover the known supersymmetric supergravity backgrounds, but it should also be possible to find as-yet unknown supersymmetric backgrounds for supergravity, as well as backgrounds which are more general than supergravity solutions. This work has begun in $D = 11$ [1, 2], $D = 4$ [3] and $D = 6$ [4], looking at particular subalgebras of the Poincaré superalgebra with minimal supersymmetry (augmented with R-symmetry in some cases), finding the Spencer cohomology and the maximally supersymmetric backgrounds. The goal of this project is to tackle the 5-dimensional case.

⁴A filtered deformation of a subalgebra, as defined in Chapter 2.

Chapter 2

Background theory

This chapter covers the background mathematical theory necessary to follow the rest of the report. It is assumed that the reader is already familiar with Lie algebras and their representation theory. Unfortunately, physical aspects of supersymmetry or supergravity, such as the transformations of field multiplets or supersymmetric Lagrangians, is beyond the scope of this project, so apart from some brief comments we will not discuss such aspects and refer the reader instead to [5].

2.1 Graded and filtered algebras

2.1.1 Basic Definitions

We begin by providing some basic definitions for the spaces in which we will work. In this section k denotes an arbitrary field but in practice we will always have $k = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. Let V be a k -vector space with a set of subspaces $\{V_i\}_{i \in I}$ indexed by a set I such that $V = \bigoplus_{i \in I} V_i$. Then V is said to be an *I -graded vector space*.

The subspaces V_i are referred to as the *graded components* of V , and elements $v \in V_i$ are called *homogenous elements of degree i* .

Definition 2.1.2. Let V be a k -vector space with a set of subspaces $\{V_i\}_{i \in I}$ indexed by a totally ordered set I such that $V_i \supseteq V_j$ for $i \leq j$ and $V = \bigcup_{i \in I} V_i$. Then V is said to be an *I -filtered vector space* (with filtration $\{V_i\}_{i \in I}$).

If V is I -filtered and I is finite, we must have $V = V_{\max I}$.

Graded and filtered vector spaces are of little interest on their own, but the concepts of filtration and grading become extremely useful when the vector space underlies an algebraic structure.

Definition 2.1.3. Let I be a totally ordered Abelian group.

- An associative k -algebra A is said to be I -graded (resp. filtered) if it has a vector space grading (resp. filtration) $\{A_i\}_{i \in I}$ such that $A_i A_j \subseteq A_{i+j}$.

- A Lie algebra $\mathfrak{g}$ over k is said to be I -graded (resp. filtered) if it has a vector space grading (resp. filtration) $\{\mathfrak{g}_i\}_{i \in I}$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

Definition 2.1.4. • A linear map between I -graded (resp. filtered) vector spaces $f : V \rightarrow W$ is said to have grading (resp. filtration) degree d if for all $i \in I$, $f(V_i) \subseteq W_{i+d}$.

- Linear maps of degree zero are said to *preserve the grading (resp. filtration)*. A *homomorphism of I -graded (resp. filtered) vector spaces* is defined to be a linear map which preserves the grading (resp. filtration).
- We define homomorphisms of graded (resp. filtered) algebras (of either type) to be homomorphisms of algebras (of the same type) which preserve the grading (resp. filtration).

The I -gradings on vector spaces V, W naturally induce an I -grading on $\text{Hom}(V, W)$ (and also on spaces of associative and Lie algebra homomorphisms), with the graded components $\text{Hom}(V, W)_d$ consisting of degree d maps, as the terminology suggests. Explicitly,

$$\text{Hom}(V, W)_d = \bigoplus_{\{i, j \in I \mid i+j=d\}} \text{Hom}(V_{-i}, W_j). \quad (2.1)$$

The tensor product $V \otimes W$ is also I -graded, with the graded components given by

$$(V \otimes W)_d = \bigoplus_{\{i, j \in I \mid i+j=d\}} V_i \otimes W_j. \quad (2.2)$$

The conditions of Definition 2.1.3 are simply that the multiplication (resp. bracket), as a linear map $A \otimes A \rightarrow A$ (resp. $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$), respects the filtration or grading.

It's interesting to briefly see how these induced gradings behave under duality. We will assume that the base field k has a standard I -grading given by $k_0 = k$ and $k_i = 0$ for $i \neq 0$. For a graded vector space $V = \bigoplus_{i \in I} V_i$, the dual space is $V^* = \text{Hom}(V, k)$, in which the degree d maps are $\text{Hom}(V_{-d}, k)$, so V^* has the induced I -grading with graded components $(V^*)_i = (V_{-i})^*$.

Note that the natural isomorphism

$$\text{Hom}(V, W) \cong V^* \otimes W \quad (2.3)$$

respects the grading for graded spaces V, W in the sense that

$$\begin{aligned} (V^* \otimes W)_i &= \bigoplus_{\{j, k \in I \mid j+k=i\}} (V^*)_j \otimes W_k \\ &= \bigoplus_{\{j, k \in I \mid j+k=i\}} (V_{-j})^* \otimes W_k \\ &\cong \bigoplus_{\{j, k \in I \mid j+k=i\}} \text{Hom}(V_{-j}, W_k). \\ &= \text{Hom}(V, W)_i \end{aligned} \quad (2.4)$$

2.1.2 $\mathbb{Z}$ -grading, filtration and filtered deformations

Many of the structures discussed below in the context of a $\mathbb{Z}$ -grading or -filtration can be generalised to other gradings or filtrations, but we will focus on the $\mathbb{Z}$ case as it is the only one required in this project. We begin with some observations about the structure of $\mathbb{Z}$ -graded Lie algebras. Each of these statements follows straightforwardly from the definition.

Proposition 2.1.5. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$ -graded Lie algebra. Then*

- *The component $\mathfrak{g}_0$ is a Lie subalgebra,*
- *For any $j \geq 0$, $\bigoplus_{i \geq j} \mathfrak{g}_i$ and $\bigoplus_{i \leq -j} \mathfrak{g}_i$ are Lie subalgebras,*
- *Each graded component $\mathfrak{g}_i$ of $\mathfrak{g}$ is a $\mathfrak{g}_0$ -module under the adjoint action,*
- *For $\mathfrak{g}_+ = \bigoplus_{i > 0} \mathfrak{g}_i$ and $\mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i$,*
 - *If one of these two subalgebras is zero, the other is an ideal subalgebra,*
 - *If $\mathfrak{g}_\pm$ is finite-dimensional, it is nilpotent.*

Similar statements hold for non-unital graded associative algebras. For a unital graded associative algebra A , $1 \in A_0$, and the same statements hold as long as subalgebras can be non-unital.

Definition 2.1.6. • A $\mathbb{Z}$ -graded vector space V has *finite depth* $h \geq 0$ if $V_{-h} \neq 0$ and $V_i = 0$ for all $i < -h$; that is, h is the minimal positive integer such that $V = \bigoplus_{i \geq -h} V_i$.

- A $\mathbb{Z}$ -filtered vector space V has *finite depth* $h \geq 0$ if its filtration has the form

$$V = V_{-h} \supset V_{-h+1} \supseteq \cdots \supseteq V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \cdots, \quad (2.5)$$

that is, $V_i = V$ for $i \leq -h$ and $V_{-h+1} \neq V$.

- A $\mathbb{Z}$ -graded or filtered vector space for which no such finite depth exists is said to have *infinite depth*.

If $\mathfrak{g}$ is a $\mathbb{Z}$ -graded Lie algebra of finite depth, $\mathfrak{g}_-$ (as defined in Proposition 2.1.5) is nilpotent.

Definition 2.1.7. The *natural filtration* on a $\mathbb{Z}$ -graded vector space V is given by $\{F_i\}_{i \in \mathbb{Z}}$, where $F_i = \bigoplus_{j \geq i} V_j$.

It is easy to see that the natural filtration on a graded vector space is indeed a $\mathbb{Z}$ -filtration and that the natural filtration on a $\mathbb{Z}$ -graded associative algebra (resp. Lie algebra) makes it into a $\mathbb{Z}$ -filtered associative algebra (resp. Lie algebra).

From any $\mathbb{Z}$ -filtered vector space, we can construct a graded vector space as follows.

Definition 2.1.8. If V is a $\mathbb{Z}$ -filtered vector space with filtration $\{V_i\}_{i \in \mathbb{Z}}$, we can define $(\text{Gr } V)_i = V_i/V_{i-1}$ for each $i \in \mathbb{Z}$. Then the external direct sum $\text{Gr } V = \bigoplus_{i \in \mathbb{Z}} (\text{Gr } V)_i$ is a graded vector space called the *associated graded vector space to V* .

If a $\mathbb{Z}$ -graded vector space has finite depth, its natural filtration has the same depth. Likewise the associated graded space of a $\mathbb{Z}$ -filtered vector space with finite depth also has finite depth.

We define a map $V \rightarrow \text{Gr } V$ given by the sum of quotient maps $V_i \rightarrow V_i/V_{i+1}$; that is, the linear extension of $v \mapsto \bar{v}$ where $\bar{v} = v + V_{i+1}$ for $v \in V_i$. This allows $\text{Gr } V$ to inherit algebra structures from V .

Proposition 2.1.9. *If V is a $\mathbb{Z}$ -filtered associative algebra (resp. Lie algebra), then $\text{Gr } V$ is a $\mathbb{Z}$ -graded associative (resp. Lie) algebra with multiplication (resp. bracket) defined by $\overline{XY} = \overline{XY}$ (resp. $[\overline{X}, \overline{Y}] = \overline{[X, Y]}$) for $X \in V_i, Y \in V_j$, extended linearly. Note that the equivalence classes on either side of the equation are taken in G_i, G_j and G_{i+j} respectively. In such cases, $\text{Gr } V$ is called the associated graded algebra.*

Proof. The multiplication or bracket operation on $\text{Gr } V$ respects the grading because the operation on V respects the filtration, and it obeys the properties of an associative multiplication or Lie bracket because the operation on V does. $\square$

We will now describe a practical way of viewing associated graded objects which will be useful for the theory of filtered deformations. Let V be a $\mathbb{Z}$ -graded vector space with a filtration

$$V \supseteq \dots V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \dots \quad (2.6)$$

For each i , we can choose (non-uniquely) a complementary subspace G_i of V_{i+1} in V_i such that $V_i = G_i \oplus V_{i+1}$ which defines a grading on V by $V = \bigoplus_{i \in \mathbb{Z}} G_i$. The quotient map $V_i \rightarrow V_i/V_{i+1} = (\text{Gr } V)_i$ restricts to an isomorphism $G_i \cong (\text{Gr } V)_i$, so then V and $\text{Gr } V$ are isomorphic as graded spaces. Different choices of complementary subspaces give different gradings on V and different graded isomorphisms.

If $G = \bigoplus_{i \in \mathbb{Z}} G_i$ is a $\mathbb{Z}$ -graded space and is given the natural filtration $F_i = \bigoplus_{j \geq i} G_j$, there clearly is a natural unique choice of complementary subspaces: $F_i = G_i \oplus \bigoplus_{j \geq i+1} G_j$. There is thus a natural isomorphism $G \cong \text{Gr } G$ of graded vector spaces, and in fact also as graded algebras of either type if G is a graded algebra.

Definition 2.1.10. If $G = \bigoplus_{i \in \mathbb{Z}} G_i$ is a $\mathbb{Z}$ -graded algebra of either type and F is a $\mathbb{Z}$ -filtered algebra

$$F \supseteq \dots F_{-1} \supseteq F_0 \supseteq F_1 \supseteq \dots \quad (2.7)$$

of the same type such that $\text{Gr } F \cong G$ (as graded algebras) then F is said to be a *filtered deformation of G* .

Let F be a filtered deformation of G with the notation of the definition above. Consider a choice of complementary subspaces V_i in F such that $F_i = V_i \oplus F_{i+1}$ that gives F a graded vector space structure (but not a graded algebra structure). Since this gives us an isomorphism of the underlying graded vector spaces $F \cong \text{Gr } F \cong G$, we will identify

those underlying spaces (that is, $V_i = (\text{Gr } F)_i = G_i$) and think about F as being an alternative algebra structure on G . We now have that $F_i = \bigoplus_{j \geq i} G_j$. Denote the linear operation (Lie bracket or multiplication) on F by μ_F and on G by μ_G . Since μ_F respects the filtration,

$$\mu_F(G_i, G_j) \subseteq F_{i+j} = \bigoplus_{k \geq i+j} G_k \quad (2.8)$$

for each $i, j \in \mathbb{Z}$, so

$$\mu_F = \sum_{i \geq 0} \mu_i \quad (2.9)$$

where each $\mu_i : G \rightarrow G$ is a linear map of grading degree i^1 . Finally, the condition for F to be a filtered deformation of G is that $\mu_0 = \mu_G$. We thus have the following.

Proposition 2.1.11. *Let $G = \bigoplus_{i \in \mathbb{Z}} G_i$ be a graded associative algebra (resp. Lie algebra) with multiplication $\cdot$ (resp. bracket $[\cdot, \cdot]$). Then any filtered deformation for G is isomorphic to an alternative algebra structure $*$ (resp. $[\cdot, \cdot]'$) on G given by a set of maps $\{\mu_i : G \rightarrow G\}_{i > 0}$ such that μ_i has grading degree i and*

$$X * Y = X \cdot Y + \sum_{i > 0} \mu_i(X, Y) \quad \left(\text{resp. } [X, Y]' = [X, Y] + \sum_{i > 0} \mu_i(X, Y) \right). \quad (2.10)$$

We can now consider μ to be a deformation of the graded operation on G into a filtered operation. Note that μ cannot be chosen arbitrarily; its components μ_i will have to satisfy some relations in order that the operation be associative (resp. antisymmetric and obey the Jacobi identity). For Lie algebras (and Lie superalgebras, which we will meet soon), it turns out that it is the Spencer cohomology which governs the deformations μ .

2.1.3 $\mathbb{Z}_2$ grading and superspaces

We will be particularly interested in graded structures where the abelian group labelling the grading is $\mathbb{Z}_2$. The prefix “super” is usually appended to mean “ $\mathbb{Z}_2$ -graded”, but what this precisely means in each case is worth discussing. A vector superspace or associative superalgebra is simply a $\mathbb{Z}_2$ -graded vector space or algebra, but Lie superalgebras are somewhat different; they are in fact a generalisation of Lie algebras which is built on top of a $\mathbb{Z}_2$ graded vector space, and we will meet them in the next section.

Definition 2.1.12. In any $\mathbb{Z}_2$ graded structure, the degree of an element is referred to as its *parity*; the elements of parity $\bar{0}$ are said to be *even* and those of parity $\bar{1}$ are *odd*.

We denote the parity of a homogeneous element by $|x|$. Note that even maps preserve grading while odd maps “flip” it, mapping odd to even and even to odd. As discussed in Subsection 2.1.1, this induces a $\mathbb{Z}_2$ -grading on spaces of homomorphisms of two vector superspaces V and W , turning Hom spaces into vector superspaces:

$$\text{Hom}(V, W) = \text{Hom}(V, W)_{\bar{0}} \oplus \text{Hom}(V, W)_{\bar{1}} \quad (2.11)$$

¹We note again that a different choice of complementary subspaces for a filtered algebra leads to a different (vector space) isomorphism to its associated graded algebra, and thus a different set of maps μ_i .

where the spaces of even and odd maps are

$$\mathrm{Hom}(V, W)_{\bar{0}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{0}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{1}}), \quad (2.12)$$

$$\mathrm{Hom}(V, W)_{\bar{1}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{0}}). \quad (2.13)$$

If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace, its endomorphism algebra $\mathrm{End}(V)$ is in particular a Hom space, so it has a vector superspace structure as above. Moreover, it's easy to see that composition of endomorphisms respects the $\mathbb{Z}_2$ -grading, so $\mathrm{End}(V)$ is an associative subalgebra. We will see some more examples of associative superalgebras in the next chapter. In particular, we'll discuss the supercommutative exterior algebra $\bigwedge V$.

Definition 2.1.13. An associative superalgebra A is said to be *supercommutative* if for all homogeneous elements $a, b \in A$,

$$ab = (-1)^{|a||b|}ba. \quad (2.14)$$

We define the *supercommutator* of two homogeneous elements a, b of a superalgebra by

$$[a, b] = ab - (-1)^{|a||b|}ba. \quad (2.15)$$

Clearly the supercommutator is bilinear, skew-symmetric on $A_{\bar{0}} \times A_{\bar{0}}$ and $A_{\bar{0}} \times A_{\bar{1}} \cup A_{\bar{1}} \times A_{\bar{0}}$ (in fact, it is a commutator) and symmetric on $A_{\bar{1}} \times A_{\bar{1}}$ (where it is an anticommutator). If, and only if, the supercommutator is identically zero, A is supercommutative.

Finally, many of the structures we are interested in have $\mathbb{Z}$ - and $\mathbb{Z}_2$ -gradings; we'll need a notion of compatibility for these. That notion is quite natural.

Definition 2.1.14. If a vector space has both a $\mathbb{Z}$ -grading and a $\mathbb{Z}_2$ -grading, the gradings are said to be *compatible* if any $\mathbb{Z}$ -homogeneous element is $\mathbb{Z}_2$ -homogeneous with its parity being its $\mathbb{Z}$ -degree modulo 2.

In such spaces “degree” will always refer to the $\mathbb{Z}$ -grading degree. Note that in particular this means that any $\mathbb{Z}$ -graded vector space or associative algebra has a unique compatible $\mathbb{Z}_2$ -grading given by its $\mathbb{Z}$ -grading modulo 2, making it into a vector superspace or superalgebra. A $\mathbb{Z}$ -grading on a Lie algebra also induces a $\mathbb{Z}_2$ grading in this way, but this does not in general endow it with the structure of a Lie superalgebra, and we will not be interested in such structures.

2.2 Lie superalgebras

In this section we discuss the algebraic structure most important to supersymmetry and to the work in this project, the Lie superalgebra. The definition of Lie superalgebras can be seen as a generalisation of associative superalgebras equipped with supercommutators in the same way as Lie algebras generalise associative algebras equipped with commutators (where we forget about the associative multiplication in both cases), but the theory is in fact much richer. Our discussion here will be brief; for a full pedagogical review of the basics of superalgebras which was extremely helpful at the beginning of this project, see [26].

Definition 2.2.1. A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a bilinear map (superbracket) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the following properties:

- Graded skew-symmetry:

$$[X, Y] = -(-1)^{|X||Y|}[Y, X],$$

- Graded Jacobi identity:

$$(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|X||Y|}[Y, [Z, X]] + (-1)^{|Y||Z|}[Z, [X, Y]] = 0,$$

- Preservation of grading:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

The “graded” signs that appear in the definition of supercommutators, Lie superalgebras and other structures built on vector superspaces are what give them their superpowers, enabling them to provide a formalism for supersymmetry in physics. Indeed, the generalisation of Lie algebras to superalgebras is necessary in order to circumvent the Coleman–Mandula Theorem, a consequence of which is that a Lie algebra which describes the continuous symmetries of a physical theory (under certain assumptions) in Minkowski space must be the direct sum of the Poincaré algebra and a Lie algebra of internal symmetries [27, 28]. Allowing for a superalgebra of symmetries lets us describe supersymmetries, which can be regarded as internal symmetries with spinorial, anticommuting generators which have non-zero brackets with the elements of the Poincaré algebra. The counterpart to the CM Theorem governing this physical situation is the Haag–Łopuszański–Sohnius Theorem [29], which essentially says that a Lie superalgebra describing the symmetries of a physical theory is a direct sum of a Poincaré *superalgebra* and an internal symmetry algebra.

As noted above, an associative superalgebra equipped with the supercommutator is a Lie superalgebra; in particular $\text{End}(V)$ of a vector superspace V is a Lie superalgebra. This observation allows us to begin discussing representations and modules of Lie superalgebras in essentially the same way as for Lie algebras. A Lie superalgebra homomorphism is of course defined as a linear map $f : \mathfrak{g} \rightarrow \mathfrak{h}$ of Lie superalgebras such that $f([X, Y]) = [f(X), f(Y)]$ for $X, Y \in \mathfrak{g}$. A representation of a Lie superalgebra $\mathfrak{g}$ is such a homomorphism $\mathfrak{g} \rightarrow \text{End } V$ for some vector superspace V . This can be used to define the equivalent notion of $\mathfrak{g}$ -modules V : the module action $\cdot$ obeys

$$[X, Y] \cdot v = X \cdot (Y \cdot v) - (-1)^{|X||Y|} Y \cdot (X \cdot v), \quad (2.16)$$

for $v \in V$. In particular, we have an adjoint representation $\text{Ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, $X \mapsto \text{Ad}_X$ defined, as for Lie algebras, by $\text{Ad}_X(Y) = [X, Y]$. There are obvious notions of subalgebras, ideals, and so on which we will not spell out explicitly.

Proposition 2.2.2. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra. Then*

- $\mathfrak{g}_0$ is a Lie algebra,
- $\mathfrak{g}_1$ is a $\mathfrak{g}_0$ module under the adjoint action,

Gradings and filtrations, as well as all of the related structures discussed in the previous section, can be defined on Lie superalgebras in the same way as Lie algebras, with the only caveat being that a $\mathbb{Z}$ -graded Lie superalgebra must have compatibility between its $\mathbb{Z}$ and $\mathbb{Z}_2$ gradings in addition to all other conditions. The discussion of filtered deformations we gave for Lie algebras previously also applies equally as well to $\mathbb{Z}$ -graded Lie superalgebras.

2.3 Generalised Spencer cohomology

2.3.1 Chain complexes, exact sequences and cohomology

We give here a short, pragmatic introduction to the basics of cohomology. This is a rich and deep subject in itself which has found utility in many areas of mathematics and we will not attempt to do the general theory justice here. We simply provide the essential definitions and properties which are required for this project. There are many sources which could be consulted to learn more; we recommend Rotman's book [30]. Throughout let R be a ring. For our purposes we can assume that R is commutative, but most the discussion here is still valid for non-commutative rings, given that we assume that R -modules are left (or right) R -modules.

Definition 2.3.1. Let $\{C^p\}_{p \in \mathbb{Z}}$ be a sequence of R -modules and let $\{\partial^p : C^p \rightarrow C^{p+1}\}_{p \in \mathbb{Z}}$ be a sequence of R -homomorphisms; diagrammatically we have

$$\dots \xrightarrow{\partial^{p-2}} C^{p-1} \xrightarrow{\partial^{p-1}} C^p \xrightarrow{\partial^p} C^{p+1} \xrightarrow{\partial^{p+1}} \dots \quad (2.17)$$

If $\partial^{p+1} \circ \partial^p = 0$ for all $p \in \mathbb{Z}$, we call this sequence a *cochain complex*. The spaces are called *cochain spaces*, their elements are *cochains*, and the maps ∂^p are *differentials*.

Where there is no risk of confusion, we can drop the superscript on the differentials and denote the conditions for the sequence (2.17) to be a cochain complex by $\partial^2 = 0$. Note that this condition is equivalent to $\text{im } \partial^p \subseteq \ker \partial^{p+1}$ in each cochain space C^{p+1} .

Definition 2.3.2. An *exact sequence* is a cochain complex such that $\text{im } \partial^p = \ker \partial^{p+1}$ for all $p \in \mathbb{Z}$.

A *short exact sequence* is an exact sequence such that all but three consecutive cochain spaces (say A, B, C) and the two differentials between them (say i, q) are zero:

$$\dots \rightarrow 0 \rightarrow A \xrightarrow{i} B \xrightarrow{q} C \rightarrow 0 \rightarrow \dots \quad (2.18)$$

The statements in the following proposition follow directly from the definition of the short exact sequence.

Proposition 2.3.3. *Consider the short exact sequence (2.18). The following hold:*

- i is injective,
- q is surjective,

- $A \cong \ker q$,
- By the First Isomorphism Theorem, $C \cong B / \ker q$.

Definition 2.3.4. Let

$$\dots \xrightarrow{\partial^{p-2}} C^{p-1} \xrightarrow{\partial^{p-1}} C^p \xrightarrow{\partial^p} C^{p+1} \xrightarrow{\partial^{p+1}} \dots \quad (2.19)$$

be a cochain complex. For each $p \in \mathbb{Z}$, we define

- The p -th space of cocycles $Z^p = \ker \partial^p$, the elements of which are p -cocycles or closed p -cochains ϕ satisfying $\partial\phi = 0$,
- The p -th space of coboundaries $B^p = \operatorname{im} \partial^{p-1}$, the elements of which are p -coboundaries or exact p -cochains ϕ satisfying $\phi = \partial\eta$ for some $(p-1)$ cochain η ,
- The p -th cohomology (group)² $H^p = Z^p / B^p$, the elements of which are cohomology classes.

Note that exact cochains are always closed ($B^p \subseteq Z^p$) since $\partial^2 = 0$. A cochain complex is an exact sequence if and only if all of its cohomologies are zero if and only if all closed cocycles are exact. Thus cohomology can be considered to be measuring the failure of a cochain complex to be an exact sequence or the failure of closed cochains to be exact.

Example 2.3.5 (De Rham cohomology). Let M be a smooth manifold. For $p \geq 0$, the space of (differential) p -forms, that is, antisymmetric (covariant) tensor fields of rank $(0, p)$, is denoted $\Omega^p(M)$. This can be thought of as sections of the exterior power of the cotangent bundle $\bigwedge^p T^*M$. This is a module of the ring of smooth functions $C^\infty = \Omega^0(M)$ with the scalar multiplication defined pointwise. Note that the space of p -forms is 0 for $p > \dim M$. The differential forms form a $\mathbb{Z}$ -graded supercommutative associative algebra over $\mathbb{R}$ $\Omega(M) = \bigoplus_{p=0}^{\dim M} \Omega^p(M)$ with the wedge product $\wedge$ which is given pointwise by the wedge product on the exterior algebra $\bigwedge T_p^*M$:

$$(\omega \wedge \eta)(x) = \omega(x) \wedge \eta(x). \quad (2.20)$$

There is an operation $d : \Omega(M) \rightarrow \Omega(M)$ satisfying the following properties:

- d is $\mathbb{R}$ -linear,
- For $f \in \Omega^0(M) = C^\infty(M)$, df is the usual derivative of smooth functions,
- For $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p d\eta$; that is, d is a (super-)derivation on $\Omega(M)$ in the algebraic sense.
- $d(d\omega) = 0$ for all $\omega \in \Omega^p(M)$.

²The terminology “cohomology group” is somewhat historical, but recall that R -modules are in particular abelian groups under addition, so it still makes sense.

In fact, these properties determine d uniquely; it is called the *exterior derivative*. There is thus a cochain complex (where the cochain spaces are 0 for $p < 0$ or $p > \dim M$)

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{\dim M}(M) \xrightarrow{d} 0 \quad (2.21)$$

called the *De Rham complex* and denoted $\Omega^\bullet(M)$. A p -cochain here is simply a p -form, while a p -cochain ω is closed if $d\omega = 0$, and it is exact if $\omega = d\eta$ for some $(p-1)$ -form η .

The de Rham cohomology is given in terms of the smooth structure on M , however, it is a remarkable fact that its cohomology is actually fully determined by its topology and not dependent on its smooth structure: De Rham's Theorem states that the de Rham cohomology of M is isomorphic to its singular cohomology, which is a topological invariant!

Example 2.3.6 (Chevalley–Eilenberg cohomology). [31] An important example of cohomology, on which the Spencer cohomology is ultimately based, is the Chevalley–Eilenberg cohomology of a Lie algebra (or superalgebra, but we will talk only of Lie algebras here).

Let $\mathfrak{g}$ be a Lie algebra over a field k , let V be a $\mathfrak{g}$ -module and for convenience let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be the representation corresponding to the module action. Then we define the *Chevalley–Eilenberg complex of $\mathfrak{g}$ with coefficients³ in V* as follows. The cochain spaces are

$$C^p(\mathfrak{g}, V) = \text{Hom}(\bigwedge^p \mathfrak{g}, V), \quad (2.22)$$

while the differentials $\partial : C^p(\mathfrak{g}, V) \rightarrow C^{p+1}(\mathfrak{g}, V)$ are given by

$$\begin{aligned} (\partial\phi)(X_1, X_2, \dots, X_{p+1}) &= \sum_{1 \leq s < t \leq p+1} (-1)^{s+t-1} \phi([X_s, X_t], X_1, \dots, \hat{X}_s, \dots, \hat{X}_t, \dots, X_{p+1}) \\ &\quad + \sum_{s=1}^p (-1)^s \rho(X_s) \phi(X_1, \dots, \hat{X}_s, \dots, X_{p+1}) \end{aligned} \quad (2.23)$$

for $\phi \in C^p(\mathfrak{g}, V)$ and $X_1, \dots, X_{p+1} \in \mathfrak{g}$. It is an exercise to check that $\partial\phi$ is totally antisymmetric (so is indeed a $(p+1)$ -cochain) and that $\partial^2 = 0$; this requires all of the properties of the definition of a Lie algebra and a Lie algebra representation.

It's useful to see what this looks like for small p . For $p = 0, 1, 2$, let $\phi_p \in C^p(\mathfrak{g}, V)$; then

$$(\partial\phi_0)(X_1) = -\rho(X_1)\phi_0, \quad (2.24)$$

$$(\partial\phi_1)(X_1, X_2) = \phi_1([X_1, X_2]) - \rho(X_1)\phi_1(X_2) + \rho(X_2)\phi_1(X_1), \quad (2.25)$$

$$\begin{aligned} (\partial\phi_2)(X_1, X_2, X_3) &= \phi([X_1, X_2], X_3) - \phi([X_1, X_3], X_2) + \phi([X_2, X_3], X_1) \\ &\quad - \rho(X_1)\phi_2(X_2, X_3) + \rho(X_2)\phi_2(X_1, X_3) - \rho(X_3)\phi_2(X_1, X_2). \end{aligned} \quad (2.26)$$

It is instructive to make some observations about a couple of particular cases. If the complex has coefficients in the base field ($V = k$), we must have $\rho = 0$ and using the notation above, $\partial\phi_0 = 0$, so all 0-cochains are cocycles, and $(\partial\phi_1)(X_1, X_2) = \phi_1([X_1, X_2])$,

³This is more historical nomenclature for which I have no justification.

so 1-cocycles are elements of g^* which act trivially on $[\mathfrak{g}, \mathfrak{g}]$. Thus $H^0(\mathfrak{g}, k) = k$ and $H^1(\mathfrak{g}, k) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. A particularly interesting example is where $\mathfrak{g}$ is the Lie algebra (over $\mathbb{R}$) of a real Lie group G . In that case, the left-invariant differential forms on G form a subcomplex of the de Rham complex $\Omega^\bullet(M)$, and the Chevalley–Eilenberg complex of $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is isomorphic to this subcomplex. We can see this as the Chevalley–Eilenberg cohomology emerging naturally as a geometric construct, rather than the rather more abstract (and general) algebraic construction we gave here.

If the complex has coefficients in the adjoint representation ($V = \mathfrak{g}$, $\rho = \text{ad}$), we have

$$(\partial\phi_0)(X_1) = [\phi_0, X_1], \quad (2.27)$$

$$(\partial\phi_1)(X_1, X_2) = \phi_1([X_1, X_2]) - [X_1, \phi^1(X_2)] - [\phi_1(X_1), X_2]. \quad (2.28)$$

Thus 0-cocycles are central elements of $\mathfrak{g}$ and 1-cocycles are derivations on $\mathfrak{g}$. The corresponding cohomologies are $H^0(\mathfrak{g}, \mathfrak{g}) = \mathfrak{Z}(\mathfrak{g})$ and $H^1(\mathfrak{g}, \mathfrak{g}) = \text{Der}(\mathfrak{g})/\text{ad}(\mathfrak{g})$.

It's possible to extend the Chevalley–Eilenberg cohomology to Lie superalgebras; in the next subsection we introduce the generalised Spencer cohomology which is a special case of this, and we defer the superalgebra discussion to there.

2.3.2 The Spencer complex and cohomology

Here we give the definition and discuss some of the features of the generalised⁴ Spencer cohomology of a $\mathbb{Z}$ -graded Lie superalgebra. Our source for this section is Cheng and Kac's paper [24, 25].

First, we briefly note that when we consider the exterior power of a vector superspace $V = V_0 \oplus V_1$, we mean it in a “super sense”:

$$\bigwedge^p V = \bigoplus_{i+j=p} \bigwedge^i V_0 \otimes \odot^j V_1. \quad (2.29)$$

In this way we can view the bracket operation on a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as a map $\bigwedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ (which additionally satisfies the Jacobi identity), just as we do with a Lie algebra.

Definition 2.3.7. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$ -graded Lie superalgebra over $\mathbb{R}$. The *Spencer complex* of $\mathfrak{g}$ is the Chevalley–Eilenberg complex of the subalgebra $\mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i$ with coefficients in $\mathfrak{g}$ under the adjoint action.

This definition somewhat belies the true nature of the Spencer complex which gives it its utility: as we will demonstrate below, the complex naturally inherits a grading from the superalgebra which is respected by the differentials. This leads to the complex breaking up into a sequence of subcomplexes, making the computation of the cohomology far simpler than it might be without the graded structure. Thus, when we refer to the

⁴We will use “Spencer cohomology” and “generalised Spencer cohomology” interchangeably. The original Spencer cohomology [32] applied only to $\mathbb{Z}$ -graded Lie algebras of depth 1; it was generalised to Lie superalgebras of arbitrary finite depth by Cheng and Kac [24, 25]. Note that Spencer's work uses a different indexing system to this work and Cheng and Kac.

Spencer complex and its cohomology, we will implicitly understand this to refer to the $\mathbb{Z}$ -indexed sequence of subcomplexes.

We define a $\mathbb{Z}$ -grading on each space of cochains $C^p(\mathfrak{g}_-, \mathfrak{g}) = \text{Hom}(\bigwedge^p \mathfrak{g}_-, \mathfrak{g})$ by the usual grading of a Hom space of $\mathbb{Z}$ -graded vector spaces: the homogeneous space of degree d consists of linear maps of degree d . We use the notation

$$C^{d,p}(\mathfrak{g}_-, \mathfrak{g}) = \text{Hom}(\bigwedge^p \mathfrak{g}_-, \mathfrak{g})_d. \quad (2.30)$$

The Chevalley–Eilenberg differential ∂ for a Lie superalgebra is a modification of the Lie algebra differential given by Equation (2.23) to include graded signs. We are interested only in the case where the representation is the adjoint. For $\phi \in C^r(\mathfrak{g}_-, \mathfrak{g})$, $X_1, \dots, X_p \in \mathfrak{g}_{\bar{0}}$ and $Y_1, \dots, Y_q \in \mathfrak{g}_{\bar{1}}$ where $p + q = r + 1$,

$$\begin{aligned} (\partial\phi)(X_1, \dots, X_p, Y_1, \dots, Y_q) &= \sum_{1 \leq s < t \leq p} (-1)^{s+t-1} \phi([X_s, X_t], X_1, \dots, \hat{X}_s, \dots, \hat{X}_t, \dots, X_p, Y_1, \dots, Y_q) \\ &+ \sum_{1 \leq s < t \leq p} (-1)^{s-1} \phi(X_1, \dots, \hat{X}_s, \dots, X_p, [X_s, Y_t], Y_1, \dots, \hat{Y}_t, \dots, Y_q) \\ &+ \sum_{1 \leq s < t \leq p} \phi([Y_s, Y_t], X_1, \dots, X_p, Y_1, \dots, \hat{Y}_s, \dots, \hat{Y}_t, \dots, Y_q) \\ &+ \sum_{s=1}^p (-1)^s \left[X_s, \phi(X_1, \dots, \hat{X}_s, \dots, X_p) \right] \\ &+ (-1)^{p-1} \sum_{s=1}^p \left[Y_s, \phi(X_1, \dots, \hat{X}_s, \dots, X_p) \right]. \end{aligned} \quad (2.31)$$

Since the superbracket respects the $\mathbb{Z}$ -grading, we can see immediately from this formula that if ϕ is a homogeneous p -cochain of degree d , $\partial\phi$ is a homogeneous $p + 1$ -cochain also of degree d . Thus ∂ respects the grading of the cochain spaces.

The Spencer cochain complex

$$\dots \rightarrow C^{p-1}(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^p(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^{p+1}(\mathfrak{g}_-, \mathfrak{g}) \rightarrow \dots \quad (2.32)$$

therefore decomposes into the subcomplexes

$$\dots \rightarrow C^{d,p-1}(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^{d,p}(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^{d,p+1}(\mathfrak{g}_-, \mathfrak{g}) \rightarrow \dots \quad (2.33)$$

labelled by the grading $d \in \mathbb{Z}$.

We can compute the cohomologies $H^{\bullet,\bullet}(\mathfrak{g}_-, \mathfrak{g})$ of each of these subcomplexes separately; the cohomology of the original complex is graded and is of course given by

$$H^{\bullet}(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}} H^{d,\bullet}(\mathfrak{g}_-, \mathfrak{g}). \quad (2.34)$$

We will say no more about the generalities of the Spencer cohomology; for a full review, see [24, 25].

2.3.3 Toy calculation - the Poincaré algebra

The goal of this subsection is to calculate the first few Spencer cohomology groups of the Poincaré algebra in order to demonstrate the technique in a relatively simple but still physically relevant setting before we go on to introduce spinors, supersymmetry and the Poincaré superalgebra which plays a central role in this dissertation. In principle it is logically possible to skip this discussion, but some of the results here will be useful to refer back to when we calculate the cohomology of the Poincaré superalgebra later.

Algebraically, the Poincaré algebra is the space $\mathfrak{p} = V \oplus \mathfrak{so}(V)$ where (V, η) is a Lorentzian vector space (we suppress the inner product η in the notation from now on) and $\mathfrak{so}(V)$ is its special orthogonal Lie algebra. The Lie bracket is given by

$$[A, B] = AB - BA, \quad (2.35)$$

$$[v, w] = 0, \quad (2.36)$$

$$[A, v] = -[v, A] = Av. \quad (2.37)$$

where $v, w \in V$ and $A, B \in \mathfrak{so}(V)$. It can be seen immediately that $\mathfrak{p}$ is a $\mathbb{Z}$ -graded Lie algebra with grading $\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$ where $\mathfrak{p}_{-2} = V$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. We could have chosen the degree of V to be -1 (or indeed $+1$) without meaningfully affecting anything in this subsection, but we choose it to be -2 so as to connect the discussion here with the description of the structure of the Poincaré superalgebra later.

Geometrically, $\mathfrak{p}$ is the Lie algebra of Killing vector fields (equivalently the Lie algebra of the group of isometries) of Minkowski spacetime. The space V can be considered to be the tangent space at any point of the spacetime, which is identified with the space $\mathfrak{p}_{-2}$ of translation generators, and $\mathfrak{p}_0$ is the Lorentz algebra, which is identified with $\mathfrak{so}(V)$. Note that this grading structure is equivalent to $\mathfrak{so}(V)$ being a subalgebra and V being an abelian ideal subalgebra - in particular $\mathfrak{p}$ is not semisimple.

The Poincaré algebra can be considered to be an $\mathfrak{so}(V)$ module via the adjoint action of the $\mathfrak{so}(V)$ subalgebra; the bracket on $\mathfrak{p}$ is then $\mathfrak{so}(V)$ -equivariant by the Jacobi identity, while V and $\mathfrak{so}(V)$ are submodules.

The Spencer cochain spaces of $\mathfrak{p}$ (also $\mathfrak{so}(V)$ -modules) are

$$C^{d,p}(\mathfrak{p}_-, \mathfrak{p}) = \text{Hom}(\bigwedge^p \mathfrak{p}_-, \mathfrak{p})_d = \text{Hom}(\bigwedge^p V, V)_d \oplus \text{Hom}(\bigwedge^p V, \mathfrak{so}(V))_d. \quad (2.38)$$

Since $\mathfrak{p}$ is a Lie algebra, the differentials

$$\partial^{d,p} : C^{d,p}(\mathfrak{p}_-, \mathfrak{p}) \rightarrow C^{d,p+1}(\mathfrak{p}_-, \mathfrak{p}) \quad (2.39)$$

are given by Equation (2.23) as

$$\begin{aligned} (\partial\phi)(X_1, X_2, \dots, X_{p+1}) &= \sum_{1 \leq s < t \leq p+1} (-1)^{s+t+1} \phi([X_s, X_t], X_1, \dots, \hat{X}_s, \dots, \hat{X}_t, \dots, X_p) \\ &\quad + \sum_{s=1}^{p+1} (-1)^s \left[X_s, \phi(X_1, \dots, \hat{X}_s, \dots, X_{p+1}) \right]. \end{aligned} \quad (2.40)$$

We'll take the complex degree by degree. Note that the cochain spaces and thus the complexes and cohomologies are all zero for odd degree d , while the lowest-degree nonzero complex has $d = -2$.

The $d = -2$ complex

The only nonzero cochain space is $C^{-2,0}(\mathfrak{p}_-, \mathfrak{p}) = \text{Hom}(\mathbb{R}, V) \cong V$, so the complex is

$$0 \rightarrow V \rightarrow 0. \quad (2.41)$$

We can immediately conclude that

$$H^{-2,p}(\mathfrak{p}_-, \mathfrak{p}) = Z^{-2,p}(\mathfrak{p}_-, \mathfrak{p}) = \begin{cases} V & \text{for } p = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (2.42)$$

The $d = 0$ complex

The nonzero cochain spaces are

$$C^{0,0}(\mathfrak{p}_-, \mathfrak{p}) = \text{Hom}(\mathbb{R}, \mathfrak{so}(V)) \cong \mathfrak{so}(V), \quad (2.43)$$

$$C^{0,1}(\mathfrak{p}_-, \mathfrak{p}) = \text{Hom}(V, V), \quad (2.44)$$

so the complex is

$$0 \rightarrow \mathfrak{so}(V) \rightarrow \text{Hom}(V, V) \rightarrow 0. \quad (2.45)$$

The nonzero differential $\partial^{0,0} : \mathfrak{so}(V) \rightarrow \text{Hom}(V, V)$ is given by $(\partial\phi)v = -[v, \phi] = \phi v$ for $\phi \in \mathfrak{so}(V)$ and $v \in V$. Since the action of $\mathfrak{so}(V)$ on V is faithful, the differential is injective (in fact it is simply the inclusion $\mathfrak{so}(V) \hookrightarrow \text{End}(V)$); thus the space of $(0, 0)$ -cocycles and cohomology are zero. The space of $(0, 1)$ -cocycles is trivially $\text{End}(V)$, so

$$H^{0,p}(\mathfrak{p}_-, \mathfrak{p}) = \begin{cases} \frac{\text{End}(V)}{\mathfrak{so}(V)} \cong \frac{\bigotimes^2 V}{\bigwedge^2 V} \cong \odot^2 V & \text{for } p = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.46)$$

where the isomorphisms are of $\mathfrak{so}(V)$ -modules.

The $d = 2$ complex

The complex here is

$$0 \rightarrow \text{Hom}(V, \mathfrak{so}(V)) \rightarrow \text{Hom}(\bigwedge^2 V, V) \rightarrow 0. \quad (2.47)$$

The nonzero differential is given by $(\partial\phi)(v, w) = -[v, \phi(w)] + [w, \phi(v)] = \phi(w)v - \phi(v)w$ for $\phi \in \mathfrak{so}(V)$ and $v, w \in V$. Using $\text{Hom}(V, \mathfrak{so}(V)) = V^* \otimes \mathfrak{so}(V)$, in components we have

$$\phi_{\mu\nu\lambda} = g(e_\mu, \phi(e_\lambda)e_\nu), \quad (2.48)$$

$$(\partial\phi)_{\lambda\mu\nu} = g(e_\lambda, (\partial\phi)(e_\mu, e_\nu)) = \phi_{\lambda\mu\nu} - \phi_{\lambda\nu\mu}. \quad (2.49)$$

The cocycle condition is then $\phi_{\lambda[\mu\nu]} = 0$. But because the codomain of ϕ is $\mathfrak{so}(V)$, $\phi_{\lambda\mu\nu} = -\phi_{\mu\lambda\nu}$, so

$$\phi_{\lambda\mu\nu} = -\phi_{\mu\lambda\nu} = -\phi_{\mu\nu\lambda} = \phi_{\nu\mu\lambda} = \phi_{\nu\lambda\mu} = -\phi_{\lambda\nu\mu}, \quad (2.50)$$

thus we also have $\phi_{\lambda(\mu\nu)} = 0$, so $(2, 1)$ -cocycles are zero and $\partial^{2,1}$ is injective. In fact, by counting dimensions, it is an isomorphism. The complex is therefore an exact sequence, so for all $p \in \mathbb{Z}$,

$$H^{2,p}(\mathfrak{p}_-, \mathfrak{p}) = 0. \quad (2.51)$$

The $d = 4$ complex

The complex is

$$0 \rightarrow \text{Hom}(\Lambda^2 V, \mathfrak{so}(V)) \rightarrow \text{Hom}(\Lambda^3 V, V) \rightarrow 0. \quad (2.52)$$

The differential $\partial^{4,2}$ is given by

$$\begin{aligned} (\partial\phi)(v_1, v_2, v_3) &= -[v_1, \phi(v_2, v_3)] + [v_2, \phi(v_1, v_3)] - [v_3, \phi(v_1, v_2)] \\ &= \phi(v_1, v_2)v_3 + \phi(v_2, v_3)v_1 + \phi(v_3, v_1)v_2, \end{aligned} \quad (2.53)$$

or in components,

$$\phi_{\mu\nu\sigma\tau} = \eta(e_\mu, \phi(e_\sigma, e_\tau)e_\nu) \quad (2.54)$$

$$(\partial\phi)_{\sigma\lambda\mu\nu} = \eta(e_\sigma, (\partial\phi)(e_\lambda, e_\mu, e_\nu)) = 3\phi_{\sigma[\lambda\mu\nu]}, \quad (2.55)$$

where we note that $\phi_{\mu\nu\sigma\tau} = -\phi_{\nu\mu\sigma\tau} = -\phi_{\mu\nu\tau\sigma}$. The cocycle condition is $\phi_{\mu[\nu\sigma\tau]} = 0$, an algebraic Bianchi identity. Together with the other symmetries of ϕ , this implies that $\phi_{\mu\nu\sigma\tau} = \phi_{\sigma\tau\mu\nu}$. Thus the space of cocycles is the space $\mathcal{B}$ of *algebraic curvature operators*, those tensors which satisfy the algebraic symmetries of the Riemann tensor. We therefore have an exact sequence

$$0 \rightarrow \mathcal{B} \rightarrow \text{Hom}(\Lambda^2 V, \mathfrak{so}(V)) \rightarrow \text{Hom}(\Lambda^3 V, V), \quad (2.56)$$

and the cohomologies are

$$H^{4,p}(\mathfrak{p}_-, \mathfrak{p}) = \begin{cases} \mathcal{B} & \text{for } p = 2 \\ \frac{\text{Hom}(\Lambda^3 V, V)}{\left\{ (v_1, v_2, v_3) \mapsto \phi(v_1, v_2)v_3 + \text{cyclic} \mid \phi \in \text{Hom}(\Lambda^2 V, \mathfrak{so}(V)) \right\}} & \text{for } p = 3 \\ 0 & \text{otherwise} \end{cases} \quad (2.57)$$

Higher-degree complexes

For even degree $d > 4$, the complex is

$$0 \rightarrow \text{Hom}(\Lambda^{\frac{d}{2}} V, \mathfrak{so}(V)) \rightarrow \text{Hom}(\Lambda^{\frac{d}{2}+1} V, V) \rightarrow 0, \quad (2.58)$$

which is of the same form as the lower-degree complexes. We will not calculate any more of the cohomologies. Note however, that $H^{d,2}(\mathfrak{p}_-, \mathfrak{p})$ vanishes for $d > 4$. This will be important when we consider filtered deformations of $\mathfrak{p}$.

2.3.4 The structure of filtered deformations

The actual utility of the Spencer cohomology is that it governs the filtered deformations of a Lie superalgebra. Indeed, Cheng and Kac's original purpose in adapting the cohomology was to explore the deformation theory of Lie superalgebras [24, 25]. As we are interested in the filtered deformations of the Poincaré superalgebra, we follow Cheng and Kac and lift a number of results from their paper for our purposes.

The proof of Theorem 2.3.11 is technical and occupies the first part of [24]. Our statement of the theorem here is a summation of a number of results from that paper, which we will adapt for our own practical purposes. The interested reader is invited to read that paper for more on filtered deformations of Lie algebras. For more on the prolongations of the Poincaré superalgebra see [33]; for more on its filtered deformations see [1].

If $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$ -graded Lie superalgebra, let $\mathfrak{g}_{\geq 0} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ and $\mathfrak{g}_{\leq 0} = \bigoplus_{i \leq 0} \mathfrak{g}_i$.

Definition 2.3.8. A $\mathbb{Z}$ -graded Lie superalgebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is called *transitive* if for all $X \in \mathfrak{g}_{\geq 0}$, $[X, \mathfrak{g}_{-}] = 0$ implies that $X = 0$.

Definition 2.3.9. If $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$ -graded Lie superalgebra, it is called a *full prolongation of $\mathfrak{g}_{\leq 0}$ of degree k* if the Spencer cohomology groups $H^{d,1}(\mathfrak{g}_{-}, \mathfrak{g})$ vanish for $d \geq k$. A full prolongation of degree 1 is also simply called a *full prolongation*.

Definition 2.3.10. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$ -graded Lie superalgebra, and let $\mathfrak{g}'$ be a filtered deformation of $\mathfrak{g}$ with bracket $[\cdot, \cdot]'$, which we consider to be supported on the same graded base space as $\mathfrak{g}$ as in Proposition 2.1.11. Then $[\cdot, \cdot]' = [\cdot, \cdot] + \mu$ for some deformation $\mu : \bigwedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$, and μ decomposes into graded components $\mu = \sum_{i \geq 0} \mu_i$, where each μ_i has degree i , and we call the sequence $\mu_1, \mu_2, \dots$ a *defining sequence of $\mathfrak{g}'$* .

Theorem 2.3.11 (Cheng–Kac [24]). *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$ -graded Lie superalgebra of finite depth. Then if $\mathfrak{g}'$ is a filtered deformation of $\mathfrak{g}$ with defining sequence $\{\mu_i\}_{i \geq k}$ for some $k > 0$,*

- *The first nonzero term μ_k in the defining sequence is an even (Chevalley–Eilenberg) 2-cocycle of $\mathfrak{g}$ with coefficients in the adjoint representation,*
- *The cohomology class $[\mu_k|_{\mathfrak{g}_{-} \times \mathfrak{g}_{-}}]$ is $\mathfrak{g}_0$ -invariant in $H^{k,2}(\mathfrak{g}_{-}, \mathfrak{g})$*
- *If $\mathfrak{g}$ is transitive and is a full prolongation of $\mathfrak{g}_{\leq 0}$, and $\mathfrak{g}''$ is another filtered deformation of $\mathfrak{g}$ with defining sequence $\{\nu_i\}_{i \geq k}$ such that*

$$[\mu_i|_{\mathfrak{g}_{-} \times \mathfrak{g}_{-}}] = [\nu_i|_{\mathfrak{g}_{-} \times \mathfrak{g}_{-}}], \quad (2.59)$$

for all i , then $\mathfrak{g}' \cong \mathfrak{g}''$,

In other words, if we have a finite-depth $\mathbb{Z}$ -graded Lie superalgebra $\mathfrak{g}$ which is transitive and is a full prolongation of $\mathfrak{g}_{\leq 0}$, then a filtered deformation of $\mathfrak{g}$ is completely determined up to isomorphism by the second Spencer cohomology $H^2(\mathfrak{g}_{-}, \mathfrak{g})_{\bar{0}} = \bigoplus_{i \text{ even}} H^{i,2}(\mathfrak{g}_{-}, \mathfrak{g})$.

To find the filtered deformations of the Poincaré superalgebra, we must therefore show that they are transitive full prolongations and find their second Spencer cohomologies.

2.3.5 Filtered deformations of the Poincaré algebra

We now use the results of Subsection 2.3.3. Since there are no nonzero odd cochains, all of the odd cohomologies vanish and in particular the Poincaré algebra is a full prolongation. Note that transitivity is equivalent to the action of $\mathfrak{so}(V)$ on V being faithful, which is the case, so Theorem 2.3.11 applies: the filtered deformations of $\mathfrak{p}$ are given by the $\mathfrak{p}_0$ -invariant $p = 2$ cohomologies. Its only nonzero such cohomology is $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = \mathcal{B} \subset \text{Hom}(\bigwedge^2 V, \mathfrak{so}(V))$, so the $\mathfrak{p}_0$ -invariants of this space determine the filtered deformations.

Explicitly, if we take a filtered deformation $\mathfrak{p}'$ of $\mathfrak{p}$ with the same underlying vector space $V \oplus \mathfrak{so}(V)$, the bracket $[\cdot, \cdot]'$ of $\mathfrak{p}'$ is, up to isomorphism, of the form

$$[A, B]' = AB - BA \quad (2.60)$$

$$[v, w]' = \phi(v, w), \quad (2.61)$$

$$[A, v]' = -[v, A]' = Av, \quad (2.62)$$

where $A, B \in \mathfrak{so}(V)$, $v, w \in V$ and $\phi : \bigwedge^2 V \rightarrow \mathfrak{so}(V)$ is an $\mathfrak{so}(V)$ -invariant algebraic curvature operator; that is, for all $A \in \mathfrak{so}(V)$ and $v, w \in V$,

$$[A, \phi(v, w)] = \phi(Av, w) + \phi(v, Aw). \quad (2.63)$$

Since $\bigwedge^2 V$ and $\mathfrak{so}(V)$ are irreducible under the $\mathfrak{so}(V)$ action, Schur's Lemma implies that ϕ is proportional to the natural isomorphism $\omega_\bullet : \bigwedge^2 V \rightarrow \mathfrak{so}(V)$ ⁵; that is, $\phi = k\omega$ for some $k \in \mathbb{R}$, so we have only a one-parameter family of Lie algebras. In fact, up to isomorphism, there are only three distinct Lie algebras corresponding to $k = 0, +1, -1$, and these are, respectively, the isometry algebras of Minkowski space, de Sitter space and anti de Sitter space! In other words, the filtered deformations of the Poincaré algebra are exactly the isometry algebras of the maximally symmetric Lorentzian manifolds.

This demonstrates the principle that underlies our approach: filtered subdeformations of the symmetry algebra of the flat model give the symmetry algebras of curved spaces, and in particular the maximal symmetry algebras are filtered deformations of the full flat model algebra.

Our goal is to realise the symmetry algebras of supersymmetric manifolds, their so-called Killing superalgebras, as filtered deformations of the Poincaré superalgebra which is the Killing superalgebra of supersymmetric Minkowski space.

2.4 Complex, quaternionic and real structures

The Clifford algebra and its modules in 5 spacetime dimensions that we meet in Chapter 3 have quaternionic and real structures which we will make heavy use of. In this last piece of background material, we discuss what these structures are.

Definition 2.4.1. A complex structure on a real vector space V is a linear map $I : V \rightarrow V$ such that $I^2 = -\mathbb{1}$.

⁵See next chapter for the definition of $\omega_\bullet$.

A complex structure on a real vector space induces on it the structure of a complex vector space as follows: for $z = x + iy \in \mathbb{C}$, where x and y are the real and imaginary parts of z , and for $v \in V$, we define $(x + iy)v = xv + yIv$. Then $\mathbb{C}$ -linear transformations on V are simply those $\mathbb{R}$ -linear transformations that commute with I , in other words $\text{End}_{\mathbb{C}}(V) = Z_{\text{End}_{\mathbb{R}}(V)}(I)$. More generally, if (V, I_V) and (W, I_W) are real vector spaces V and W with complex structures I_V and I_W respectively and we make them into complex vector spaces as described above, $\text{Hom}_{\mathbb{C}}(V, W)$ are simply elements $f \in \text{Hom}_{\mathbb{R}}(V, W)$ such that $I_W \circ f = f \circ I_V$.

Conversely, if V is a vector space over $\mathbb{C}$, then in particular it is a vector space over $\mathbb{R}$ on which multiplication by i is linear; this defines a complex structure on V considered as a real vector space. The action of $\mathbb{C}$ then induced by the complex structure coincides with the original $\mathbb{C}$ -vector space structure of V .

Recall that an $\mathbb{R}$ -linear map $f : V \rightarrow W$ of $\mathbb{C}$ -vector spaces V, W is called antilinear (or conjugate-linear) if $f(zv) = \bar{z}v$ for all $z \in \mathbb{C}$.

Definition 2.4.2. A real structure on a complex vector space V is an antilinear map $C : V \rightarrow V$ such that $C^2 = \mathbb{1}$.

In the same way that a complex structure on a real vector space plays the role of i without explicit mention of i or $\mathbb{C}$, a real structure on a complex vector space plays the role of complex conjugation. Indeed, if $V = \mathbb{C}^n$, then complex conjugation is a real structure on V . We can decompose V into two real vector spaces, its real and imaginary parts, $V = \mathbb{R}^n \oplus i\mathbb{R}^n$, where the direct sum is internal. This decomposition can be thought of as being induced by complex conjugation; since conjugation is an $\mathbb{R}$ -linear map whose square is the identity, it has eigenvalue ± 1 , thus the real and imaginary parts of V are the two eigenspaces of conjugation.

If V is an arbitrary $\mathbb{C}$ -vector space, complex conjugation is not defined. We could of course choose a basis $\{b^i\}_{i=1}^{\dim_{\mathbb{C}} V}$ for V and define a real structure C by $C(z_i b^i) = \bar{z}_i b^i$, but this is clearly basis-dependent and far from unique. It will transpire, however, that real structures do arise in a natural way in situations that we will discuss later on. By the same argument we gave for complex conjugation, a real structure on V decomposes it into a direct sum of real eigenspaces corresponding to eigenvalues ± 1 , which we will still call the “real” and “imaginary” subspaces V_R and V_I respectively. Finally, we note that V is then just the complexification of its real subspace, $V \cong V_R \otimes_{\mathbb{R}} \mathbb{C}$; in particular $\mathbb{C}^n \cong \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ where the real structure is complex conjugation.

Definition 2.4.3. • A quaternionic structure on a complex vector space is an antilinear map $J : V \rightarrow V$ such that $J^2 = -\mathbb{1}$.

- A quaternionic structure on a real vector space is a pair of complex structures (I, J) such that $I \circ J = -J \circ I$.

First, note that these two notions are equivalent. If V is a complex vector space with a quaternionic structure J and the complex structure I induced by the action of i on V considered as a real vector space, then (I, J) is a quaternionic structure on the real vector space V . Conversely, if V is a real vector space with a quaternionic structure

(I, J) , and V is made into a complex vector space using the complex structure I , then J is a quaternionic structure on the complex vector space V . Note that the roles of I and J could be swapped in the converse case, leading to different but isomorphic quaternionic structures on the complex V .

A quaternionic structure on a real or complex vector V space is equivalent to V being given the structure of a right vector space over $\mathbb{H}$: over $\mathbb{C}$, J plays the role of right multiplication by the quaternion j , and over $\mathbb{R}$, I and J play the role of right multiplication by quaternions i and j respectively. Note that in this latter case, there is another complex structure $K = JI$ which corresponds to right multiplication by $k = ij$.

Chapter 3

Conventions, Clifford algebras, representations

This chapter gives a brief introduction to Clifford algebras, the algebraic structures required to deal with spinors, in particular laying out conventions we will use throughout the remainder of this work. It also lays out some extra structures and conventions we use to deal with spinors in 5 dimensions and a number of results which will be useful in calculations later.

For a field or skew-field k (which will be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), we denote the $n \times n$ matrix algebra over k by $k(n)$. Unless otherwise stated, the ground field here is always $\mathbb{R}$.

3.1 General dimension

3.1.1 Clifford algebras

Throughout, let D be the spacetime dimension and let (V, η) be a Lorentzian inner product space with mostly minus signature; this can be thought of as the tangent space at a point in spacetime but for our purposes it is usually to be thought of as the space of generators of translations in Minkowski space. We let $\{e_0, e_1, \dots, e_D\}$ be an orthonormal basis for (V, η) such that

$$\eta_{\mu\nu} = \eta(e_\mu, e_\nu) = \text{diag}(1, -1, \dots, -1), \quad (3.1)$$

where the Greek indices run over $0, 1, \dots, D$.

The Clifford algebra $\mathcal{CL}(V, \eta)$ is generated by the e_μ subject to the relation

$$v^2 - \eta(v, v) = 0 \quad (3.2)$$

for all $v \in V$, or equivalently

$$e_\mu \cdot e_\nu + e_\nu \cdot e_\mu = 2\eta_{\mu\nu}. \quad (3.3)$$

In the remainder of this report, we will suppress the inner product in the notation and write $\mathcal{CL}(V)$ for the Clifford algebra.

The Clifford algebra is $\mathbb{Z}$ -graded as a vector space (though not as an algebra); the grading is given by $\mathcal{CL}(V) = \bigoplus_{p=0}^D \mathcal{CL}(V)_p$ where

$$\mathcal{CL}(V)_p = \text{span}\{e_{\mu_1} \cdot e_{\mu_2} \cdot \dots \cdot e_{\mu_p} \mid \mu_1 < \mu_2 < \dots < \mu_p\}. \quad (3.4)$$

Note that the $e_{\mu_1} \cdot e_{\mu_2} \cdot \dots \cdot e_{\mu_p}$ with $\mu_1 < \mu_2 < \dots < \mu_p$ form a basis for $\mathcal{CL}(V)_p$. As an algebra, $\mathcal{CL}(V)$ is a filtered with respect to the natural vector space filtration induced by this grading, and it is a superalgebra with parity given by degree modulo 2.

3.1.2 The Clifford algebra as a filtered deformation of the exterior algebra

The filtration of $\mathcal{CL}(V)$ resembles the exterior algebra $\bigwedge V$ of V ; we have

$$\bigwedge V = \bigoplus_{p=0}^D \bigwedge^p V \quad \text{where} \quad \bigwedge^p V = \text{span}\{e_{\mu_1} \wedge e_{\mu_2} \wedge \dots \wedge e_{\mu_p} \mid \mu_1 < \mu_2 < \dots < \mu_p\}; \quad (3.5)$$

the exterior algebra (with multiplication given by wedge product) is a $\mathbb{Z}$ -graded supercommutative superalgebra (with parity given by degree modulo 2), and clearly as a graded vector space it is isomorphic to $\mathcal{CL}(V)$.

In fact $\mathcal{CL}(V)$ is a filtered deformation of $\bigwedge V$ with the isomorphism of algebras $\text{Gr } \mathcal{CL}(V) \cong \bigwedge V$ given by the composition of vector space isomorphisms $\text{Gr } \mathcal{CL}(V) \cong \mathcal{CL}(V) \cong \bigwedge V$. Using these isomorphisms, we may identify $\bigwedge V$ as the underlying vector space of $\mathcal{CL}(V)$, writing elements of the Clifford algebra as elements of the exterior algebra and using $\cdot$ to denote Clifford multiplication and $\wedge$ to denote the wedge product as two different multiplications on the same space. With either multiplication, V (and indeed any basis for V) generates $\mathcal{CL}(V)$ or $\bigwedge V$ as an algebra.

3.1.3 Embedding of the Lorentz algebra

The Lie algebra $\mathfrak{so}(V, \eta)$ of the Lorentz group $SO(V, \eta)$ (hereafter we suppress η in the notation) can be regarded as the set of endomorphisms of V which are skew-symmetric with respect to η :

$$\mathfrak{so}(V) = \{A \in \text{End}(V) \mid \eta(Av, w) = -\eta(v, Aw)\} \quad (3.6)$$

or equivalently those $A \in \text{End}(V)$ whose components with respect to the orthonormal basis, with indices lowered by η , are skew-symmetric:

$$A_{\mu\nu} = \eta_{\mu\rho} A^\rho{}_\nu = \eta(e_\mu, Ae_\nu) = -\eta(Ae_\mu, e_\nu) = -A_{\nu\mu}. \quad (3.7)$$

This is a realisation of the natural isomorphism $\omega^\bullet : \mathfrak{so}(V) \rightarrow \bigwedge^2 V^*$ given by

$$\omega^A(v, w) = \eta(v, Aw). \quad (3.8)$$

In components, $\omega_{\mu\nu}^A = \omega^A(e_\mu, e_\nu) = A_{\mu\nu}$ so that $\omega^A = \frac{1}{2} A_{\mu\nu} f^\mu \wedge f^\nu$, where $\{f^\mu\}$ is the dual basis of $\{e_\mu\}$ for V^* . Finally, we may use the natural isomorphism $V \cong V^*$

induced by η to identify $\bigwedge^p V^* \cong \bigwedge^p V$, giving an isomorphism $\omega_\bullet : \mathfrak{so}(V) \rightarrow \bigwedge^2 V$ with $\omega_A = \frac{1}{2} A^{\mu\nu} e_\mu \wedge e_\nu$. From now on we will not distinguish between $\bigwedge V$ and $\bigwedge V^*$, nor between $\omega_\bullet$ and $\omega^\bullet$. It will still be useful, however, to distinguish $\mathfrak{so}(V)$ and $\bigwedge^2 V$, due to the following observation.

We can obtain a Lie algebra representation of $\mathfrak{so}(V)$ on $\mathcal{CL}(V)$ as follows. Let $\rho : \mathfrak{so}(V) \rightarrow \text{End}(\mathcal{CL}(V))$ such that, for $A \in \mathfrak{so}(V)$ and $\phi \in \bigwedge V$

$$\rho(A)(\phi) = \frac{1}{2} [\omega_A, \phi] = \frac{1}{4} A^{\mu\nu} [e_\mu \wedge e_\nu, \phi], \quad (3.9)$$

where the bracket is the commutator for Clifford multiplication. In particular, if $v \in V$, we have $\rho(A)(v) = A^\mu_\nu v^\nu e_\mu = Av$; the ρ -action of $\mathfrak{so}(V)$ on V considered as a subspace of $\mathcal{CL}(V)$ is the same as the action of $\mathfrak{so}(V)$ on V . Another way of viewing this is by defining a Lie algebra homomorphism $\Omega : \mathfrak{so}(V) \rightarrow \mathcal{CL}(V)$ (the latter with its bracket given by the commutator) by $A \mapsto \frac{1}{2} \omega_A$; then the representation above is simply $\rho = \text{Ad}_{\mathcal{CL}(V)} \circ \Omega$.

We will from now on suppress all three maps ω , Ω and ρ in the notation, writing for instance $A \cdot \phi$ for $\Omega_A \cdot \phi = \frac{1}{2} \omega_A \cdot \phi$ where $A \in \mathfrak{so}(V)$ and $\phi \in \bigwedge V$. If M is a module of the Clifford algebra, we will also write $A \cdot m$ for $\Omega_A \cdot m = \frac{1}{2} \omega_A \cdot m$ for $m \in M$. In this way, $\mathcal{CL}(V)$ (associative algebra) modules and representations become $\mathfrak{so}(V)$ (Lie algebra) representations.

3.2 In 5 dimensions

3.2.1 Structure of the Clifford algebra

In this section, we endeavour when possible to choose conventions and notation consistent with [4], since the spinor structure is similar in 5 and 6 dimensions.

Proposition 3.2.1. *If V is a 5-dimensional Lorentzian inner product space,*

$$\mathcal{CL}(V) \cong \mathbb{H}(2) \oplus \mathbb{H}(2). \quad (3.10)$$

To prove this statement, some Clifford algebra theory is required. A *Clifford map* is a linear map $f : W \rightarrow A$ from a quadratic vector space (W, Q) to an associative algebra A such that $f(w)f(w) = Q(w)\mathbb{1}_A$ for all $w \in W$. The Clifford algebra $\mathcal{CL}(W, Q)$ has the following universal property: if $f : W \rightarrow A$ is a Clifford map, then there exists a unique algebra homomorphism $\tilde{f} : \mathcal{CL}(W, Q) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{\quad} & \mathcal{CL}(W), \\ & \searrow f & \swarrow \tilde{f} \\ & & A \end{array} \quad (3.11)$$

where the map at the top of the diagram is the natural embedding of V into $\mathcal{CL}(V)$. Since V generates $\mathcal{CL}(V)$, the homomorphism $\tilde{f}$ is completely defined by specifying that $\tilde{f}|_V = f(V)$. Note that if f is Clifford, $-\tilde{f}$ is too.

Proof of Proposition 3.2.1. We can take $V \cong \mathbb{R}^{1,4} \cong \mathbb{R} \oplus \mathbb{H}$ as vector spaces. Furthermore, defining the quadratic form on $\mathbb{R} \oplus \mathbb{H}$,

$$Q(t, q) = t^2 - |q|^2 = t^2 - q\bar{q}, \quad (3.12)$$

where $(t, q) \in \mathbb{R} \oplus \mathbb{H}$, we have an isomorphism $V \cong \mathbb{R} \oplus \mathbb{H}$ of quadratic vector spaces. For the remainder of this proof, we identify V with $\mathbb{R} \oplus \mathbb{H}$. Now let $\phi : V \rightarrow \mathbb{H}(2)$ be the map

$$\phi(t, q) = \begin{pmatrix} t & q \\ -\bar{q} & -t \end{pmatrix}. \quad (3.13)$$

This map is Clifford and injective:

$$\phi(t, q)\phi(t, q) = \begin{pmatrix} t & q \\ -\bar{q} & -t \end{pmatrix} \begin{pmatrix} t & q \\ -\bar{q} & -t \end{pmatrix} = \begin{pmatrix} t^2 - q\bar{q} & tq - qt \\ -\bar{q}t + t\bar{q} & -\bar{q}q + t^2 \end{pmatrix} = Q(t, q)\mathbb{1}_2. \quad (3.14)$$

and clearly $\phi(t, q) = 0$ if and only if $t = q = 0$. Since ϕ and $-\phi$ are both Clifford and injective, the map $\Phi : V \rightarrow \mathbb{H}(2) \oplus \mathbb{H}(2)$ defined by $\Phi(t, q) = (\phi(t, q), -\phi(t, q))$ is also Clifford and injective.

We claim that the map $\tilde{\Phi} : \mathcal{CL}(V, Q) \rightarrow \mathbb{H}(2) \oplus \mathbb{H}(2)$ induced by the universal property of Clifford algebras is an isomorphism. Indeed, since $\tilde{\Phi}$ is an algebra homomorphism and the image of V under the natural embedding generates $\mathcal{CL}(V, Q)$ as an associative algebra, injectivity of Φ implies that $\tilde{\Phi}$ is injective. We can show that $\tilde{\Phi}$ is surjective by dimension counting:

$$\begin{aligned} \dim_{\mathbb{R}}(\mathcal{CL}(V, Q)) &= \dim_{\mathbb{R}} \left(\bigwedge V \right) = \sum_{p=0}^5 \binom{5}{p} = 2^5 = 32 \\ &= 2 \times 2^2 \times 4 = \dim_{\mathbb{R}}(\mathbb{H}(2)), \end{aligned} \quad (3.15)$$

hence the claim. □

3.2.2 Clifford modules and Γ -endomorphisms

Consider now $\mathbb{H}^2$ regarded as a right- $\mathbb{H}$ vector space and a left $\mathbb{H}(2)$ -module (by left matrix multiplication). Then $\mathbb{H}^2$ is a left $\mathcal{CL}(V) \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$ -module, with one copy of $\mathbb{H}(2)$ acting via left matrix multiplication and the other copy acting trivially. There are therefore two choices of $\mathcal{CL}(V)$ module structure on $\mathbb{H}^2$ distinguished by the action of the volume element $\text{vol} = e_1 \wedge e_2 \wedge \cdots \wedge e_5 = e_1 \cdot e_2 \cdots e_5$; we can show (by simply writing it out explicitly) that under the isomorphism $\mathcal{CL}(V) \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$, $\text{vol} \sim (-\mathbb{1}, \mathbb{1})$, so vol acts as $\pm \mathbb{1}$ on either module. Thus we have two inequivalent left modules with $\mathbb{H}^2$ as the underlying vector space. We will work with the module on which vol acts as $\mathbb{1}$, which we denote Σ , and consider it as $\mathbb{C}^4$ with a quaternionic structure which is invariant under the Clifford action - we will spell this out in more detail shortly.

We denote the image in $\text{End}(\Sigma)$ of $e_{\mu_1} \wedge e_{\mu_2} \cdots \wedge e_{\mu_p}$ by $\Gamma_{\mu_1 \mu_2 \dots \mu_p}$. In keeping with physics notation, we could denote the image of vol by $\Gamma_6 = \Gamma_{01234}$, but this is just $\mathbb{1}$. Notice that

$$\Gamma_{\mu_1 \mu_2 \dots \mu_p} = \Gamma_{[\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_p]}. \quad (3.16)$$

We'll call this a Γ -endomorphism of rank p . This expression, along with the image of the Clifford relation (which follows from Equation (3.3))

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} \mathbb{1} \quad (3.17)$$

is sufficient to prove the following relations via some combinatorial arguments (the process is similar to Wick's theorem):

Lemma 3.2.2. *The products of the Γ -endomorphisms of rank 1 or 2 are*

$$\Gamma^\mu \Gamma_\sigma = \Gamma^\mu_\sigma + \delta^\mu_\sigma, \quad (3.18)$$

$$\Gamma^{\mu\nu} \Gamma_\sigma = \Gamma^{\mu\nu}_\sigma + 2\delta^{[\nu}_\sigma \Gamma^{\mu]}, \quad (3.19)$$

$$\Gamma^\mu \Gamma_{\sigma\tau} = \Gamma^\mu_{\sigma\tau} + 2\delta^{[\mu}_{[\sigma} \Gamma^{\tau]}, \quad (3.20)$$

$$\Gamma^{\mu\nu} \Gamma_{\sigma\tau} = \Gamma^{\mu\nu}_{\sigma\tau} + 4\delta^{[\nu}_\sigma \Gamma^{\mu]}_\tau + 2\delta^{[\nu}_\sigma \delta^{\mu]}_\tau. \quad (3.21)$$

The following combinatorial identity is also useful when evaluating multiple products.

Lemma 3.2.3. *For $0 < r < D$,*

$$\epsilon^{\mu_1 \mu_2 \dots \mu_D} \epsilon_{\nu_1 \nu_2 \dots \nu_r \mu_{r+1} \dots \mu_D} = (D-r)! r! \delta^{[\mu_1}_{[\nu_1} \delta^{\mu_2}_{\nu_2} \dots \delta^{\mu_r}_{\nu_r]} = (D-r)! r! \delta^{[\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \dots \delta^{\mu_r}_{\nu_r]}. \quad (3.22)$$

The two lemmas above are valid for the relevant Clifford representation in any dimension; the following only holds for the chosen module Σ in $D = 5$.

Lemma 3.2.4. *Since vol acts as $\mathbb{1}$, Hodge duality in $\bigwedge V$ induces the following identities in $\text{End}(\Sigma)$:*

$$\Gamma_{\alpha\beta\gamma} = -\frac{1}{2} \epsilon_{\alpha\beta\gamma\sigma\tau} \Gamma^{\sigma\tau}, \quad (3.23)$$

$$\Gamma_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta\sigma} \Gamma^\sigma, \quad (3.24)$$

$$\Gamma_{\alpha\beta\gamma\delta\epsilon} = \epsilon_{\alpha\beta\gamma\delta\epsilon} \mathbb{1}. \quad (3.25)$$

These three lemmas are sufficient to evaluate any product of Γ endomorphisms. For exposition, let's see how a general p -vector $\Phi \in \bigwedge^p V$ acts via the identification with the Clifford algebra on $\sigma \in \Sigma$:

$$\Phi \cdot \sigma = \frac{1}{p!} \Phi^{\mu_1 \dots \mu_p} (e_{\mu_1} \wedge \dots \wedge e_{\mu_p}) \cdot \sigma = \frac{1}{p!} \Phi^{\mu_1 \dots \mu_p} \Gamma_{\mu_1 \mu_2 \dots \mu_p} \sigma. \quad (3.26)$$

One last property of the Γ -endomorphisms to note is that they are traceless.

Lemma 3.2.5. *The rank- p Γ -endomorphisms for $0 < p < D$ are traceless.*

Proof. We follow the argument given in [5]. We can use Equation (3.19) to evaluate the commutator

$$[\Gamma^{\mu\nu}, \Gamma_\sigma] = \Gamma^{\mu\nu}_\sigma - \Gamma_\sigma^{\mu\nu} + 2\delta^{[\nu}_\sigma \Gamma^{\mu]} - 2\delta^{[\mu}_\sigma \Gamma^{\nu]} = 4\delta^{[\nu}_\sigma \Gamma^{\mu]}. \quad (3.27)$$

Using the trace property $\text{Tr}(AB) = \text{Tr}(BA)$, we see that the trace of the left hand side vanishes, giving $\delta^{[\nu}_\sigma \text{Tr} \Gamma^{\mu]} = 0$. Contracting σ and ν gives us $\text{Tr} \Gamma^\mu = 0$. The vanishing of the other Γ -endomorphisms follow from symmetry arguments. $\square$

3.2.3 Quaternionic and real structures on Clifford modules

The module Σ could be taken as the space of spinors, however it is 8-dimensional over $\mathbb{R}$ so $\text{End}_{\mathbb{R}}(\Sigma)$ is 64-dimensional. This is a problem because $\mathcal{CL}(V)$ is only 32-dimensional and the representation of $\mathcal{CL}(V)$ on Σ is not faithful by Lemma 3.2.4; its image in $\text{End}_{\mathbb{R}}(\Sigma)$ is only 16-dimensional, so not all endomorphisms can be expressed in terms of Γ -endomorphisms. This would cause practical problems later on when we want to work explicitly with endomorphisms of the spinor space.

To deal with this, we will introduce another module with a quaternionic and symplectic structure, tensor it with Σ and then use a real structure to obtain another 8-dimensional module which will be endowed with enough structure to make explicit calculations possible.

Consider $\Delta = \mathbb{H}$ as a trivial left $\mathcal{CL}(V)$ -module thought of as $\mathbb{C}^2$ with an invariant quaternionic structure (more on this shortly). It also has a symplectic form $\epsilon(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{C}$ defined by

$$\epsilon((u_1, u_2), (v_1, v_2)) = u_1 v_2 - u_2 v_1 \quad (3.28)$$

for $(u_1, u_2), (v_1, v_2) \in \Delta \cong \mathbb{C}^2$. We will work in components using the Einstein convention; denoting the standard basis over $\mathbb{C}$ on Δ by $\{e_A\}_{A=1,2}$, we have $\epsilon(u, v) = \epsilon_{AB} u^A v^B$, where ϵ_{AB} is the totally skew-symmetric symbol determined by $\epsilon_{12} = 1$. We also define the totally skew-symmetric symbol ϵ^{AB} with $\epsilon^{12} = 1$, which similarly defines a symplectic form on Δ^* in the dual basis to $\{e_A\}_{A=1,2}$.

The symplectic form induces an isomorphism $\Delta \cong \Delta^*$ denoted $u \rightarrow u^b$, where $u^b(w) = \epsilon(w, u)$ (note that there is an alternative choice of isomorphism given by $u \rightarrow u^{-b}$ where $u^{-b}(w) = \epsilon(u, w)$).

As a result we can identify Δ with Δ^* and $\bigotimes_{\mathbb{C}}^2 \Delta$ with $\Delta^* \otimes_{\mathbb{C}} \Delta \cong \text{End}_{\mathbb{C}}(\Delta)$. We will work mainly with Δ and $\text{End}_{\mathbb{C}} \Delta \cong \bigotimes_{\mathbb{C}}^2 \Delta = (\odot_{\mathbb{C}}^2 \Delta) \oplus (\wedge_{\mathbb{C}}^2 \Delta)$. We will suppress the $\mathbb{C}$ subscript on all structure on Δ and make repeated use of the fact that $\epsilon(\cdot, \cdot)$ spans $\wedge^2 \Delta$.

These identifications are implemented at the level of components by using the symbols ϵ_{AB} and ϵ^{AB} to raise and lower indices; since these symbols are skew-symmetric this requires a choice of convention. If $\phi_{\dots A \dots}$ (resp. $\theta_{\dots A \dots}$) are objects with index A up (resp. down), we lower (resp. raise) the index as follows:

$$\phi_{\dots A \dots} = \epsilon_{AB} (\phi_{\dots}^{B \dots}) \quad \text{and} \quad \theta_{\dots}^{A \dots} = (\theta_{\dots B \dots}) \epsilon^{BA}, \quad (3.29)$$

and this choice of convention agrees with the choice of isomorphism $\Delta \cong \Delta^*$. Finally, it is useful to note the following.

Lemma 3.2.6. *We have the following identities for Δ -indices.*

$$\epsilon^A_B = \delta_B^A, \quad (3.30)$$

$$\epsilon_A^B = -\delta_A^B, \quad (3.31)$$

$$(\phi_{\dots}^{A \dots})(\theta_{\dots A \dots}) = -(\phi_{\dots A \dots})(\theta_{\dots}^{A \dots}), \quad (3.32)$$

where ϕ and θ are some tensors of Δ .

Proof.

$$\epsilon^A{}_B = \epsilon_{CB} \epsilon^{CA} = \delta_B^A, \quad (3.33)$$

$$\epsilon_A{}^B = \epsilon_{AC} \epsilon^{CB} = -\delta_A^B, \quad (3.34)$$

$$\begin{aligned} (\phi^{\dots A \dots})(\theta^{\dots A \dots}) &= (\phi^{\dots B \dots}) \epsilon^{BA} \epsilon_{AC} (\theta^{\dots C \dots}) \\ &= (\phi^{\dots B \dots})(-\delta_C^B)(\theta^{\dots C \dots}) \\ &= -(\phi^{\dots B \dots})(\theta^{\dots B \dots}) \\ &= -(\phi^{\dots A \dots})(\theta^{\dots A \dots}). \end{aligned} \quad (3.35)$$

□

We defined Σ and Δ as having underlying spaces $\mathbb{H}^2$ and $\mathbb{H}$ respectively. We consider these to be right $\mathbb{H}$ -vector spaces, with $\mathcal{CL}(V)$ acting as $\mathbb{H}$ -matrices (Γ -endomorphisms) on the left. We prefer to describe them as complex spaces with quaternionic structures J_Σ and J_Δ which are invariant under the $\mathcal{CL}(V)$ action. We will now demonstrate this concretely.

Complex scalar multiplication is also defined to take place *on the right* even if we write it on the left, so that it does not get in the way of quaternionic matrix multiplication; in particular if $w, z \in \mathbb{C}$ and $\sigma \in \Sigma$,

$$w(z\sigma) = w(\sigma z) = \sigma(zw) = \sigma(wz). \quad (3.36)$$

Since $\mathcal{CL}(V)$ acts on the left via matrix multiplication, it commutes with scalar multiplication (by $\mathbb{H}$ or $\mathbb{C}$). If we define $J_\Sigma \sigma = \sigma j$, clearly $J_\Sigma^2 = -1$ and

$$J_\Sigma(\sigma z) = (\sigma z)j = \sigma(zj) = \sigma(jz^*) = (\sigma j)z^* = J_\Sigma(\sigma)z^*, \quad (3.37)$$

so J_Σ is also antilinear and thus a quaternionic structure. Similarly J_Σ commutes with the $\mathcal{CL}(V)$ -action:

$$\Phi \cdot (J_\Sigma(\sigma)) = \Phi \cdot (\sigma j) = (\Phi \cdot \sigma)j = J_\Sigma(\Phi \cdot \sigma), \quad (3.38)$$

so σ has a $Cl(V)$ -invariant quaternionic structure as claimed. The same argument holds for J_Δ .

Proposition 3.2.7. *The tensor product $\Sigma \otimes_{\mathbb{C}} \Delta$ has a $\mathcal{CL}(V)$ -invariant real structure which induces on it a decomposition*

$$\Sigma \otimes_{\mathbb{C}} \Delta \cong S \otimes \mathbb{C} \quad (3.39)$$

into the complexification of an 8-dimensional module S of $\mathcal{CL}(V)$. In the language of the physics literature, S is the space of symplectic Majorana spinors.

Proof. We follow an argument given in the appendices of [34]. We consider Σ and Δ as complex spaces with invariant quaternionic structures J_Σ and J_Δ , and we will show that

$J_\Sigma \otimes J_\Delta$ gives us a real structure on $\Sigma \otimes_{\mathbb{C}} \Delta$. Since J_Σ and J_Δ are C -antilinear, for $\sigma \in \Sigma$ and $u \in \Delta$ we have

$$\begin{aligned}
(J_\Sigma \otimes J_\Delta)((\sigma \otimes u)z) &= (J_\Sigma \otimes J_\Delta)(\sigma \otimes (uz)) \\
&= (\sigma j) \otimes (uzj) \\
&= (\sigma j) \otimes (ujz^*) \\
&= ((\sigma j) \otimes (uj))z^* \\
&= ((J_\Sigma \otimes J_\Delta)(\sigma \otimes u))z^*
\end{aligned} \tag{3.40}$$

so $J_\Sigma \otimes J_\Delta$ is antilinear, and since $J_\Sigma^2 = -\mathbb{1}$ and $J_\Delta^2 = -\mathbb{1}$,

$$(J_\Sigma \otimes J_\Delta)^2(\sigma \otimes u) = (J_\Sigma^2 \sigma) \otimes (J_\Delta^2 u) = (-1)^2 \sigma \otimes u = \sigma \otimes u. \tag{3.41}$$

Since J_Σ and J_Δ are invariant under the action of $\mathcal{CL}(V)$, so is $J_\Sigma \otimes J_\Delta$. Thus we have an invariant real structure as claimed. Then S is simply the real part of $\Sigma \otimes_{\mathbb{C}} \Delta$; that is, the $\mathbb{R}$ -subspace consisting of elements $s \in \Sigma \otimes_{\mathbb{C}} \Delta$ such that $(J_\Sigma \otimes J_\Delta)s = s$. By the invariance of the real structure, S is preserved by the $\mathcal{CL}(V)$ -action: for all $s \in S$ and $\Phi \in \mathcal{CL}(V)$,

$$(J_\Sigma \otimes J_\Delta)(\Phi \cdot s) = \Phi \cdot ((J_\Sigma \otimes J_\Delta)s) = \Phi \cdot s \tag{3.42}$$

so S is a representation of $\mathcal{CL}(V)$. $\square$

The module S (which is faithful and irreducible) is 8-dimensional over $\mathbb{R}$, so $\text{End}(S)$ is 64-dimensional, just like the endomorphism algebra $\text{End}_{\mathbb{R}}(\Sigma)$. However, consider its complexification $\text{End}(S) \otimes_{\mathbb{R}} \mathbb{C} \cong \text{End}_{\mathbb{C}}(\Sigma \otimes_{\mathbb{C}} \Delta)$; this has complex dimension 64. On the other hand, denoting by Γ the representation of $\mathcal{CL}(V)$ on Σ , we argued before that $\dim_{\mathbb{R}}(\Gamma(\mathcal{CL}(V))) = 16$, so

$$\dim_{\mathbb{C}}((\Gamma(\mathcal{CL}(V)) \otimes \mathbb{C}) \otimes_{\mathbb{C}} \text{End}(\Delta)) = 16 \times 4 = 64. \tag{3.43}$$

We also have

$$(\Gamma(\mathcal{CL}(V)) \otimes \mathbb{C}) \otimes_{\mathbb{C}} \text{End}(\Delta) \cong \Gamma(\mathcal{CL}(V)) \otimes (\mathbb{C} \otimes_{\mathbb{C}} \text{End}(\Delta)) \cong \Gamma(\mathcal{CL}(V)) \otimes \text{End}(\Delta). \tag{3.44}$$

It therefore follows that $\Gamma(\mathcal{CL}(V)) \otimes \text{End}(\Delta) \cong \text{End}_{\mathbb{C}}(\Sigma \otimes_{\mathbb{C}} \Delta)$, so any $\mathbb{C}$ -endomorphism of $\Sigma \otimes_{\mathbb{C}} \Delta$ (or $\mathbb{R}$ -endomorphism of its real part S) can be written in terms of Γ -endomorphisms and endomorphisms of Δ .

3.2.4 Notation for the spinor modules

We now introduce the notation we will practically use to work with this structure. Let $s \in \Sigma \otimes_{\mathbb{C}} \Delta$. All of the following discussion follows in particular if $s \in S$, but we will say more about this special case at the end. Since all structures we deal with are multi-linear, we assume without loss of generality that s is a pure tensor; that is, $s = \sigma \otimes u$ where $\sigma \in \Sigma$ and $u \in \Delta$. Then, recalling the $\mathbb{C}$ -basis $\{e_A\}_{A=1,2}$ for Δ , we can write $u = u^A e_A$ for a pair of complex numbers u^1, u^2 , and so

$$s = \sigma \otimes u = \sigma \otimes (u^A e_A) = (u^A \sigma) \otimes e_A = s^A \otimes e_A \tag{3.45}$$

where $s^A = u^A \sigma$ for $A = 1, 2$, so s is specified in the given basis for Δ by the pair of elements $s^1, s^2 \in \Sigma$. We could, of course, pick a basis for Σ with some indices $a, b, \dots$ and fully write s in components s^{Aa} , but this won't be necessary; it is sufficient for us to specify s by the Σ -elements s^A . Furthermore, since Δ is trivial as a $\mathcal{CL}(V)$ -module, for $\Phi \in \bigwedge^p V$ we have

$$\Phi \cdot s = \Phi \cdot (s^A \otimes e_A) = (\Phi \cdot s^A) \otimes e_A = \frac{1}{p!} \Phi^{\mu_1 \dots \mu_p} (\Gamma_{\mu_1 \mu_2 \dots \mu_p} s^A) \otimes e_A. \quad (3.46)$$

As we have previously noted, the modules of $\mathcal{CL}(V)$ automatically become modules of $\mathfrak{so}(V)$. Since we will be using it extensively, we will demonstrate the module action explicitly now. With the notation of the discussion following Equation (3.9), we have, for $A \in \mathfrak{so}(V)$,

$$A \cdot s = \frac{1}{2} \omega_A \cdot s = \frac{1}{4} A^{\mu\nu} \Gamma_{\mu\nu} s = \frac{1}{4} A^{\mu\nu} (\Gamma_{\mu\nu} s^A) \otimes e_A. \quad (3.47)$$

If $\theta \in \text{End}(\Delta)$ and $u \in \Delta$, in (complex) components we of course have $\theta u = \theta^A_B u^B e_A$, or $(\theta u)^A = \theta^A_B u^B$. By the comment following Equation (3.44) any endomorphism of $\Sigma \otimes_{\mathbb{C}} \Delta$ is a $\mathbb{C}$ -linear combination of elements of the form $\Gamma_{\mu_1 \dots \mu_p} \otimes \theta$ for some $\theta \in \text{End}(\Delta)$, and its action is

$$(\Gamma_{\mu_1 \dots \mu_p} \otimes \theta) s = (\Gamma_{\mu_1 \dots \mu_p} s^B) \otimes (\theta e_B) = \theta^A_B (\Gamma_{\mu_1 \dots \mu_p} s^B) \otimes e_A. \quad (3.48)$$

In all of these calculations, the basis of Δ always ends up simply being a spectator. We can therefore write actions on $\Sigma \otimes_{\mathbb{C}} \Delta$ in their “components” in Σ , for example, Equations (3.46), (3.47) and (3.48) become

$$(\Phi s)^A = \Phi s^A = \frac{1}{p!} \Phi^{\mu_1 \dots \mu_p} \Gamma_{\mu_1 \mu_2 \dots \mu_p} s^A, \quad (3.49)$$

$$(A \cdot s)^A = A \cdot s^A = \frac{1}{4} A^{\mu\nu} \Gamma_{\mu\nu} s^A, \quad (3.50)$$

$$[(\Gamma_{\mu_1 \dots \mu_p} \otimes \theta) s]^A = \theta^A_B \Gamma_{\mu_1 \dots \mu_p} s^B. \quad (3.51)$$

Now, if $s \in S$, all of the above holds, but (perhaps with an adjustment of Δ -basis) the components θ^A_B can be assumed to be real.

3.2.5 Additional structure on the spinor module

There are a number of additional structures on S which either play directly into the Poincaré superalgebra structure or are otherwise useful for performing calculations with it. We begin by defining some bilinear forms.

An $\mathbb{H}$ -Hermitian inner product on a right $\mathbb{H}$ vector space V is a nondegenerate $\mathbb{R}$ -bilinear map $h : V \times V \rightarrow \mathbb{H}$ such that $h(v_1, v_2 \lambda) = h(v_1, v_2) \lambda$ and $h(v_1, v_2) = \overline{h(v_2, v_1)}$ where $v_1, v_2 \in V$ and $\lambda \in \mathbb{H}$. Together, these imply that $h(v_1 \lambda, v_2) = \lambda h(v_1, v_2)$. Recall that if $u \in \mathbb{H}$ then $u = z + jw$ for some unique $z, w \in \mathbb{C}$. Thus for any $v_1, v_2 \in V$ we can write

$$h(v_1, v_2) = H(v_1, v_2) + jB(v_1, v_2) \quad (3.52)$$

where $H(v_1, v_2), B(v_1, v_2) \in \mathbb{C}$. This defines a pair of maps $H, B : V \times V \rightarrow \mathbb{C}$. It can be shown that H is $\mathbb{C}$ -Hermitian (nondegenerate but not necessarily positive-definite) and B is $\mathbb{C}$ -bilinear and symplectic. We then have the following.

Proposition 3.2.8. *There is an $\mathbb{H}$ -hermitian inner product $h : \Sigma \times \Sigma \rightarrow \mathbb{H}$ on Σ such that*

$$h(\sigma_1, v \cdot \sigma_2) = h(v \cdot \sigma_1, \sigma_2). \quad (3.53)$$

for $\sigma_1, \sigma_2 \in \Sigma$ and $v \in V$. There thus exist a $\mathbb{C}$ -Hermitian form H and a $\mathbb{C}$ -bilinear symplectic form B on Σ with the same symmetry.

Proof. Considering Σ as $\mathbb{H}^2$, we define

$$h(\sigma_1, \sigma_2) = \sigma_1^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma_2. \quad (3.54)$$

It can be directly checked that this $\mathbb{H}$ -Hermitian. The symmetry under the Clifford action can be checked by explicitly using the map $V \cong \mathbb{R} \oplus \mathbb{H} \hookrightarrow \mathcal{CL}(V) \cong \mathbb{H}(2)\mathbb{H}(2) \rightarrow \mathbb{H}(2)$ from the proof of Proposition 3.2.1 to obtain a matrix representation for $v \in V$. The forms H and B have the same symmetry by linear independence of 1 and j in $\mathbb{H}$. $\square$

Definition 3.2.9. On $\Sigma \otimes_{\mathbb{C}} \Delta$, we define the following three Dirac bilinears. For two spinors $s_1, s_2 \in \Sigma \otimes_{\mathbb{C}} \Delta$, we have

1. The Dirac scalar, a $\mathbb{C}$ -bilinear symmetric form, is given by

$$\mu(s_1, s_2) = (B(\cdot, \cdot) \otimes \epsilon(\cdot, \cdot))(s_1, s_2) = \epsilon_{AB} B(s_1^A, s_2^B). \quad (3.55)$$

2. The Dirac 1-form, $\kappa^\flat : \odot^2(\Sigma \otimes_{\mathbb{C}} \Delta) \rightarrow (V \otimes \mathbb{C})^*$, is given by

$$\kappa^\flat(s_1, s_2)(X) = \mu(s_1, X \cdot s_2), \quad (3.56)$$

for all $X \in V \otimes \mathbb{C}$.

3. The Dirac current $\kappa : \odot^2(\Sigma \otimes_{\mathbb{C}} \Delta) \rightarrow V \otimes \mathbb{C}$ is the vector associated to the Dirac one-form under the natural isomorphism $V \otimes \mathbb{C} \cong (V \otimes \mathbb{C})^*$; that is, κ is given by

$$\eta(\kappa(s_1, s_2), X) = \mu(s_1, X \cdot s_2) \quad (3.57)$$

for all $X \in V \otimes \mathbb{C}$.

4. The Dirac 2-form with values in $\otimes^2 \Delta$, $\omega : \otimes^2(\Sigma \otimes_{\mathbb{C}} \Delta) \rightarrow (\wedge^2 V \otimes \mathbb{C})^* \otimes_{\mathbb{C}} (\otimes^2 \Delta)$, is given by

$$\omega(s_1, s_2)^{AB}(X, Y) = B(s_1^A, (X \wedge Y) \cdot s_2^B), \quad (3.58)$$

for all $X, Y \in V \otimes \mathbb{C}$.

We will not show it here, but on S , the Dirac norm and current are real:

$$\mu : \odot^2 S \rightarrow \mathbb{R}, \quad \kappa : \odot^2 S \rightarrow V. \quad (3.59)$$

There is also a reality condition for ω when restricted to S , but it is more complicated to state and we will not use it.

For notational convenience, we introduce the notation

$$B(\sigma_1, \sigma_2) = \bar{\sigma}_1 \sigma_2 \quad \text{so} \quad \bar{\sigma}_1 = B(\sigma_1, \cdot). \quad (3.60)$$

Then in components,

$$\mu(s, s) = \epsilon_{AB} \bar{s}^A s^B, \quad (3.61)$$

$$\kappa_\mu(s, s) = \epsilon_{AB} \bar{s}^A \Gamma_\mu s^B, \quad (3.62)$$

$$\omega_{\mu\nu}^{AB}(s, s) = \bar{s}^A \Gamma_{\mu\nu} s^B. \quad (3.63)$$

The next lemma demonstrates the basic symmetry properties of these bilinears.

Lemma 3.2.10. *For $s_1, s_2 \in \Sigma \otimes_{\mathbb{C}} \Delta$, we have the following:*

$$\bar{s}_1^A s_2^B = -\bar{s}_2^B s_1^A, \quad \bar{s}_1^A \Gamma_\mu s_2^B = -\bar{s}_2^B \Gamma_\mu s_1^A, \quad \bar{s}_1^A \Gamma_{\mu\nu} s_2^B = \bar{s}_2^B \Gamma_{\mu\nu} s_1^A. \quad (3.64)$$

Consequently, $\omega_{\mu\nu}^{AB}(s_1, s_2) = \omega_{\mu\nu}^{BA}(s_2, s_1)$; in particular, $\omega(s, s)$ is a $\odot^2 \Delta$ -valued 2-form. Furthermore, for $s \in \Sigma \otimes_{\mathbb{C}} \Delta$,

$$\bar{s}^A s^B = \frac{1}{2} \epsilon^{AB} \mu(s, s), \quad \bar{s}^A \Gamma_\mu s^B = \frac{1}{2} \epsilon^{AB} \kappa_\mu(s, s). \quad (3.65)$$

Proof. All of these follow from symmetries of B . We have

$$B(s_1^A, s_2^B) = -B(s_2^B, s_1^A), \quad (3.66)$$

$$B(s_1^A, \Gamma_\mu s_2^B) = B(\Gamma_\mu s_1^A, s_2^B) = -B(s_2^B, \Gamma_\mu s_1^A), \quad (3.67)$$

$$B(s_1^A, \Gamma_{[\mu} \Gamma_{\nu]} s_2^B) = B(\Gamma_{[\mu} s_1^A, \Gamma_{\nu]} s_2^B) = B(\Gamma_{[\nu} \Gamma_{\mu]} s_1^A, s_2^B) = B(s_2^B, \Gamma_{[\mu} \Gamma_{\nu]} s_1^A), \quad (3.68)$$

hence Equations (3.64). For Equations (3.65), fixing $s \in \Sigma \otimes_{\mathbb{C}} \Delta$, $\bar{s}^A s^B = -\bar{s}^B s^A$, so we must have $\bar{s}^A s^B = \lambda \epsilon^{AB}$ for some $\lambda \in \mathbb{C}$. Contracting with ϵ_{AB} gives $\mu(s, s) = 2\lambda$. The same argument follows for the other equation. $\square$

The next proposition is important for the structure of the Poincaré superalgebra; it ensures that the Jacobi identity is satisfied.

Proposition 3.2.11. *The Dirac current is $\mathfrak{so}(V)$ -equivariant; that is, for $A \in \mathfrak{so}(V)$ and $s \in S$,*

$$A\kappa(s, s) = 2\kappa(s, A \cdot s). \quad (3.69)$$

Proof. In components,

$$\begin{aligned}
2\kappa^\mu(s, A \cdot s) &= \frac{2}{4} A^{\alpha\beta} \epsilon_{AB} \bar{s}^A \Gamma^\mu \Gamma_{\alpha\beta} s^B \\
&= \frac{1}{2} A^{\alpha\beta} \epsilon_{AB} \bar{s}^A \left(\Gamma_{\alpha\beta}^\mu + 2\delta_{[\alpha}^\mu \Gamma_{\beta]} \right) s^B \\
&= A^{\mu\beta} \kappa(s, s)_\beta,
\end{aligned} \tag{3.70}$$

hence the claim. $\square$

The next couple of results are very useful tools for performing calculations with the spinor module.

Proposition 3.2.12 (Fierz identity). *For $s_1, s_2 \in \Sigma \otimes \Delta$, the endomorphism $s_1^A \bar{s}_2^B$ defined by $s_3^C \rightarrow (\bar{s}_2^B s_3^C) s_1^A$ can be written*

$$\begin{aligned}
s_1^A \bar{s}_2^B &= -\frac{1}{4} \left[(\bar{s}_1^A s_2^B) \mathbb{1} + (\bar{s}_1^A \Gamma_\mu s_2^B) \Gamma^\mu + \frac{1}{2} (\bar{s}_1^A \Gamma_{\mu\nu} s_2^B) \Gamma^{\mu\nu} \right] \\
&= -\frac{1}{8} \left[\epsilon^{AB} (\mu(s_1, s_2) \mathbb{1} + \kappa_\mu(s_1, s_2) \Gamma^\mu) + \omega_{\mu\nu}^{AB}(s_1, s_2) \Gamma^{\mu\nu} \right].
\end{aligned} \tag{3.71}$$

The Fierz identity is actually far more general than this and is extremely useful for manipulating expressions involving spinors by rearranging the factors in products, but this is the only version of it we'll require. See [5] for the more general case. When we deploy it later on, we'll refer to its use as *Fierz rearrangement*.

Proof of Proposition 3.2.12. Since $s_1^A \bar{s}_2^B \in \text{End}_{\mathbb{C}}(\Sigma \otimes_{\mathbb{C}} \Delta)$, we can write

$$s_1^A \bar{s}_2^B = \alpha^{AB} \mathbb{1} + \beta_\mu^{AB} \Gamma^\mu + \frac{1}{2} \gamma_{\mu\nu}^{AB} \Gamma^{\mu\nu} \tag{3.72}$$

for some $\alpha^{AB}, \beta_\mu^{AB}, \gamma_{\mu\nu}^{AB} \in \mathbb{C}$. Using Lemmas 3.2.2 and 3.2.5, we take the following traces over $\mathbb{C}$ of endomorphisms of $\Sigma \otimes_{\mathbb{C}} \Delta$:

$$\text{Tr}(s_1^A \bar{s}_2^B) = 4\alpha^{AB}, \tag{3.73}$$

$$\text{Tr}(\Gamma_\sigma s_1^A \bar{s}_2^B) = 4\beta_\sigma^{AB}, \tag{3.74}$$

$$\text{Tr}(\Gamma_{\sigma\tau} s_1^A \bar{s}_2^B) = -4\gamma_{\sigma\tau}^{AB}. \tag{3.75}$$

On the other hand,

$$\text{Tr}(s_1^A \bar{s}_2^B) = \bar{s}_2^B s_1^A = -\bar{s}_1^A s_2^B = -\frac{1}{2} \epsilon^{AB} \mu(s_1, s_2), \tag{3.76}$$

$$\text{Tr}(\Gamma_\sigma s_1^A \bar{s}_2^B) = \bar{s}_2^B \Gamma_\sigma s_1^A = -\bar{s}_1^A \Gamma_\sigma s_2^B = -\frac{1}{2} \epsilon^{AB} \kappa_\sigma(s_1, s_2), \tag{3.77}$$

$$\text{Tr}(\Gamma_{\sigma\tau} s_1^A \bar{s}_2^B) = \bar{s}_2^B \Gamma_{\sigma\tau} s_1^A = \bar{s}_1^A \Gamma_{\sigma\tau} s_2^B = \omega_{\sigma\tau}^{AB}(s_1, s_2). \tag{3.78}$$

The first equality in each of these expressions is seen to be true by writing in components with respect to any basis, then we use Lemma 3.2.10. Hence the result. $\square$

Lemma 3.2.13. *The following identities hold for a spinor $s \in S$:*

$$\kappa(s, s) \cdot s^A = \mu(s, s) s^A, \quad (3.79)$$

$$\omega^{AB}(s, s) \cdot s^C = 2\mu(s, s) \epsilon^{C(A} s^{B)}, \quad (3.80)$$

$$\omega^{(AB}(s, s) \cdot s^{C)} = 0. \quad (3.81)$$

Proof. Let us fix $s \in S$ and then write μ , κ and ω for its Dirac bilinears, suppressing their arguments (s, s) . We can write

$$\mu s^A = s^A \mu = \epsilon_{BC} s^A \bar{s}^B s^C, \quad (3.82)$$

$$\kappa \cdot s^A = \Gamma^\mu s^A \kappa_\mu = \epsilon_{BC} \Gamma^\mu s^A \bar{s}^B \Gamma_\mu s^C, \quad (3.83)$$

$$\omega^{AB} \cdot s^C = \frac{1}{2} \Gamma^{\mu\nu} s^C \omega_{\mu\nu}^{AB} = \frac{1}{2} \Gamma^{\mu\nu} s^C \bar{s}^A \Gamma_{\mu\nu} s^B. \quad (3.84)$$

Then, using the Fierz identity on each of these, evaluating some products of Γ endomorphisms and collecting like terms, we eventually obtain

$$7\mu s^A = \kappa \cdot s^A + 2\omega^A_B \cdot s^B, \quad (3.85)$$

$$11\kappa \cdot s^A = 5\mu s^A + 2\omega^A_B \cdot s^B, \quad (3.86)$$

$$4\omega^{AB} \cdot s^C + 2\omega^{CA} \cdot s^B = \epsilon^{CA} (5\mu s^B + \kappa \cdot s^B). \quad (3.87)$$

Now Equation (3.79) follows from Equations (3.85) and (3.86), while Equation (3.81) follows from Equation (3.87) by completely symmetrising. Substituting Equation (3.79) into Equation (3.87) gives us

$$2\omega^{AB} \cdot s^C + \omega^{CA} \cdot s^B = 3\mu \epsilon^{CA} s^B. \quad (3.88)$$

Note that since $\omega^{AB} = \omega^{BA}$, Equation (3.81) is equivalent to

$$\omega^{AB} \cdot s^C + \omega^{CA} \cdot s^B + \omega^{BC} \cdot s^A = 0. \quad (3.89)$$

Using this to substitute for the second term in Equation (3.88) yields

$$\omega^{AB} \cdot s^C - \omega^{BC} \cdot s^A = 3\mu \epsilon^{CA} s^B. \quad (3.90)$$

Finally, symmetrising Equations (3.88) and (3.90) on AB and adding them together yields Equation (3.80). $\square$

Chapter 4

Generalised Spencer cohomology of the Poincaré superalgebra

Throughout this chapter, all spaces are considered with $\mathbb{R}$ as the ground field unless otherwise stated.

The first two sections of this chapter hold in principle in any spacetime dimension, however we are interested in particular in $D = 5$, which the remainder of the dissertation focusses on. Here we assume the existence, in any spacetime dimension, of a faithful, irreducible spinor representation S of the Lorentz algebra $\mathfrak{so}(V)$ and an $\mathfrak{so}(V)$ -equivariant symmetric bilinear map (the Dirac current) $\kappa : S \times S \rightarrow V$. We have proven the existence of these structures in $D = 5$ in the previous chapter.

4.1 The Poincaré superalgebra

The Poincaré superalgebra $\mathfrak{s}$ is an extension of the Poincaré algebra $\mathfrak{p}$ by generators of supersymmetry in Minkowski space. It is a Lie superalgebra, the even subspace of which is simply the Poincaré algebra $\mathfrak{s}_0 = \mathfrak{p}$, while the odd subspace $\mathfrak{s}_1$ consists of the supersymmetry generators. We will not deal with extended supersymmetry here, so for the purposes of this dissertation $\mathfrak{s}_1 = S$, where S is the chosen spinor representation of the Lorentz algebra $\mathfrak{so}(V)$. The bracket operation on $\mathfrak{s}_0$ is simply the Lie bracket of the Poincaré algebra; on $\mathfrak{s}_1$ it is the Dirac current, and the remaining brackets are zero.

In summary, the Poincaré superalgebra has the underlying $\mathbb{Z}$ -graded vector space

$$\mathfrak{s} = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0, \quad (4.1)$$

where

$$\mathfrak{s}_{-2} = V, \quad \mathfrak{s}_{-1} = S, \quad \mathfrak{s}_0 = \mathfrak{so}(V). \quad (4.2)$$

It has a $\mathbb{Z}_2$ -grading given by its $\mathbb{Z}$ -grading modulo 2, and its superbracket is given by

$$[A, B] = AB - BA, \quad [s, s] = \kappa(s, s), \quad (4.3)$$

$$[A, v] = -[v, A] = Av, \quad [A, s] = -[s, A] = A \cdot s, \quad (4.4)$$

$$[v, w] = 0, \quad [v, s] = -[s, v] = 0, \quad (4.5)$$

Table 4.1: Low-degree cochains for the Poincaré superalgebra $\mathfrak{s}$

	$p = 0$	$p = 1$	$p = 2$	$p = 3$
$d = -2$	V			
$d = -1$	S	$\text{Hom}(S, V)$		
$d = 0$	$\mathfrak{so}(V)$	$\text{Hom}(V, V) \oplus \text{Hom}(S, S)$	$\text{Hom}(\odot^2 S, V)$	
$d = 1$		$\text{Hom}(V, S) \oplus \text{Hom}(S, \mathfrak{so}(V))$	$\text{Hom}(V \otimes S, V) \oplus \text{Hom}(\odot^2 S, S)$	$\text{Hom}(\odot^3 S, V)$
$d = 2$		$\text{Hom}(V, \mathfrak{so}(V))$	$\text{Hom}(\wedge^2 V, V) \oplus \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\odot^2 S, \mathfrak{so}(V))$	$\text{Hom}(V \otimes \odot^2 S, V) \oplus \text{Hom}(\odot^3 S, S)$
$d = 3$			$\text{Hom}(\wedge^2 V, S) \oplus \text{Hom}(V \otimes S, \mathfrak{so}(V))$	$\text{Hom}(\wedge^2 V \otimes S, V) \oplus \text{Hom}(V \otimes \odot^2 S, S) \oplus \text{Hom}(\odot^3 S, \mathfrak{so}(V))$
$d = 4$			$\text{Hom}(\wedge^2 V, \mathfrak{so}(V))$	$\text{Hom}(\wedge^3 V, V) \oplus \text{Hom}(\wedge^2 V \otimes S, S) \oplus \text{Hom}(V \otimes \odot^2 S, \mathfrak{so}(V))$
$d = 5$				$\text{Hom}(\wedge^3 V, S) \oplus \text{Hom}(\wedge^2 V \otimes S, \mathfrak{so}(V))$
$d = 6$				$\text{Hom}(\wedge^3 V, \mathfrak{so}(V))$

where $A, B \in \mathfrak{so}(V)$, $v, w \in V$, $s \in S$ and κ is the Dirac current (note that since $[\cdot, \cdot] = \kappa$ is symmetric on S , it is determined by its values on the diagonal $[s, s]$). With these definitions, $\mathfrak{s}$ is a $\mathbb{Z}$ -graded Lie superalgebra. Indeed, it can be seen from the definition that the bracket preserves both the $\mathbb{Z}$ -grading and the $\mathbb{Z}_2$ -grading and that it is super skew-symmetric, while its Jacobi identity can be divided into a number of cases. We use the shorthand $[ijk]$ for the Jacobi identity with one homogeneous element with degree $-i$, one with degree $-j$ and one with degree $-k$.

- $[000]$ follows from the Jacobi identity on $\mathfrak{so}(V)$,
- $[002]$ follows because the action of $\mathfrak{so}(V)$ on V is a Lie algebra representation,
- $[001]$ follows because the action of $\mathfrak{so}(V)$ on S is a Lie algebra representation,
- $[011]$ follows from $\mathfrak{so}(V)$ -equivariance of the Dirac current (Proposition 3.2.11),
- $[012]$, $[022]$, $[111]$, $[112]$, $[022]$ and $[222]$ are trivial.

The subspace $\mathfrak{s}_- = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} = V \oplus S$ is called the *supertranslation ideal*; it is a nilpotent ideal subalgebra.

4.2 The Spencer complex and cohomologies in general dimension

The Spencer cochain spaces are given by

$$C^{d,p}(\mathfrak{s}_-, \mathfrak{s}) = \text{Hom}(\wedge^p \mathfrak{s}_-, \mathfrak{s})_d. \quad (4.6)$$

Table 4.2 give the cochain spaces of low grading degree (d) and complex degree (p). The cochain degree runs left to right in the table, while the $\mathbb{Z}$ -grading degree inherited from $\mathfrak{s}$ runs top to bottom so that each row in the table is a subcomplex of a particular degree. The complexes for $d \geq 2$ are incomplete since the table stops at $p = 3$, but we will

not need the missing cochain spaces. Blank spaces in the table and cochain spaces with $d < -2$ or $p < 0$ are zero.

We will now examine the cochain complex more closely and find some of the cohomology groups, recalling that the Spencer complex splits into subcomplexes indexed by grading degree d and working degree by degree. Since the filtered deformations of $\mathfrak{s}$ are governed by the even-degree Spencer cohomologies with $p = 2$, we will focus on these and only go up to the $d = 4$ cochain complex.

The $d = -2$ complex

$$0 \rightarrow V \rightarrow 0 \quad (4.7)$$

Immediately we can see that

$$H^{-2,p}(\mathfrak{s}_-, \mathfrak{s}) = Z^{-2,p}(\mathfrak{s}_-, \mathfrak{s}) = \begin{cases} V & \text{for } p = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4.8)$$

The $d = -1$ complex

$$0 \rightarrow S \rightarrow \text{Hom}(S, V) \rightarrow 0 \quad (4.9)$$

The differential $\partial^{-1,0} : S \rightarrow \text{Hom}(S, V)$ is given by $(\partial s_1)(s_2) = [s_1, s_2] = \kappa(s_1, s_2)$ for $s_1, s_2 \in S$, so the $(-1, 0)$ -cocycles are spinors s_1 such that $\kappa(s_1, s_2) = 0$ for all $s_2 \in S$. The Dirac current is nondegenerate so the cocycles and thus the cohomology vanish and the differential is injective. Moving one place to the right, we see that

$$Z^{-1,1}(\mathfrak{s}_-, \mathfrak{s}) = C^{-1,1}(\mathfrak{s}_-, \mathfrak{s}) = \text{Hom}(S, V), \quad (4.10)$$

while

$$B^{-1,1}(\mathfrak{s}_-, \mathfrak{s}) = \{\phi \in \text{Hom}(S, V) \mid \exists s_0 \in S : \phi(s) = \kappa(s_0, s)\}, \quad (4.11)$$

thus

$$H^{-1,p}(\mathfrak{s}_-, \mathfrak{s}) = \begin{cases} \text{Hom}(S, V) / \{\kappa(s_0, \cdot) \mid s_0 \in S\} & \text{for } p = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (4.12)$$

The $d = 0$ complex

$$0 \rightarrow \mathfrak{so}(V) \rightarrow \text{Hom}(S, S) \oplus \text{Hom}(V, V) \rightarrow \text{Hom}(\odot^2 S, V) \rightarrow 0 \quad (4.13)$$

The first nonzero differential $\partial^{0,0}$ has two components. We can define projections of the $(0, 1)$ -cochain spaces

$$\pi_1^{0,1} : \text{Hom}(V, V) \oplus \text{Hom}(S, S) \rightarrow \text{Hom}(V, V), \quad (4.14)$$

$$\pi_2^{0,1} : \text{Hom}(V, V) \oplus \text{Hom}(S, S) \rightarrow \text{Hom}(S, S), \quad (4.15)$$

and then let $\partial_1^{0,0} = \pi_1^{0,0} \circ \partial^{0,0}$ and $\partial_2^{0,0} = \pi_2^{0,0} \circ \partial^{0,0}$ be the corresponding components of $\partial^{0,0}$. They are given by

$$(\partial_1^{0,0} A)(s) = -[s, A] = A \cdot s = \frac{1}{4} A_{\mu\nu} \Gamma^{\mu\nu} s \quad (4.16)$$

$$(\partial_2^{0,0} A)(v) = -[v, A] = Av \quad (4.17)$$

for $A \in \mathfrak{so}(V)$, $s \in S$ and $v \in V$. Since the actions of $\mathfrak{so}(V)$ on S and V are faithful, both components are injective, so $Z^{0,0}(\mathfrak{s}_-, \mathfrak{s}) = 0$. The next differential is

$$\partial^{0,1} : \text{Hom}(S, S) \oplus \text{Hom}(V, V) \rightarrow \text{Hom}(\odot^2 S, V). \quad (4.18)$$

Let $\alpha : V \rightarrow V$, $\beta : S \rightarrow S$ be endomorphisms, so that $\alpha + \beta$ is a typical $(0, 1)$ -cochain. Then (noting that we only need to check values on the diagonal by symmetry)

$$\partial(\alpha + \beta)(s, s) = \alpha([s, s]) - 2[s, \beta(s)] = \alpha(\kappa(s, s)) - 2\kappa(s, \beta(s)) \quad (4.19)$$

so the cocycle condition is

$$\alpha(\kappa(s, s)) = 2\kappa(s, \beta(s)). \quad (4.20)$$

The $d = 1$ complex

$$\begin{aligned} 0 \rightarrow \text{Hom}(V, S) \oplus \text{Hom}(S, \mathfrak{so}(V)) &\rightarrow \text{Hom}(V \otimes S, V) \oplus \text{Hom}(\odot^2 S, S) \\ &\rightarrow \text{Hom}(\odot^3 S, V) \rightarrow 0 \end{aligned} \quad (4.21)$$

To the differential $\partial^{1,2}$, let

$$\pi_1^{1,2} : \text{Hom}(V \otimes S, V) \oplus \text{Hom}(\odot^2 S, S) \rightarrow \text{Hom}(V \otimes S, V), \quad (4.22)$$

$$\pi_2^{1,2} : \text{Hom}(V \otimes S, V) \oplus \text{Hom}(\odot^2 S, S) \rightarrow \text{Hom}(\odot^2 S, S) \quad (4.23)$$

be the projections of the codomain to the two components and let $\partial_1^{1,1} = \pi_1^{1,2} \circ \partial^{1,1}$ and $\partial_2^{1,1} = \pi_2^{1,2} \circ \partial^{1,1}$ be the corresponding components of the differential. Taking a typical $(1, 1)$ -cochain $\alpha + \beta \in \text{Hom}(V, S) \oplus \text{Hom}(S, \mathfrak{so}(V))$, we have

$$\partial_1^{1,1}(\alpha + \beta)(v, s) = [s, \alpha(v)] - [v, \beta(s)] = \kappa(s, \alpha(v)) + \beta(s)v, \quad (4.24)$$

and for the other component,

$$\partial_2^{1,1}(\alpha + \beta)(s, s) = \alpha([s, s]) - 2[s, \beta(s)] = \alpha(\kappa(s, s)) + 2\beta(s) \cdot s. \quad (4.25)$$

The cocycle conditions are then

$$\kappa(s, \alpha v) + \beta(s)v = 0, \quad (4.26)$$

$$\alpha(\kappa(s, s)) + 2\beta(s) \cdot s = 0. \quad (4.27)$$

It's possible to show directly, using the same methods we use to compute the $(2, 2)$ -cohomology in Section 4.3, (at least in 5 dimensions)¹ that the only solution to these equations is $\alpha = 0$, $\beta = 0$ so

$$H^{1,1}(\mathfrak{s}_-, \mathfrak{s}) = 0. \quad (4.28)$$

¹There is a higher-level argument, which holds in any dimension, given in the proof of Theorem 9 in [1] that $H^{d,1}(\mathfrak{s}_-, \mathfrak{s}) = 0$ for $d \geq 1$.

Turning to the $p = 2$ differential, there is only one component. Let $\gamma + \delta \in \text{Hom}(V \otimes S, V) \oplus \text{Hom}(\odot^2 S, S)$ be a typical cocycle. Then for $s \in S$,

$$\partial(\gamma + \delta) = 3\gamma([s, s], s) - 3[s, \delta(s, s)] = \gamma(\kappa(s, s), s) - 3(\kappa(s, \delta(s))), \quad (4.29)$$

so the cocycle condition is

$$\gamma(\kappa(s, s), s) = \kappa(s, \delta(s, s)). \quad (4.30)$$

The $d = 2$ complex

$$\begin{aligned} 0 \rightarrow \text{Hom}(V, \mathfrak{so}(V)) &\rightarrow \text{Hom}(\bigwedge^2 V, V) \oplus \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\odot^2 S, \mathfrak{so}(V)), \\ &\rightarrow \text{Hom}(V \otimes \odot^2 S, V) \oplus \text{Hom}(\odot^3 S, S) \\ &\rightarrow \text{Hom}(\odot^4 S, V) \rightarrow 0 \end{aligned} \quad (4.31)$$

Note that the last cochain space is not on the table. We define projections

$$\pi_1^{2,2} : \text{Hom}(\bigwedge^2 V, V) \oplus \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\odot^2 S, \mathfrak{so}(V)) \rightarrow \text{Hom}(\bigwedge^2 V, V), \quad (4.32)$$

$$\pi_2^{2,2} : \text{Hom}(\bigwedge^2 V, V) \oplus \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\odot^2 S, \mathfrak{so}(V)) \rightarrow \text{Hom}(V \otimes S, S), \quad (4.33)$$

$$\pi_3^{2,2} : \text{Hom}(\bigwedge^2 V, V) \oplus \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\odot^2 S, \mathfrak{so}(V)) \rightarrow \text{Hom}(\odot^2 S, \mathfrak{so}(V)), \quad (4.34)$$

and then let $\partial_i^{2,1} = \pi_i^{2,2} \circ \partial^{2,1}$ be the corresponding components of the $p = 1$ differential. The first component $\partial_1^{2,1}$ is the $\partial^{2,1}$ differential of the Poincaré algebra, which is an isomorphism (see the complex in Equation (2.47), which we showed is exact). If a $(2, 1)$ -cochain $\phi : V \rightarrow \mathfrak{so}(V)$ is coclosed, all three components of $\partial^{2,1}\phi$ must vanish separately; thus by the vanishing of the first component, the $(2, 1)$ -cocycles are zero and so

$$H^{2,1}(\mathfrak{s}_-, \mathfrak{s}) = 0. \quad (4.35)$$

Let the projections of the $p = 3$ cochain spaces be

$$\pi_1^{2,3} : \text{Hom}(V \otimes \odot^2 S, V) \oplus \text{Hom}(\odot^3 S, S) \rightarrow \text{Hom}(V \otimes \odot^2 S, V), \quad (4.36)$$

$$\pi_2^{2,3} : \text{Hom}(V \otimes \odot^2 S, V) \oplus \text{Hom}(\odot^3 S, S) \rightarrow \text{Hom}(\odot^3 S, S), \quad (4.37)$$

and denote the components of the differential by $\partial_i^{2,2} = \pi_i^{2,3} \circ \partial^{2,2}$. Let $\alpha + \beta + \gamma \in$ be typical $(2, 2)$ -cochain. Then, for $v \in V, s \in S$ (only checking values on the diagonal for $\odot^n S$ arguments),

$$\begin{aligned} (\partial_1^{2,2}(\alpha + \beta + \gamma))(v, s, s) &= \alpha([s, s], v) + 2[s, \beta(v, s)] - [v, \gamma(s, s)] \\ &= \alpha(\kappa(s, s), v) + 2\kappa(s, \beta(v, s)) + \gamma(s, s)v, \end{aligned} \quad (4.38)$$

$$\begin{aligned} (\partial_1^{2,2}(\alpha + \beta + \gamma))(s, s, s) &= 3\beta([s, s], s) - 3[s, \gamma(s, s)] \\ &= 3\beta(\kappa(s, s), s) + 3\gamma(s, s) \cdot s. \end{aligned} \quad (4.39)$$

The cocycle conditions are thus

$$\alpha(\kappa(s, s), v) + 2\kappa(s, \beta(v, s)) + \gamma(s, s)v = 0, \quad (4.40)$$

$$\beta(\kappa(s, s), s) + \gamma(s, s) \cdot s = 0. \quad (4.41)$$

The next section will be concerned with finding the $(2, 2)$ -cohomology in $D = 5$; it is useful now to show that this calculation can be simplified by taking $\alpha = 0$ without loss of generality.

Proposition 4.2.1. *Every cohomology class in $H^{2,2}(\mathfrak{s}_-, \mathfrak{s})$ has a unique representative cocycle with zero $\text{Hom}(\bigwedge^2 V, V)$ component, and we have*

$$H^{2,2}(\mathfrak{s}_-, \mathfrak{s}) \cong \left\{ \beta + \gamma \in \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\odot^2 S, \mathfrak{so}(V)) \left| \begin{array}{l} 2\kappa(s, \beta(v, s)) + \gamma(s, s)v = 0, \\ \beta(\kappa(s, s), s) + \gamma(s, s) \cdot s = 0 \end{array} \right. \right\}; \quad (4.42)$$

that is, the cohomology is simply the space of cocycles with $\alpha = 0$.

Proof. Let $\alpha + \beta + \gamma$ be a cocycle, where we use the same notation as above, which of course has cohomology class $[\alpha + \beta + \gamma]$. Since $\partial_1^{2,1} : \text{Hom}(V, \mathfrak{so}(V)) \rightarrow \text{Hom}(\bigwedge^2 V, V)$ is an isomorphism, there is a unique $(2, 1)$ -cocycle $\phi \in \text{Hom}(V, \mathfrak{so}(V))$ such that $\partial_1^{2,1}\phi = \alpha$. Since adding a coboundary does not change the cohomology class, we then have

$$\begin{aligned} [\alpha + \beta + \gamma] &= [\alpha + \beta + \gamma - \partial\phi] \\ &= [\alpha + \beta + \gamma - \alpha - \partial_2^{2,1}\phi - \partial_3^{2,1}\phi] \\ &= [(\beta - \partial_2^{2,1}\phi) + (\gamma - \partial_3^{2,1}\phi)] \\ &= [\tilde{\beta} + \tilde{\gamma}], \end{aligned} \quad (4.43)$$

where $\tilde{\beta} = (\beta - \partial_2^{2,1}\phi)$ and $\tilde{\gamma} = (\gamma - \partial_3^{2,1}\phi)$, and $\beta + \gamma$ is a cocycle. Since ϕ is unique, $\tilde{\beta} + \tilde{\gamma}$ is the unique representative of $[\alpha + \beta + \gamma]$ with first component zero, hence the first claim. The isomorphism in the second claim is given by $[\alpha + \beta + \gamma] \mapsto \tilde{\beta} + \tilde{\gamma}$. $\square$

Finally, for a typical $(2, 3)$ -cochain $\phi + \psi \in \text{Hom}(V \otimes \odot^2 S, V) \oplus \text{Hom}(\odot^3 S, S)$ and for $s \in S$,

$$\begin{aligned} (\partial^{2,3}(\phi + \psi))(s, s, s, s) &= 3\phi([s, s], s, s) - 4[s, \psi(s, s, s)] \\ &= 3\phi(\kappa(s, s), s, s) - 4\kappa(s, \psi(s, s, s)), \end{aligned} \quad (4.44)$$

so the cocycle condition is

$$3\phi(\kappa(s, s), s, s) = 4\kappa(s, \psi(s, s, s)). \quad (4.45)$$

The $d = 3$ complex

$$0 \rightarrow \text{Hom}(\bigwedge^2 V, S) \oplus \text{Hom}(V \otimes S, \mathfrak{so}(V)) \quad (4.46)$$

$$\rightarrow \text{Hom}(\bigwedge^2 V \otimes S, V) \oplus \text{Hom}(V \otimes \odot^2 S, S) \oplus \text{Hom}(\odot^3 S, \mathfrak{so}(V)) \rightarrow \dots \quad (4.47)$$

We now look only at the cochain complex up to $p = 3$. Let

$$\begin{aligned} \pi_1^{3,3} : \text{Hom}(\bigwedge^2 V \otimes S, V) \oplus \text{Hom}(V \otimes \odot^2 S, S) \oplus \text{Hom}(\odot^3 S, \mathfrak{so}(V)) \\ \rightarrow \text{Hom}(\bigwedge^2 V \otimes S, V), \end{aligned} \quad (4.48)$$

$$\begin{aligned} \pi_2^{3,3} : \text{Hom}(\bigwedge^2 V \otimes S, V) \oplus \text{Hom}(V \otimes \odot^2 S, S) \oplus \text{Hom}(\odot^3 S, \mathfrak{so}(V)) \\ \rightarrow \text{Hom}(V \otimes \odot^2 S, S), \end{aligned} \quad (4.49)$$

$$\begin{aligned} \pi_3^{3,3} : \text{Hom}(\bigwedge^2 V \otimes S, V) \oplus \text{Hom}(V \otimes \odot^2 S, S) \oplus \text{Hom}(\odot^3 S, \mathfrak{so}(V)) \\ \rightarrow \text{Hom}(\odot^3 S, \mathfrak{so}(V)) \end{aligned} \quad (4.50)$$

be the $p = 3$ cochain projections and let $\partial_i^{3,2}$ be the corresponding differentials as usual. For typical cochains $\alpha + \beta \in \text{Hom}(\bigwedge^2 V, S) \oplus \text{Hom}(V \otimes S, \mathfrak{so}(V))$,

$$\begin{aligned} (\partial_1^{3,2}(\alpha + \beta))(v, w, s) &= -[s, \alpha(v, w)] - [v, \beta(w, s)] + [w, \beta(v, s)] \\ &= -\kappa(s, \alpha(v, w)) + \beta(w, s)v - \beta(v, s)w, \end{aligned} \quad (4.51)$$

$$\begin{aligned} (\partial_2^{3,2}(\alpha + \beta))(v, s, s) &= \alpha([s, s], v) + 2[s, \beta(v, s)] \\ &= \alpha(\kappa(s, s), v) - 2\beta(v, s) \cdot s, \end{aligned} \quad (4.52)$$

$$\begin{aligned} (\partial_3^{3,2}(\alpha + \beta))(s, s, s) &= 3\beta([s, s], s) \\ &= 3\beta(\kappa(s, s), s), \end{aligned} \quad (4.53)$$

so the cochain conditions are

$$\kappa(s, \alpha(v, w)) + \beta(v, s)w - \beta(w, s)v = 0, \quad (4.54)$$

$$\alpha(v, \kappa(s, s)) - 2\beta(v, s) \cdot s = 0, \quad (4.55)$$

$$\beta(\kappa(s, s), s) = 0. \quad (4.56)$$

Note that the third condition is redundant; it can be obtained from the second by setting $v = \kappa(s, s)$ and using the fact that the $\mathfrak{so}(V)$ -action on S is faithful. The $p = 2$ cohomology is thus

$$H^{3,2}(\mathfrak{s}_-, \mathfrak{s}) = \{\alpha + \beta \in \text{Hom}(\bigwedge^2 V, S) \oplus \text{Hom}(V \otimes S, \mathfrak{so}(V)) \mid \text{Eqs. (4.54) and (4.55)}\}. \quad (4.57)$$

The $d = 4$ complex

$$0 \rightarrow \text{Hom}(\bigwedge^2 V, \mathfrak{so}(V)) \quad (4.58)$$

$$\rightarrow \text{Hom}(\bigwedge^3 V, V) \oplus \text{Hom}(\bigwedge^2 V \otimes S, S) \oplus \text{Hom}(V \otimes \odot^2 S, \mathfrak{so}(V)) \rightarrow \dots \quad (4.59)$$

Again, we only look at $p \leq 3$ in this complex. Define projections

$$\begin{aligned} \pi_1^{4,3} : \text{Hom}(\bigwedge^3 V, V) \oplus \text{Hom}(\bigwedge^2 V \otimes S, S) \oplus \text{Hom}(V \otimes \odot^2 S, \mathfrak{so}(V)) \\ \rightarrow \text{Hom}(\bigwedge^3 V, V), \end{aligned} \quad (4.60)$$

$$\begin{aligned} \pi_2^{4,3} : \text{Hom}(\bigwedge^3 V, V) \oplus \text{Hom}(\bigwedge^2 V \otimes S, S) \oplus \text{Hom}(V \otimes \odot^2 S, \mathfrak{so}(V)) \\ \rightarrow \text{Hom}(\bigwedge^2 V \otimes S, S), \end{aligned} \quad (4.61)$$

$$\begin{aligned} \pi_3^{4,3} : \text{Hom}(\bigwedge^3 V, V) \oplus \text{Hom}(\bigwedge^2 V \otimes S, S) \oplus \text{Hom}(V \otimes \odot^2 S, \mathfrak{so}(V)) \\ \rightarrow \text{Hom}(V \otimes \odot^2 S, \mathfrak{so}(V)) \end{aligned} \quad (4.62)$$

and components of the $(4, 2)$ -differential $\partial_i^{4,2}$. Then for a $(4, 3)$ -cochain $\phi \in \text{Hom}(\bigwedge^2 V, \mathfrak{so}(V))$,

$$\begin{aligned} (\partial_1^{4,2}\phi)(v_1, v_2, v_3) &= -[v_1, \phi(v_2, v_3)] + [v_2, \phi(v_1, v_3)] - [v_3, \phi(v_1, v_2)] \\ &= \phi(v_1, v_2)v_3 + \phi(v_2, v_3)v_1 + \phi(v_3, v_1)v_2, \end{aligned} \quad (4.63)$$

$$\begin{aligned} (\partial_2^{4,2}\phi)(v_1, v_2, s) &= -[s, \phi(v_1, v_2)] \\ &= \phi(v_1, v_2) \cdot s, \end{aligned} \quad (4.64)$$

$$(\partial_3^{4,2}\phi)(v_1, s, s) = \phi([s, s], v_1) = \phi(\kappa(s, s), v_1). \quad (4.65)$$

Since the action of $\mathfrak{so}(V)$ on S is faithful and κ is surjective on V , the second and third components of the differential are injective, thus the differential $\partial^{4,2}$ is injective and the cocycles are zero. Note that the first component is simply the $(4, 2)$ -differential of the Poincaré algebra, so its kernel is the space of algebraic curvature operators $\mathcal{B}$, as in Equations (2.51) and (2.56). We thus have

$$H^{4,2}(\mathfrak{s}_-, \mathfrak{s}) = 0. \quad (4.66)$$

4.3 The $(2, 2)$ -cohomology group in $D = 5$

In the previous section, we saw that the only even-degree 2-cohomology (that is, the cohomology which governs filtered deformations) which does not vanish identically in all dimensions is the degree-2 cohomology. This section makes heavy use of the identities and notation of Section 3.2.

Theorem 4.3.1. *The $(2, 2)$ -cohomology group in 5 dimensions is given by the isomorphism of $\mathfrak{so}(V) \otimes \text{End}(\Delta)$ -modules*

$$H^{2,2}(\mathfrak{s}_-, \mathfrak{s}) \cong (\bigwedge^2 V \otimes \bigwedge^2 \Delta) \oplus (V \otimes \odot^2 \Delta). \quad (4.67)$$

The goal of this section is to prove this result.

4.3.1 Rewriting the cocycle conditions

The cocycle conditions on $C^{2,2}(\mathfrak{s}_-, \mathfrak{s})$ are

$$[\gamma(s, s), v] + 2[s, \beta(v, s)] = 0, \quad (4.68)$$

$$[\gamma(s, s), s] + \beta([s, s], s) = 0, \quad (4.69)$$

for all $v \in V$ and $s \in S$. To begin solving these, we show that they can be rewritten as one equation simply expressing γ in terms of β , and two conditions on β only.

Lemma 4.3.2. *In the orthonormal basis for V , the cocycle conditions are equivalent to the following:*

$$\gamma_{\mu\nu}(s, s) = -2\bar{s}^A \Gamma_\mu \beta_{\nu AB} s^B, \quad (4.70)$$

$$\bar{s}^A \Gamma_{(\mu} \beta_{\nu)AB} s^B = 0, \quad (4.71)$$

$$-\frac{1}{2}(\bar{s}^A \Gamma_\mu \beta_{\nu AB} s^B) \Gamma^{\mu\nu} s^C + \kappa^\mu(s, s) \beta_\mu{}^C{}_A s^A = 0. \quad (4.72)$$

Proof. Recall that $\gamma(s, s) \in \mathfrak{so}(V)$ and that the adjoint action of $\mathfrak{s}_0 = \mathfrak{so}(V)$ on $\mathfrak{s}_{-1} = V$ is simply the natural action. Setting $v = e_\nu$ and taking the inner product with e_μ , the first term of Equation (4.68) yields

$$\eta(e_\mu, [\gamma(s, s), e_\nu]) = \eta(e_\mu, \gamma(s, s)e_\nu) = \gamma_{\mu\nu}(s, s). \quad (4.73)$$

To evaluate the second term, we make use of the natural isomorphism

$$\text{Hom}(V \otimes S, S) \cong \text{Hom}(V, \text{End}(S)), \quad (4.74)$$

defined as follows: to $\beta : V \otimes S \rightarrow S$ we associate a map $\bar{\beta} : V \rightarrow \text{End}(S)$ defined by $\bar{\beta}(v)s = \beta(v, s)$ for all $v \in V, s \in S$. To ease notation, we will write β_v for $\bar{\beta}(v)$, and in components $\beta_\mu = \beta_{e_\mu} = \bar{\beta}(e_\mu)$. Now recall that the Lie superbracket restricted to S simply yields the Dirac current, so returning to Equation (4.69), we again set $v = e_\nu$ and take the inner product of the second term with e_μ :

$$\eta(e_\mu, 2[s, \beta(e_\nu, s)]) = 2\eta(e_\mu, \kappa(s, \beta(e_\nu, s))) = 2\kappa_\mu(s, \beta_\nu s) = 2\epsilon_{AB}\bar{s}^A\Gamma_\mu\beta_\nu{}^B{}_C s^C. \quad (4.75)$$

From this, we have Equation (4.70) and since $\gamma(s, s) \in \mathfrak{so}(V)$, its symmetric part vanishes, hence Equation (4.71). Turning now to Equation (4.69) and using the action of $\mathfrak{so}(V)$ on S and Equation (4.70), the first term is

$$[\gamma(s, s), s] = \gamma(s, s)s = \frac{1}{4}\gamma_{\mu\nu}(s, s)\Gamma^{\mu\nu}s = -\frac{1}{2}(\bar{s}^A\Gamma_\mu\beta_{\nu AB}s^B)\Gamma^{\mu\nu}s, \quad (4.76)$$

while the second term is

$$\beta([s, s], s) = \beta(\kappa(s, s), s) = \beta(\kappa^\mu(s, s)e_\mu, s) = \kappa^\mu(s, s)\beta(e_\mu, s) = \kappa^\mu(s, s)\beta_\mu s. \quad (4.77)$$

Making symplectic indices explicit, we thus have Equation (4.72). $\square$

By Proposition 3.2.7 and the discussion following it, $\text{End}(S)$ is given by the real part of $\Gamma(\mathcal{CL}(V)) \otimes \text{End}(\Delta)$ and $\text{End}(\Delta) \cong \bigotimes^2 \Delta$, so we can write

$$\beta_\mu = \beta_\mu^0 \mathbb{1} + \beta_{\mu\nu}^1 \Gamma^\nu + \frac{1}{2}\beta_{\mu\nu\rho}^2 \Gamma^{\nu\rho}, \quad (4.78)$$

where $\beta_\mu^i \in \bigwedge^i V \otimes \bigotimes^2 \Delta$. But we can do better; since $\bigotimes^2 \Delta = \odot^2 \Delta \oplus \bigwedge^2 \Delta$ and $\bigwedge^2 \Delta$ is one-dimensional and spanned by the symplectic form ϵ , β can be further expanded:

$$\beta_{\mu AB} = (A_\mu \epsilon_{AB} + B_{\mu AB})\mathbb{1} + (C_{\mu\alpha} \epsilon_{AB} + D_{\mu\alpha AB})\Gamma^\alpha + \frac{1}{2}(E_{\mu\alpha\beta} \epsilon_{AB} + F_{\mu\alpha\beta AB})\Gamma^{\alpha\beta}, \quad (4.79)$$

where $A \in V, B \in V \otimes \odot^2 \Delta, C \in V \otimes V, D \in V \otimes V \otimes \odot^2 \Delta, E \in V \otimes \bigwedge^2 V, F \in V \otimes \bigwedge^2 V \otimes \odot^2 \Delta$.

4.3.2 Solving the first cocycle condition

We solve the first cocycle condition Equation (4.71) by substituting in Equation (4.79).

Proposition 4.3.3. *The general solution to Equation (4.71) is given by*

$$\beta_{\mu AB} = B_{\mu AB} \mathbf{1} + C_{\mu\alpha} \epsilon_{AB} \Gamma^\alpha + d_{AB} \Gamma_\mu + \frac{1}{2} E_{\mu\alpha\beta} \epsilon_{AB} \Gamma^{\alpha\beta} + \frac{1}{4} \phi^\alpha_{AB} \Gamma_{\mu\alpha}, \quad (4.80)$$

where B and ϕ are $\odot^2 \Delta$ -valued 1-forms, C is a 2-form, $d \in \odot^2 \Delta$ and E is a 3-form.

Proof. Using Lemmas 3.2.2 and 3.2.4 to evaluate Γ products and then Lemma 3.2.10 to deal with the Dirac bilinears, we compute

$$\begin{aligned} & \bar{s}^A \Gamma_\mu \beta_{\nu AB} s^B \\ &= (A_\nu \epsilon_{AB} + B_{\nu AB}) \bar{s}^A \Gamma_\mu s^B + (C_{\nu\alpha} \epsilon_{AB} + D_{\nu\alpha AB}) \bar{s}^A \Gamma_\mu \Gamma^\alpha s^B \\ & \quad + \frac{1}{2} (E_{\nu\alpha\beta} \epsilon_{AB} + F_{\nu\alpha\beta AB}) \bar{s}^A \Gamma_\mu \Gamma^{\alpha\beta} s^B \\ &= (A_\nu \epsilon_{AB} + B_{\nu AB}) \bar{s}^A \Gamma_\mu s^B + (C_{\nu\alpha} \epsilon_{AB} + D_{\nu\alpha AB}) \bar{s}^A (\Gamma_\mu^\alpha + \delta_\mu^\alpha) s^B \\ & \quad + \frac{1}{2} (E_{\nu\alpha\beta} \epsilon_{AB} + F_{\nu\alpha\beta AB}) \bar{s}^A (\Gamma_\mu^{\alpha\beta} + 2\delta_\mu^\alpha \Gamma^\beta) s^B \\ &= (A_\nu \epsilon_{AB} + B_{\nu AB}) \bar{s}^A \Gamma_\mu s^B + (C_{\nu\alpha} \epsilon_{AB} + D_{\nu\alpha AB}) \bar{s}^A (\Gamma_\mu^\alpha + \delta_\mu^\alpha) s^B \\ & \quad + \frac{1}{2} (E_{\nu\alpha\beta} \epsilon_{AB} + F_{\nu\alpha\beta AB}) \bar{s}^A \left(-\frac{1}{2} \epsilon_\mu^{\alpha\beta\sigma\tau} \Gamma_{\sigma\tau} + 2\delta_\mu^\alpha \Gamma^\beta \right) s^B \\ &= (A_\nu \epsilon_{AB} + B_{\nu AB}) \left(\frac{1}{2} \epsilon^{AB} \kappa_\mu(s, s) \right) \\ & \quad + (C_{\nu\alpha} \epsilon_{AB} + D_{\nu\alpha AB}) \left(\omega_\mu^{\alpha AB}(s, s) + \frac{1}{2} \epsilon^{AB} \mu(s, s) \delta_\mu^\alpha \right) \\ & \quad + \frac{1}{2} (E_{\nu\alpha\beta} \epsilon_{AB} + F_{\nu\alpha\beta AB}) \left(-\frac{1}{2} \epsilon_\mu^{\alpha\beta\sigma\tau} \omega_{\sigma\tau}^{AB}(s, s) + \epsilon^{AB} \delta_\mu^\alpha \kappa^\beta(s, s) \right) \\ &= C_{\nu\mu} \mu(s, s) + (A_\nu \eta_{\mu\beta} + E_{\nu\mu\beta}) \kappa^\beta(s, s) \\ & \quad + \left(D_\nu{}^\tau{}_{AB} \delta_\mu^\sigma - \frac{1}{4} \epsilon_\mu^{\alpha\beta\sigma\tau} F_{\nu\alpha\beta AB} \right) \omega_{\sigma\tau}^{AB}(s, s). \end{aligned} \quad (4.81)$$

Equation (4.71) is satisfied if and only if the coefficients of each of the three Dirac bilinears vanish separately; this gives the following system of equations:

$$C_{(\mu\nu)} = 0, \quad (4.82)$$

$$A_\mu \eta_{\nu\beta} + A_\nu \eta_{\mu\beta} + E_{\mu\nu\beta} + E_{\nu\mu\beta} = 0, \quad (4.83)$$

$$\delta_{(\mu}^{[\sigma} D_{\nu)}^{\tau]}{}_{AB} - \frac{1}{4} \epsilon_{(\mu}^{\alpha\beta\sigma\tau} F_{\nu)\alpha\beta AB} = 0. \quad (4.84)$$

Equation (4.82) simply gives $C \in \wedge^2 V$. Contracting Equation (4.83) with $\eta^{\mu\nu}$ or with $\eta^{\mu\beta}$ yields

$$2A_\beta + 2E^\mu_{\mu\beta} = 0, \quad (4.85)$$

$$6A_\nu + E^\mu_{\nu\mu} + E_\nu{}^\mu{}_\mu = 0. \quad (4.86)$$

Relabelling some indices and recalling that E is skew-symmetric in its last two indices,

$$A_\nu + E^\mu_{\mu\nu} = 0, \quad (4.87)$$

$$6A_\nu - E^\mu_{\mu\nu} = 0, \quad (4.88)$$

so $A = 0$, and then (4.83) implies that $E \in \bigwedge^3 V$. Turning to the remaining Equation (4.84), we contract it with $\eta^{\mu\nu}$ to yield

$$D^{[\sigma\tau]}_{AB} = \frac{1}{4}\epsilon^{\mu\alpha\beta\sigma\tau}F_{\mu\alpha\beta AB}. \quad (4.89)$$

To proceed further, it will be helpful to expand the symmetrised expressions in Equation (4.84) (while suppressing the Δ indices):

$$\delta^\sigma_\mu D_\nu{}^\tau - \delta^\tau_\mu D_\nu{}^\sigma + \delta^\sigma_\nu D_\mu{}^\tau - \delta^\tau_\nu D_\mu{}^\sigma - \frac{1}{2}\epsilon_\mu{}^{\alpha\beta\sigma\tau}F_{\nu\alpha\beta} - \frac{1}{2}\epsilon_\nu{}^{\alpha\beta\sigma\tau}F_{\mu\alpha\beta} = 0. \quad (4.90)$$

Now contracting the σ and μ indices gives

$$5D_\nu{}^\tau - \delta^\tau_\nu D^\mu{}_\mu - \frac{1}{2}\epsilon_\nu{}^{\alpha\beta\mu\tau}F_{\mu\alpha\beta} = 0, \quad (4.91)$$

so that the symmetric and skew-symmetric parts of D are given by

$$D^{(\sigma\tau)} = \frac{1}{5}\eta^{\sigma\tau}D^\mu{}_\mu, \quad (4.92)$$

$$D^{[\sigma\tau]} = -\frac{1}{10}\epsilon^{\mu\alpha\beta\sigma\tau}F_{\mu\alpha\beta}, \quad (4.93)$$

and together with Equation (4.89), this means that the skew-symmetric part of D must vanish. On the other hand, taking the trace of Equation (4.92) shows that is consistent for any choice of $D^\mu{}_\mu$. We thus have $D_{\mu\nu AB} = \eta_{\mu\nu}d_{AB}$ where $d_{AB} = \frac{1}{5}D^\mu{}_{\nu AB}$. Substituting this into Equation (4.84) gives $\epsilon_{\sigma\tau\alpha\beta(\mu}F_{\nu)}{}^{\alpha\beta}{}_{AB} = 0$ and contracting with $\epsilon^{\sigma\tau\theta\phi\psi}$ results in

$$0 = \epsilon^{\sigma\tau\theta\phi\psi}\epsilon_{\sigma\tau\alpha\beta(\mu}F_{\nu)}{}^{\alpha\beta}{}_{AB} = 3!2!\delta_\alpha^{[\theta}\delta_\beta^{\phi}\delta_{(\mu}^{\psi]}F_{\nu)}{}^{\alpha\beta}{}_{AB} = 12\delta_{(\mu}^{[\psi}F_{\nu)}{}^{\theta\phi]}. \quad (4.94)$$

Contracting with $\eta^{\mu\nu}$ gives $F^{[\phi\theta\psi]} = 0$. On the other hand, we can expand this expression and use the skew-symmetry of the last two indices of F to get

$$0 = \delta^\psi_\mu F_\nu{}^{\theta\phi}{}_{AB} + \delta^\theta_\mu F_\nu{}^{\phi\psi}{}_{AB} + \delta^\phi_\mu F_\nu{}^{\psi\theta}{}_{AB} + \delta^\psi_\nu F_\mu{}^{\theta\phi}{}_{AB} + \delta^\theta_\nu F_\mu{}^{\phi\psi}{}_{AB} + \delta^\phi_\nu F_\mu{}^{\psi\theta}{}_{AB}, \quad (4.95)$$

and contraction of μ and ψ gives

$$0 = 4F_\nu{}^{\theta\phi}{}_{AB} + \delta^\theta_\nu F_\mu{}^{\phi\mu}{}_{AB} + \delta^\phi_\nu F_\mu{}^{\mu\theta}{}_{AB}, \quad (4.96)$$

so that finally

$$F_\nu{}^{\theta\phi}{}_{AB} = -\frac{1}{4}\left(\delta^\theta_\nu F_\mu{}^{\phi\mu}{}_{AB} + \delta^\phi_\nu F_\mu{}^{\mu\theta}{}_{AB}\right) = \frac{1}{4}\left(\delta^\theta_\nu F_\mu{}^{\mu\phi}{}_{AB} - \delta^\phi_\nu F_\mu{}^{\theta\mu}{}_{AB}\right) = \frac{1}{2}\delta^{[\theta}{}_\nu\phi^{\phi]}{}_{AB}, \quad (4.97)$$

where $\phi_{\mu AB} = F_\mu{}^\nu{}_{\nu AB}$. Note that this automatically satisfies the condition $F_{[\mu\nu\rho]} = 0$ found before. $\square$

In the next section, we will also require the following expression (which determines γ):

$$\bar{s}^A\Gamma_\mu\beta_{\nu AB}s^B = -C_{\mu\nu}\mu(s, s) - E_{\mu\nu\alpha}\kappa^\alpha(s, s) + \left(d_{AB}\delta^\sigma_\mu\delta^\tau_\nu - \frac{1}{8}\epsilon_{\mu\nu}{}^{\alpha\sigma\tau}\phi_{\alpha AB}\right)\omega_{\sigma\tau}{}^{AB}(s, s). \quad (4.98)$$

4.3.3 Solving the second cocycle condition

We now turn to finding the subspace of solutions given in Proposition 4.3.3 which also satisfy the remaining cocycle condition, Equation (4.72).

Lemma 4.3.4. *Provided Equation (4.71) is satisfied, Equation (4.72) is equivalent to the following system of equations:*

$$\Theta_{AB}\Gamma_\alpha + 5\Phi_{\alpha AB} + \Phi_{\mu AB}\Gamma_\alpha{}^\mu + \frac{1}{2}\Psi_{\mu\nu AB}\Gamma_\alpha{}^{\mu\nu} + \Psi_{\alpha\nu AB}\Gamma^\nu = 0, \quad (4.99)$$

$$4\Psi_{\alpha\beta AB}\delta_D^C - 2(\Theta_{(A}^C\epsilon_{B)D}\Gamma_{\alpha\beta} + \Phi_{\mu(A}^C\epsilon_{B)D}\Gamma_{\alpha\beta}\Gamma^\mu) - \Psi_{\mu\nu D(A}\delta_{B)}^C\Gamma_{\alpha\beta}\Gamma^{\mu\nu} = 0, \quad (4.100)$$

where

$$\Theta_{AB} = 3d_{AB} - \frac{1}{4}\phi_{\sigma AB}\Gamma^\sigma + \frac{1}{2}C_{\sigma\tau}\Gamma^{\sigma\tau}\epsilon_{AB}, \quad (4.101)$$

$$\Phi_{\mu AB} = \left(B_{\mu AB} + \frac{1}{4}\phi_{\mu AB}\right)\mathbb{1} + (C_{\mu\sigma}\Gamma^\sigma + E_{\mu\sigma\tau}\Gamma^{\sigma\tau})\epsilon_{AB}, \quad (4.102)$$

$$\Psi_{\mu\nu AB} = \frac{1}{8}\epsilon_{\mu\nu\alpha\sigma\tau}\phi_{AB}^\alpha\Gamma^{\sigma\tau}. \quad (4.103)$$

Proof. First we substitute Equations (4.98) and (4.80) into Equation (4.72):

$$\begin{aligned} 0 &= -\frac{1}{2}\left(-C_{\mu\nu}\mu(s, s) - E_{\mu\nu\alpha}\kappa^\alpha(s, s) + \left(d_{AB}\delta_\mu^\sigma\delta_\nu^\tau - \frac{1}{8}\epsilon_{\mu\nu}{}^{\alpha\sigma\tau}\phi_{\alpha AB}\right)\omega_{\sigma\tau}{}^{AB}(s, s)\right)\Gamma^{\mu\nu}s^C \\ &\quad + \kappa^\mu(s, s)\left(B_\mu{}^C{}_A\mathbb{1} + C_{\mu\alpha}\epsilon^C{}_A\Gamma^\alpha + d^C{}_A\Gamma_\mu + \frac{1}{2}E_{\mu\alpha\beta}\epsilon^C{}_A\Gamma^{\alpha\beta} + \frac{1}{4}\phi^{\alpha C}{}_A\Gamma_{\mu\alpha}\right)s^A \\ &= \mu(s, s)\left(\frac{1}{2}C_{\mu\nu}\right)\Gamma^{\mu\nu}s^C + \frac{1}{2}\omega_{\sigma\tau}{}^{AB}(s, s)\left(\frac{1}{8}\epsilon_{\mu\nu}{}^{\alpha\sigma\tau}\phi_{\alpha AB} - d_{AB}\delta_\mu^\sigma\delta_\nu^\tau\right)\Gamma^{\mu\nu}s^C \\ &\quad + \kappa^\mu(s, s)\left(B_\mu{}^C{}_A\mathbb{1} + C_{\mu\alpha}\delta_A^C\Gamma^\alpha + d^C{}_A\Gamma_\mu + E_{\mu\alpha\beta}\delta_A^C\Gamma^{\alpha\beta} + \frac{1}{4}\phi^{\alpha C}{}_A\Gamma_{\mu\alpha}\right)s^A \\ &= \mu(s, s)\left(\frac{1}{2}C_{\mu\nu}\right)\Gamma^{\mu\nu}s^C + \left(d^C{}_A - \frac{1}{4}\phi^{\alpha C}{}_A\Gamma_\alpha\right)\kappa(s, s) \cdot s^A \\ &\quad + \kappa^\mu(s, s)\left(\left(B_\mu{}^C{}_A + \frac{1}{4}\phi_\mu{}^C{}_A\right)\mathbb{1} + C_{\mu\alpha}\delta_A^C\Gamma^\alpha + E_{\mu\alpha\beta}\delta_A^C\Gamma^{\alpha\beta}\right)s^A \\ &\quad + \frac{1}{2}\omega_{\sigma\tau}{}^{AB}(s, s)\left(\frac{1}{8}\epsilon_{\mu\nu}{}^{\alpha\sigma\tau}\phi_{\alpha AB}\right)\Gamma^{\mu\nu}s^C - d_{AB}\omega^{AB}(s, s) \cdot s^C \\ &= \mu(s, s)\left(\frac{1}{2}C_{\mu\nu}\delta_A^C\Gamma^{\mu\nu} + d^C{}_A - \frac{1}{4}\phi^{\alpha C}{}_A\Gamma_\alpha - 2d_{AB}\epsilon^{CB}\right)s^A \\ &\quad + \kappa^\mu(s, s)\left(\left(B_\mu{}^C{}_A + \frac{1}{4}\phi_\mu{}^C{}_A\right)\mathbb{1} + C_{\mu\alpha}\delta_A^C\Gamma^\alpha + E_{\mu\alpha\beta}\delta_A^C\Gamma^{\alpha\beta}\right)s^A \\ &\quad + \frac{1}{2}\omega_{\sigma\tau}{}^{AB}(s, s)\left(\frac{1}{8}\epsilon_{\mu\nu}{}^{\alpha\sigma\tau}\phi_{\alpha AB}\Gamma^{\mu\nu}\right)s^C \\ &= \mu(s, s)\Theta^C{}_As^A + \kappa^\mu(s, s)\Phi_\mu{}^Cs^A + \frac{1}{2}\omega_{\sigma\tau}{}^{AB}(s, s)\Psi^{\sigma\tau}{}_{AB}s^C, \end{aligned} \quad (4.104)$$

where in the second equality we used $\Gamma_{\mu\alpha} = -\Gamma_\alpha\Gamma_\mu + \eta_{\mu\alpha}$ and the definitions of κ and ω , in the second to last we used Lemma 3.2.13 and the last defines Θ, Φ and Ψ as per Equations (4.101), (4.102) and (4.103).

Equation (4.104) is cubic in s ; our goal now is to express it as a quadratic in s , which will be written in terms of the bilinears μ, κ, ω , giving us three equations in B, C, d, E, ϕ . This will be achieved by first polarising Equation (4.104) then rearranging the polarised expression into the form $W(s_1, s_2)s_3 = 0$ for some $W : \odot^2 S \rightarrow \text{End}(S)$, which is equivalent to $W(s, s) = 0$. To this end, define the map $Z : \odot^2 S \rightarrow \text{End}(S)$ by

$$Z(s_1, s_2)s_3^C = \mu(s_1, s_2)\Theta_A^C s_3^A + \kappa^\mu(s_1, s_2)\Phi_\mu^C s_3^A + \frac{1}{2}\omega^{\mu\nu}_{AB}(s_1, s_2)\Psi_{\mu\nu}^{AB} s_3^C. \quad (4.105)$$

Equation (4.104) is then simply $Z(s, s)s^C = 0$, which is equivalent to its polarised form

$$Z(s_1, s_2)s_3^C + Z(s_2, s_3)s_1^C + Z(s_3, s_1)s_2^C = 0 \quad (4.106)$$

since the left hand side of the latter expression is completely symmetric and therefore determined by its value on the diagonal. This is the expression that must be rearranged to the form $W(s_1, s_2)s_3 = 0$. Using the definition and symmetry of Z , Equation (4.106) expands to

$$\begin{aligned} 0 &= \mu(s_1, s_2)\Theta_A^C s_3^A + \kappa^\mu(s_1, s_2)\Phi_\mu^C s_3^A + \frac{1}{2}\omega^{\mu\nu}_{AB}(s_1, s_2)\Psi_{\mu\nu}^{AB} s_3^C \\ &\quad + \Theta_A^C (s_1^A \mu(s_2, s_3) + s_2^A \mu(s_1, s_3)) + \Phi_\mu^C (s_1^A \kappa^\mu(s_2, s_3) + s_3^A \kappa^\mu(s_3, s_2)) \\ &\quad + \frac{1}{2}\Psi_{\mu\nu}^{AB} (s_1^C \omega^{\mu\nu}_{AB}(s_2, s_3) + s_2^C \omega^{\mu\nu}_{AB}(s_1, s_3)). \end{aligned} \quad (4.107)$$

Recall Lemma 3.2.10; the μ and κ terms are symmetric in s_1, s_2 , and for the ω terms we have $\omega^{\mu\nu}_{AB}(s_1, s_2) = \omega^{\mu\nu}_{BA}(s_2, s_1)$. We now Fierz rearrange the last three terms using Proposition 3.2.12; we have

$$\begin{aligned} s_1^A \mu(s_2, s_3) &= \epsilon_{BC} s_1^A \bar{s}_2^B s_3^C \\ &= -\frac{1}{8}\epsilon_{BC} [\epsilon^{AB}(\mu(s_1, s_2) + \kappa^\alpha(s_1, s_2)\Gamma_\alpha) + \omega^{\alpha\beta AB}(s, s)\Gamma_{\alpha\beta}] s_3^C \\ &= \frac{1}{8}(\mu(s_1, s_2) + \kappa^\alpha(s_1, s_2)\Gamma_\alpha) s_3^A + \frac{1}{8}\omega^{\alpha\beta A}{}_C(s_1, s_2)\Gamma_{\alpha\beta} s_3^C, \end{aligned} \quad (4.108)$$

and similarly

$$\begin{aligned} s_1^A \kappa^\mu(s_2, s_3) &= \epsilon_{BC} s_1^A \bar{s}_2^B \Gamma^\mu s_3^C \\ &= \frac{1}{8}(\mu(s_1, s_2)\Gamma^\mu + \kappa^\alpha(s_1, s_2)\Gamma_\alpha\Gamma^\mu) s_3^A + \frac{1}{8}\omega^{\alpha\beta A}{}_C(s_1, s_2)\Gamma_{\alpha\beta}\Gamma^\mu s_3^C, \end{aligned} \quad (4.109)$$

and for the 2-form,

$$\begin{aligned} s_1^C \omega^{\mu\nu AB}(s_2, s_3) &= s_1^C \bar{s}_2^A \Gamma^{\mu\nu} s_3^B \\ &= -\frac{1}{8}\epsilon^{CA}(\mu(s_1, s_2)\Gamma^{\mu\nu} + \kappa^\alpha(s_1, s_2)\Gamma_\alpha\Gamma^{\mu\nu}) s_3^B - \frac{1}{8}\omega^{\alpha\beta CA}(s_1, s_2)\Gamma_{\alpha\beta}\Gamma^{\mu\nu} s_3^B. \end{aligned} \quad (4.110)$$

Continuing on from Equation (4.107),

$$\begin{aligned}
0 &= \mu(s_1, s_2) \Theta^C{}_A s_3^A + \kappa^\mu(s_1, s_2) \Phi_\mu{}^C{}_A s_3^A + \frac{1}{2} \omega^{\mu\nu}{}_{AB}(s_1, s_2) \Psi_{\mu\nu}{}^{AB} s_3^C \\
&\quad + \frac{1}{4} \Theta^C{}_A [(\mu(s_1, s_2) + \kappa^\alpha(s_1, s_2) \Gamma_\alpha) s_3^A + \epsilon_{DB} \omega^{\alpha\beta(AB)}(s_1, s_2) \Gamma_{\alpha\beta} s_3^D] \\
&\quad + \frac{1}{4} \Phi_\mu{}^C{}_A [(\mu(s_1, s_2) \Gamma^\mu + \kappa^\alpha(s_1, s_2) \Gamma_\alpha \Gamma^\mu) s_3^A \\
&\quad \quad \quad + \epsilon_{DB} \omega^{\alpha\beta(AB)}(s_1, s_2) \Gamma_{\alpha\beta} \Gamma^\mu s_3^D] \\
&\quad - \frac{1}{8} \Psi_{\mu\nu AB} [\epsilon^{CA} (\mu(s_1, s_2) \Gamma^{\mu\nu} + \kappa^\alpha(s_1, s_2) \Gamma_\alpha \Gamma^{\mu\nu}) s_3^B \\
&\quad \quad \quad + \omega^{\alpha\beta(CA)}(s_1, s_2) \Gamma_{\alpha\beta} \Gamma^{\mu\nu} s_3^B] \quad (4.111) \\
&= \mu(s_1, s_2) \left[\Theta^C{}_D + \frac{1}{4} \Theta^C{}_D + \frac{1}{4} \Phi_\mu{}^C{}_D \Gamma^\mu + \frac{1}{8} \Psi_{\mu\nu}{}^C{}_D \Gamma^{\mu\nu} \right] s_3^D \\
&\quad + \kappa^\alpha(s_1, s_2) \left[\Phi_\alpha{}^C{}_D + \frac{1}{4} \Theta^C{}_D \Gamma_\alpha + \frac{1}{4} \Phi_\mu{}^C{}_D \Gamma_\alpha \Gamma^\mu + \frac{1}{8} \Psi_{\mu\nu}{}^C{}_D \Gamma_\alpha \Gamma^{\mu\nu} \right] s_3^D \\
&\quad + \frac{1}{2} \omega^{\alpha\beta(AB)}(s_1, s_2) \left[\Psi_{\alpha\beta AB} \delta_D^C + \frac{1}{2} \epsilon_{DB} (\Theta^C{}_A \Gamma_{\alpha\beta} + \Phi_\mu{}^C{}_A \Gamma_{\alpha\beta} \Gamma^\mu) \right. \\
&\quad \quad \quad \left. - \frac{1}{4} \Psi_{\mu\nu AD} \delta_B^C \Gamma_{\alpha\beta} \Gamma^{\mu\nu} \right] s_3^D.
\end{aligned}$$

This is the $W(s_1, s_2) s_3 = 0$ expression we sought, equivalent to $W(s, s) = 0$. The coefficients of μ, κ and ω must therefore vanish separately:

$$5\Theta_{AB} + \Phi_{\mu AB} \Gamma^\mu + \frac{1}{2} \Psi_{\mu\nu AB} \Gamma^{\mu\nu} = 0, \quad (4.112)$$

$$4\Phi_{\alpha AB} + \Theta_{AB} \Gamma_\alpha + \Phi_{\mu AB} \Gamma_\alpha \Gamma^\mu + \frac{1}{2} \Psi_{\mu\nu AB} \Gamma_\alpha \Gamma^{\mu\nu} = 0, \quad (4.113)$$

$$4\Psi_{\alpha\beta AB} \delta_D^C - 2(\Theta^C{}_{(A} \epsilon_{B)D} \Gamma_{\alpha\beta} + \Phi_\mu{}^C{}_{(A} \epsilon_{B)D} \Gamma_{\alpha\beta} \Gamma^\mu) - \Psi_{\mu\nu D(A} \delta_{B)}^C \Gamma_{\alpha\beta} \Gamma^{\mu\nu} = 0. \quad (4.114)$$

Contracting with Equation (4.113) Γ^α on the right gives

$$\begin{aligned}
0 &= 4\Phi_{\alpha AB} \Gamma^\alpha + \Theta_{AB} (5) + \Phi_{\mu AB} (-3\Gamma^\mu) + \frac{1}{2} \Psi_{\mu\nu AB} \Gamma^{\mu\nu} \\
&= \Phi_{\mu AB} \Gamma^\mu + 5\Theta_{AB} + \frac{1}{2} \Psi_{\mu\nu AB} \Gamma^{\mu\nu},
\end{aligned} \quad (4.115)$$

but this is just Equation (4.112), so that equation is redundant. Evaluating the gamma matrix products in Equation (4.113) yields

$$\begin{aligned}
0 &= 4\Phi_{\alpha AB} + \Theta_{AB} \Gamma_\alpha + \Phi_{\mu AB} (\Gamma_\alpha{}^\mu + \delta_\alpha^\mu) + \frac{1}{2} \Psi_{\mu\nu AB} (\Gamma_\alpha{}^{\mu\nu} + 2\delta_\alpha^{[\mu} \Gamma^{\nu]}) \\
&= \Theta_{AB} \Gamma_\alpha + 5\Phi_{\alpha AB} + \Phi_{\mu AB} \Gamma_\alpha{}^\mu + \frac{1}{2} \Psi_{\mu\nu AB} \Gamma_\alpha{}^{\mu\nu} + \Psi_{\alpha\nu AB} \Gamma^\nu,
\end{aligned} \quad (4.116)$$

hence the result. $\square$

Theorem (4.3.1) is a direct corollary of the following result.

Proposition 4.3.5. *The solution to Equations (4.70), (4.71) and (4.72) is*

$$\beta_{\mu AB} = \phi_{AB}^{\alpha} \left(\frac{1}{4} \Gamma_{\mu\alpha} - \frac{1}{2} \eta_{\mu\alpha} \mathbb{1} \right) + C^{\alpha\beta} \epsilon_{AB} \left(\frac{1}{8} \epsilon_{\mu\alpha\beta\sigma\tau} \Gamma^{\sigma\tau} + \eta_{\mu\alpha} \Gamma_{\beta} \right), \quad (4.117)$$

$$\gamma_{\mu\nu}(s, s) = \frac{1}{4} \epsilon_{\mu\nu\alpha\sigma\tau} \phi_{AB}^{\alpha} \omega^{\sigma\tau AB}(s, s) + C^{\alpha\beta} \left(2\eta_{\mu\alpha} \eta_{\nu\beta} \mu(s, s) + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta\sigma} \kappa^{\sigma}(s, s) \right), \quad (4.118)$$

where $\phi \in V \otimes \odot^2 \Delta$ and $C \in \wedge^2 V$.

Proof. First, note that in Equations (4.101)-(4.103) we have written Θ, Φ and Ψ explicitly as sums of their symmetric and skew-symmetric (in the symplectic indices) parts; Ψ is symmetric in the symplectic indices and skew-symmetric in the spacetime indices. The symmetric part of Equation (4.99) is

$$\begin{aligned} 0 &= \left(3d_{AB} - \frac{1}{4} \phi_{\sigma} \Gamma^{\sigma} \right) \Gamma_{\alpha} + 5 \left(B_{\alpha AB} + \frac{1}{4} \phi_{\alpha AB} \right) \mathbb{1} + \left(B_{\mu AB} + \frac{1}{4} \phi_{\mu AB} \right) \Gamma_{\alpha}^{\mu} \\ &\quad + \frac{1}{16} \epsilon_{\mu\nu\beta\sigma\tau} \phi_{AB}^{\beta} \Gamma^{\sigma\tau} \Gamma_{\alpha}^{\mu\nu} + \frac{1}{8} \epsilon_{\alpha\nu\beta\sigma\tau} \phi_{AB}^{\beta} \Gamma^{\sigma\tau} \Gamma^{\nu} \\ &= 3d_{AB} \Gamma_{\alpha} - \frac{1}{4} \phi_{\sigma AB} (\Gamma^{\sigma}_{\alpha} + \delta_{\alpha}^{\sigma}) \\ &\quad + \left(5B_{\alpha AB} + \frac{5}{4} \phi_{\alpha AB} \right) \mathbb{1} + \left(B_{\mu AB} + \frac{1}{4} \phi_{\mu AB} \right) \Gamma_{\alpha}^{\mu} \\ &\quad - \frac{1}{32} \epsilon_{\mu\nu\beta\sigma\tau} \epsilon_{\alpha}^{\mu\nu\lambda\rho} \phi_{AB}^{\beta} \Gamma^{\sigma\tau} \Gamma_{\lambda\rho} - \frac{1}{8} \epsilon_{\alpha\beta\sigma\tau\nu} \phi_{AB}^{\beta} (\Gamma^{\sigma\tau\nu} + 2\Gamma^{[\sigma} \delta^{\tau]\nu}) \\ &= 3d_{AB} \Gamma_{\alpha} + \frac{1}{4} \phi_{\sigma} \Gamma_{\alpha}^{\sigma} + (5B_{\alpha AB} + \phi_{\alpha AB}) \mathbb{1} + \left(B_{\mu AB} + \frac{1}{4} \phi_{\mu AB} \right) \Gamma_{\alpha}^{\mu} \\ &\quad - \frac{1}{32} \left(3!2! \delta_{[\alpha}^{[\beta} \delta_{\sigma}^{\lambda} \delta_{\tau]}^{\rho]} \right) \phi_{\beta AB} \left(\Gamma^{\sigma\tau}_{\lambda\rho} + 4\delta_{[\lambda}^{[\tau} \Gamma_{\rho]}^{\sigma]} + 2\delta_{[\lambda}^{[\tau} \delta_{\rho]}^{\sigma]} \right) \\ &\quad + \frac{1}{16} \epsilon_{\alpha\beta\sigma\tau\nu} \epsilon^{\sigma\tau\nu\lambda\rho} \phi_{AB}^{\beta} \Gamma_{\lambda\rho} \\ &= (5B_{\alpha AB} + \phi_{\alpha AB}) \mathbb{1} + 3d_{AB} \Gamma_{\alpha} + \left(B_{\mu AB} + \frac{1}{2} \phi_{\mu AB} \right) \Gamma_{\alpha}^{\mu} \\ &\quad + \frac{3}{8} \phi_{\beta AB} \left(4\delta_{[\alpha}^{[\beta} \delta_{\lambda}^{\lambda} \Gamma_{\rho]}^{\rho]} + 2\delta_{[\alpha}^{[\beta} \delta_{\lambda}^{\lambda} \delta_{\rho]}^{\rho]} \right) + \frac{1}{16} \left(3!2! \delta_{[\alpha}^{[\lambda} \delta_{\beta]}^{\rho]} \right) \phi_{AB}^{\beta} \Gamma_{\lambda\rho} \\ &= (5B_{\alpha AB} + \phi_{\alpha AB}) \mathbb{1} + 3d_{AB} \Gamma_{\alpha} + \left(B_{\mu AB} + \frac{1}{2} \phi_{\mu AB} \right) \Gamma_{\alpha}^{\mu} \\ &\quad - \frac{3}{4} \phi_{\mu AB} \Gamma_{\alpha}^{\mu} + \frac{3}{2} \phi_{\alpha AB} + \frac{3}{4} \phi_{AB}^{\beta} \Gamma_{\alpha\beta} \\ &= 5 \left(B_{\alpha AB} + \frac{1}{2} \phi_{\alpha AB} \right) \mathbb{1} + 3d_{AB} \Gamma_{\alpha} + \left(B_{\mu AB} + \frac{1}{2} \phi_{\mu AB} \right) \Gamma_{\alpha}^{\mu}, \end{aligned} \quad (4.119)$$

so that

$$d_{AB} = 0, \quad (4.120)$$

$$B_{\mu AB} = -\frac{1}{2} \phi_{\mu AB}. \quad (4.121)$$

The skew-symmetric part (after removing the ϵ_{AB} factor) is

$$\begin{aligned}
0 &= \frac{1}{2}C_{\sigma\tau}\Gamma^{\sigma\tau}\Gamma_\alpha + 5(C_{\alpha\sigma}\Gamma^\sigma + E_{\alpha\sigma\tau}\Gamma^{\sigma\tau}) + (C_{\mu\sigma}\Gamma^\sigma + E_{\mu\sigma\tau}\Gamma^{\sigma\tau})\Gamma_\alpha{}^\mu \\
&= \frac{1}{2}C_{\sigma\tau}(\Gamma^{\sigma\tau}{}_\alpha + 2\Gamma^{[\sigma}\delta_{\alpha}^{\tau]}) + 5C_{\alpha\sigma}\Gamma^\sigma + 5E_{\alpha\sigma\tau}\Gamma^{\sigma\tau} \\
&\quad + C^\mu{}_\sigma(\Gamma^\sigma{}_{\alpha\mu} + 2\delta_{[\alpha}^\sigma\Gamma_{\mu]}) + E^\mu{}_{\sigma\tau}(\Gamma^{\sigma\tau}{}_{\alpha\mu} + 4\delta_{[\alpha}^{[\tau}\Gamma_{\mu]}^{\sigma]} + 2\delta_{[\alpha}^{[\tau}\delta_{\mu]}^{\sigma]}) \\
&= \frac{3}{2}C_{\sigma\tau}\Gamma^{\sigma\tau}{}_\alpha + C_{\sigma\alpha}\Gamma^\sigma + 5C_{\alpha\sigma}\Gamma^\sigma + 5E_{\alpha\sigma\tau}\Gamma^{\sigma\tau} \\
&\quad + C^\mu{}_\alpha\Gamma_\mu + E^\mu{}_{\sigma\tau}\epsilon^{\sigma\tau}{}_{\alpha\mu\lambda}\Gamma^\lambda + 2E^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_\mu \\
&= (3C_{\alpha\lambda} - \epsilon_{\alpha\lambda\mu\sigma\tau}E^{\mu\sigma\tau})\Gamma^\lambda + 3\left(E_{\alpha\sigma\tau} - \frac{1}{4}\epsilon_{\alpha\mu\nu\sigma\tau}C^{\mu\nu}\right)\Gamma^{\sigma\tau},
\end{aligned} \tag{4.122}$$

thus we have the (equivalent) expressions

$$C_{\alpha\beta} = \frac{1}{3}\epsilon_{\alpha\beta\lambda\mu\nu}E^{\lambda\mu\nu} \quad \text{or} \quad E_{\lambda\mu\nu} = \frac{1}{4}\epsilon_{\lambda\mu\nu\alpha\beta}C^{\alpha\beta}. \tag{4.123}$$

The solution is thus determined by ϕ and C (equivalently ϕ and E). It remains to be checked that Equation (4.100) is satisfied. To this end, note that, using Equations (4.120) and (4.123),

$$\Theta^C{}_A = \frac{1}{2}\left(-\frac{1}{2}\phi_\sigma{}^C{}_A\Gamma^\sigma + C_{\sigma\tau}\Gamma^{\sigma\tau}\delta_A^C\right), \tag{4.124}$$

$$\Phi_\mu{}^C{}_A = -\frac{1}{4}\phi_\mu{}^C{}_A\mathbb{1} + C^{\sigma\tau}\delta_A^C\left(\eta_{\mu\sigma}\Gamma_\tau + \frac{1}{4}\epsilon_{\mu\sigma\tau\alpha\beta}\Gamma^{\alpha\beta}\right), \tag{4.125}$$

$$\Psi_{\mu\nu AB} = \frac{1}{8}\epsilon_{\mu\nu\alpha\sigma\tau}\phi^\alpha{}_{AB}\Gamma^{\sigma\tau}. \tag{4.126}$$

We do not explicitly show the calculations here, but using similar (though more arduous) manipulations to those already used in this proof, we can evaluate the main expression in each of the terms of Equation (4.100) in turn:

$$4\Psi_{\alpha\beta AB} = \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma\tau}\phi^\mu{}_{AB}\Gamma^{\sigma\tau}, \tag{4.127}$$

$$\begin{aligned}
-2\Theta^C{}_A\Gamma_{\alpha\beta} &= -\frac{1}{4}\epsilon_{\alpha\beta\mu\sigma\tau}\phi^{\mu C}{}_A\Gamma^{\sigma\tau} + \phi_{[\alpha}{}^C{}_A\Gamma_{\beta]} \\
&\quad + (4C_{\sigma[\alpha}\Gamma_{\beta]}{}^\sigma - \epsilon_{\alpha\beta\mu\nu\sigma}C^{\mu\nu}\Gamma^\sigma + 2C_{\alpha\beta})\delta_A^C,
\end{aligned} \tag{4.128}$$

$$\begin{aligned}
-2\Phi_\mu{}^C{}_A\Gamma_{\alpha\beta}\Gamma^\mu &= -\frac{1}{4}\epsilon_{\alpha\beta\mu\sigma\tau}\phi^{\mu C}{}_A\Gamma^{\sigma\tau} - \phi_{[\alpha}{}^C{}_A\Gamma_{\beta]} \\
&\quad - (4C_{\sigma[\alpha}\Gamma_{\beta]}{}^\sigma - \epsilon_{\alpha\beta\mu\nu\sigma}C^{\mu\nu}\Gamma^\sigma + 2C_{\alpha\beta})\delta_A^C,
\end{aligned} \tag{4.129}$$

$$-\Psi_{\mu\nu DA}\Gamma_{\alpha\beta}\Gamma^{\mu\nu} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\sigma\tau}\phi^\mu{}_{DA}\Gamma^{\sigma\tau}. \tag{4.130}$$

Substituting these expressions into Equation (4.100), we have

$$\begin{aligned}
0 &= 4\Psi_{\alpha\beta AB}\delta_D^C - 2(\Theta^C{}_{(A}\epsilon_{B)D}\Gamma_{\alpha\beta} + \Phi_\mu{}^C{}_{(A}\epsilon_{B)D}\Gamma_{\alpha\beta}\Gamma^\mu) - \Psi_{\mu\nu D(A}\delta_{B)}^C\Gamma_{\alpha\beta}\Gamma^{\mu\nu} \\
&= \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma\tau}\left(\phi^\mu{}_{AB}\delta_D^C - \phi^{\mu C}{}_{(A}\epsilon_{B)D} - \phi^\mu{}_{D(A}\delta_{B)}^C\right)\Gamma^{\sigma\tau},
\end{aligned} \tag{4.131}$$

or equivalently,

$$\phi^\mu_{AB} \epsilon^{DC} + \phi^{\mu C}_{(A} \delta^D_{B)} - \phi^{\mu D}_{(A} \delta^C_{B)} = 0. \quad (4.132)$$

Clearly this is skew-symmetric in CD . It is seen to be identically satisfied by contracting the left hand side with ϵ_{DC} (this is sufficient since $\bigwedge^2 \Delta$ is spanned by ϵ):

$$\begin{aligned} \epsilon_{DC} \left(\phi^\mu_{AB} \epsilon^{DC} + \phi^{\mu C}_{(A} \delta^D_{B)} - \phi^{\mu D}_{(A} \delta^C_{B)} \right) &= 2\phi^\mu_{AB} - \phi^\mu_{D(A} \delta^D_{B)} - \phi^{\mu D}_{(A} \epsilon_{|D|B)} \\ &= 2\phi^\mu_{AB} - \phi^\mu_{(AB)} - \phi^\mu_{(AB)} \\ &= 0. \end{aligned} \quad (4.133)$$

We therefore have the exact solution β to the cocycle conditions given by Equation (4.117). We then simply use Equations (4.70) and (4.98) to obtain Equation (4.118) explicitly. $\square$

We can also write, for $X \in V$, $s \in S$,

$$\beta_X s = \frac{1}{4}(X \cdot C - 3C \cdot X) \cdot s - \frac{1}{8}(X \cdot \phi + 3\phi \cdot X) \cdot s. \quad (4.134)$$

4.3.4 Subalgebras differing only in zero degree

We now provide a slight generalisation of the results of this section to some subalgebras $\mathfrak{t}$ of $\mathfrak{s}$; namely those differing in zero degree: $\mathfrak{t} = \mathfrak{s}_- \oplus \mathfrak{t}_0$ where $\mathfrak{t}_0$ is a subalgebra of $\mathfrak{s}_0 = \mathfrak{so}(V)$. We do this by adapting the method in [1] to find $H^{2,2}(\mathfrak{t}_-, \mathfrak{t})$ from $H^{2,2}(\mathfrak{s}_-, \mathfrak{s})$.

The low-degree cochains are the same as in Table 4.2, except $\mathfrak{so}(V)$ is now replaced with its subalgebra $\mathfrak{t}_0$. We will not dwell much on the $p \neq 2$ cohomologies (indeed, very little changes from the discussion in Section 4.2). It may be helpful at this point to review the proof of Theorem 4.2.1, as we will be adapting it in what follows.

Theorem 4.3.6. *Let $\beta^{C,\phi} + \gamma^{C,\phi}$ be the cocycle representative of the cohomology class in $H^{2,2}(\mathfrak{s}_-, \mathfrak{s})$ associated to $C \in \bigwedge^2 V$ and $\phi \in V \otimes \bigodot^2 \Delta$ as per Proposition 4.3.5. Then for even $d > 2$, $H^{d,2}(\mathfrak{t}_-, \mathfrak{t}) = 0$, and*

$$\begin{aligned} H^{2,2}(\mathfrak{t}_-, \mathfrak{t}) = & \frac{\left\{ \partial \tilde{\alpha} + \beta^{C,\phi} + \gamma^{C,\phi} \left| \begin{array}{l} C \in \bigwedge^2 V, \phi \in V \otimes \bigodot^2 \Delta, \tilde{\alpha} \in \text{Hom}(V, \mathfrak{so}(V)) \\ \text{such that } \gamma^{C,\phi} + \tilde{\alpha}(\kappa(s, s)) \in \mathfrak{t}_0 \end{array} \right. \right\}}{\{\partial \tilde{\alpha} \mid \tilde{\alpha} \in \text{Hom}(V, \mathfrak{t}_0)\}}. \end{aligned} \quad (4.135)$$

Proof. For the first statement, the argument for $d = 4$ is the same as for Equation (4.66), while for $d > 4$, the cochain spaces are zero, so in particular the cohomologies are zero.

Let $\alpha + \beta + \gamma \in \text{Hom}(\bigwedge^2 V, V) \oplus \text{Hom}(V \otimes S, S) \oplus \text{Hom}(\bigodot^2 S, \mathfrak{so}(V))$ be a typical $(2, 2)$ -cocycle. Then, lifting notation from the discussion of the $d = 2$ complex for $\mathfrak{s}$, there exists a unique $\tilde{\alpha} : V \rightarrow \mathfrak{so}(V)$ such that $\alpha = \partial_1^{2,2} \tilde{\alpha}$ (note that $\tilde{\alpha}$ and γ take values in $\mathfrak{so}(V)$, not necessarily in $\mathfrak{t}_0$). Then

$$\begin{aligned} \alpha + \beta + \gamma &= \alpha + \beta + \gamma - \partial \tilde{\alpha} + \partial \tilde{\alpha} \\ &= (\beta - \partial_2^{2,2} \tilde{\alpha}) + (\gamma + \partial_3^{2,2} \tilde{\alpha}) + \partial \tilde{\alpha}. \end{aligned} \quad (4.136)$$

Since $\alpha + \beta + \gamma$ is a cocycle and $\partial^2 = 0$, $(\beta - \partial_2^{2,2}\tilde{\alpha}) + (\gamma - \partial_3^{2,2}\tilde{\alpha})$ is also a cocycle, so we must in fact have $(\beta - \partial_2^{2,2}\tilde{\alpha}) = \beta^{C,\phi}$ and $(\gamma - \partial_3^{2,2}\tilde{\alpha}) = \gamma^{C,\phi}$ for some C, ϕ . It now only remains to check the conditions under which a cocycle $\partial\tilde{\alpha} + \beta^{C,\phi} + \gamma^{C,\phi}$ takes values in $\mathfrak{t}$; all that is required is that the $\text{Hom}(\bigodot^2 S, \mathfrak{so}(V))$ component takes values in $\mathfrak{t}_0$. Thus, since

$$(\partial\tilde{\alpha} + \beta^{C,\phi} + \gamma^{C,\phi})(s, s) = \alpha(\kappa(s, s)) - \gamma^{C,\phi}(s, s), \quad (4.137)$$

we have $\partial\tilde{\alpha} + \beta^{C,\phi} + \gamma^{C,\phi} \in Z^{2,2}(\mathfrak{s}_-, \mathfrak{t})$ if and only if $\alpha(\kappa(s, s)) + \gamma^{C,\phi}(s, s) \in \mathfrak{t}_0$. Hence the result. $\square$

4.4 A comment on filtered deformations

We ultimately did not have time to find the filtered deformations of $\mathfrak{s}$, but we leave some comments about how this is to be done here. From the results of Section 4.2, the Spencer cohomologies with $p = 1$ all vanish, so $\mathfrak{s}$ is a full prolongation. Since the actions of $\mathfrak{so}(V)$ on V and S are both faithful, $\mathfrak{s}$ is also transitive. Theorem 2.3.11 of Cheng and Kac therefore applies: the $\mathfrak{s}_0 = \mathfrak{so}(V)$ -invariant $p = 2$ cohomology classes govern the filtered deformations of $\mathfrak{s}$. The same arguments also apply to a subalgebra $\mathfrak{t}$ which differs only in degree 0.

We showed that the only nonzero $p = 2$ cohomology group is $H^{2,2}(\mathfrak{s}_-, \mathfrak{s})$, given by Theorem 4.3.1 for $\mathfrak{s}$ and by Theorem 4.3.6 for $\mathfrak{t}$. In the former case, the filtered deformations are parametrised by $\mathfrak{so}(V)$ -invariants in

$$(\bigwedge^2 V \otimes \bigwedge^2 \Delta) \oplus (V \otimes \bigodot^2 \Delta), \quad (4.138)$$

and the latter case depends on the choice of $\mathfrak{t}_0 \subset \mathfrak{so}(V)$. Following the methods of [1], we would first determine the $\mathfrak{so}(V)$ -invariants and use them to define a first-order deformation of the bracket. A second-order deformation is required in general to ensure that the Jacobi identity is satisfied, but this is uniquely determined by the choice of first-order deformation.

The determination of the filtered deformations is not necessary in order to define Killing spinors, but ultimately if we seek to find all of the supersymmetric backgrounds and their Killing superalgebras (discussed in the next chapter), the filtered deformations should be found.

Chapter 5

Killing spinors and the Killing superalgebra

5.1 Setup

5.1.1 Spinor bundles

Throughout, let (M, g, C, ϕ) (referred to as a *background*) be a connected Lorentzian spin manifold (M, g) , equipped with a $\odot^2 \Delta$ -valued one-form ϕ and a $\bigwedge^2 \Delta$ -valued 2-form C . Borrowing the notation of [4], we denote by $\mathbb{S}$ the spinor bundle (with fibre Σ), by $\mathcal{H}$ the trivial bundle with fibre Δ and by $\mathbb{S}_{\mathbb{C}}$ the tensor bundle $\mathbb{S} \otimes \mathcal{H}$ which of course has fibre $S \otimes \mathbb{C} = \Sigma \otimes_{\mathbb{C}} \Delta$. In practice we work in this latter complex bundle but keep in mind that ultimately we are interested in the real subbundle $\mathbb{S}$ of $\mathbb{S}_{\mathbb{C}}$ which has fibre S . We will denote the space of sections of a bundle $B \rightarrow M$ by $\Gamma(B)$, in particular the space of vector fields is denoted by $\mathfrak{X}(M) = \Gamma(TM)$. The Levi-Civita connection (and its spin lift) will be denoted ∇ .

Definition 5.1.1. We define two different kinds of spinor fields as follows:

- A (*complex*) *spinor field* is a section of the bundle $\mathbb{S}_{\mathbb{C}}$.
- A *symplectic Majorana spinor field* is a section of the bundle $\mathbb{S}$.

Recall that for any vector field X , ∇X defines an endomorphism on the $\mathcal{C}^\infty(M)$ -module of vector fields $\mathfrak{X}(M)$ by $Y \mapsto \nabla_Y X$; on the tangent space at any point it is a linear map. In other words, ∇X is a section of $\text{End}(TM)$; its components are $(\nabla X)^\mu{}_\nu = \nabla_\nu X^\mu$. Furthermore, ∇X is actually a section of $\mathfrak{so}(TM)$ if and only if X is a Killing vector field (this follows directly from the definition), thus if X is a Killing vector, ∇X has the action on spinors $(\nabla X) \cdot s = -\frac{1}{4} \nabla_\mu X_\nu \Gamma^{\mu\nu} s$. This allows us to make the following definition.

Definition 5.1.2. The *spinorial Lie derivative of the spinor field s along the Killing vector field X* is given by

$$\mathcal{L}_X s = \nabla_X s - (\nabla X) \cdot s = \nabla_X s + \frac{1}{4} \nabla_\mu X_\nu \Gamma^{\mu\nu} s \quad (5.1)$$

where $\cdot$ denotes the action of $\mathfrak{so}(TM)$ on spinor fields.

This obeys the Leibniz rule: for a smooth function f and a spinor field s ,

$$\begin{aligned}\mathcal{L}_X(fs) &= \nabla_X(fs) - (\nabla X) \cdot (fs) \\ &= X(f)s + f\nabla_X s - f((\nabla X) \cdot s) \\ &= X(f)s + f\mathcal{L}_X s.\end{aligned}\tag{5.2}$$

We will not prove the following lemma; we refer the reader to [35, 36].

Lemma 5.1.3. *The spinorial Lie derivative obeys the following properties:*

1. *Compatibility with the Lie derivative of vector fields (or representation of the Lie algebra of vector fields on spinor fields):*

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \quad \text{or} \quad \mathcal{L}_X \mathcal{L}_Y s = \mathcal{L}_{[X,Y]} s + \mathcal{L}_Y \mathcal{L}_X s \tag{5.3}$$

for all $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathbb{S}_{\mathbb{C}})$,

2. *Compatibility with the Clifford action*

$$[\mathcal{L}_X, \Phi] = \mathcal{L}_X(\Phi) \quad \text{or} \quad \mathcal{L}_X(\Phi \cdot s) = (\mathcal{L}_X \Phi) \cdot s + \Phi \cdot (\mathcal{L}_X s) \tag{5.4}$$

for all $\Phi \in \Omega^\bullet(M)$ and $s \in \Gamma(\mathbb{S}_{\mathbb{C}})$,

3. *Compatibility with the Levi-Civita connection:*

$$[\mathcal{L}_X, \nabla_Y] = \nabla_{[X,Y]} \quad \text{or} \quad \mathcal{L}_X \nabla_Y s = \nabla_{[X,Y]} s + \nabla_Y \mathcal{L}_X s \tag{5.5}$$

for all $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathbb{S}_{\mathbb{C}})$.

5.1.2 Killing spinors and supersymmetric backgrounds

Using the results of the previous chapter, we can upgrade the Spencer cocycle β to a geometric one-form with values in $\text{End}(\mathbb{S}_{\mathbb{C}})$ defined by

$$\beta_X s = \frac{1}{4}(X \cdot C - 3C \cdot X) \cdot s - \frac{1}{8}(X \cdot \phi + 3\phi \cdot X) \cdot s, \tag{5.6}$$

where s is a complex spinor field and X is a vector field. In components this is

$$(\beta_\mu s)^A = \frac{1}{4}\phi^{\alpha A}{}_{\mu} \Gamma_{\mu\alpha} s^B - \frac{1}{2}\phi_\mu{}^A{}_{\mu} s^B + \frac{1}{8}\epsilon_{\mu\alpha\beta\sigma\tau} C^{\alpha\beta} \Gamma^{\sigma\tau} s^A + C_{\mu\alpha} \Gamma^\alpha s^A. \tag{5.7}$$

We define the *superconnection* on $\mathbb{S}_{\mathbb{C}}$ by

$$\begin{aligned}\mathcal{D}_X s &= \nabla_X s - \beta_X s \\ &= \nabla_X s - \frac{1}{4}(X \cdot C - 3C \cdot X) \cdot s + \frac{1}{8}(X \cdot \phi + 3\phi \cdot X) \cdot s,\end{aligned}\tag{5.8}$$

where X is a vector field and s is a spinor field. The utility of using the Spencer cohomology in this definition will become clear later on when we show that it ensures that the supersymmetry generators form a Lie superalgebra.

Definition 5.1.4. A *Killing spinor field* is a symplectic Majorana spinor field $s \in \Gamma(\mathbb{S})$ which is parallel with respect to the superconnection $\mathcal{D}$; that is, if it is a solution to the *Killing spinor equation*

$$\mathcal{D}_X s = \nabla_X s - \beta_X s = 0, \quad (5.9)$$

or in components,

$$\nabla_\mu s^A - \frac{1}{4} \phi^{\alpha A}{}_B (\Gamma_{\mu\alpha} - 2\eta_{\mu\alpha}) s^B - \frac{1}{8} C^{\alpha\beta} (\epsilon_{\mu\alpha\beta\sigma\tau} \Gamma^{\sigma\tau} + 8\eta_{\mu\alpha} \Gamma_\beta) s^A = 0. \quad (5.10)$$

We take the opportunity now to note that by setting $\phi = 0$, we can obtain the 5-dimensional minimal supergravity spinor equation¹. Our later discussion of maximally supersymmetric backgrounds should thus contain *at least* those of supergravity, but since our Killing spinor equation has extra ϕ -dependent terms, we will find more. We also do not impose any equations of motion, so even the set of backgrounds corresponding to $\phi = 0$ may be larger than that of supergravity.

The Killing spinor equation is linear, so the Killing spinors form a vector space. Since they are the parallel sections of a connection on a vector bundle, Killing spinor fields are completely determined by their value (a choice of spinor) at any point on a connected manifold. The fibre at any point is S , so the maximal dimension of the space of Killing spinor fields is $\dim_{\mathbb{R}} S$. Note that this gives the *maximal* possible number of linearly independent Killing spinor fields on (M, g, C, ϕ) ; the actual number may be less than this or even zero. We therefore make the following definitions

Definition 5.1.5. The background (M, g, C, ϕ) is called *supersymmetric* if it supports a nonzero Killing spinor. It is called *maximally supersymmetric* if the space of Killing spinor fields has maximal dimension, $\dim_{\mathbb{R}} S$.

The supersymmetric backgrounds of 5-dimensional minimal supergravity have been classified [21]. Our goal will be slightly more modest; we will classify the *maximally* supersymmetric backgrounds with respect to our more general notion of Killing spinors. Our starting point is the following observation

Lemma 5.1.6. *A background is maximally supersymmetric if and only if it is flat with respect to the superconnection; that is, if the curvature tensor $R^{\mathcal{D}}$, given by*

$$R^{\mathcal{D}}(X, Y) = \mathcal{D}_{[X, Y]} - [\mathcal{D}_X, \mathcal{D}_Y] \quad (5.11)$$

where X, Y are vector fields, vanishes.

Proof. An *integrability condition* for parallel sections of a vector bundle (a condition which must be satisfied for a locally parallel section to analytically continue to a globally-defined one) is that the curvature tensor annihilates the section. In other words, if s is a Killing spinor, $R^{\mathcal{D}}(X, Y)s = 0$ for all vector fields X, Y . If the space of Killing spinor fields is maximal, this means that $R^{\mathcal{D}}$ must annihilate all spinors; thus it must vanish. $\square$

¹See Equation (2.3) of [21]. That paper uses the notation D for the spin lift of the Levi-Civita connection which we denote by ∇ , and their 2-form field $F = \sqrt{3}C$.

5.2 Conditions for flatness of the superconnection

In this section, we derive a set of conditions on (M, G, C, ϕ) equivalent to flatness of the superconnection. These conditions will later allow us to classify the maximally supersymmetric backgrounds and show that their supersymmetry algebras form Lie superalgebras.²

5.2.1 Rewriting the flatness equation

The vanishing of the curvature Equation (5.11) is equivalent to

$$R(X, Y) \cdot s - \beta_{[X, Y]}s - [\beta_X, \beta_Y]s + [\nabla_X, \beta_Y]s - [\nabla_Y, \beta_X]s = 0 \quad (5.12)$$

for all vector fields X, Y and spinor fields s , where R is the Riemann curvature (so $R(X, Y)$ is a Riemann 2-form which has the Clifford action on spinors). We begin by simplifying this expression. We have

$$\begin{aligned} [\nabla_X, \beta_Y]s &= \nabla_X(\beta_Y s) - \beta_Y \nabla_X s \\ &= \nabla_X(\beta_Y)s + \beta_Y \nabla_X s - \beta_Y \nabla_X s \\ &= \nabla_X(\beta_Y)s \\ &= \nabla_X(\beta)_Y s + \beta_{\nabla_X Y} s, \end{aligned} \quad (5.13)$$

and since ∇ is the Levi-Civita connection, we have $\nabla_X Y - \nabla_Y X = [X, Y]$ so Equation (5.12) is equivalent to

$$R(X, Y) \cdot s - [\beta_X, \beta_Y]s + \nabla_X(\beta)_Y s - \nabla_Y(\beta)_X s = 0. \quad (5.14)$$

Finally, in order to begin computation, we write this in components (abstracting away the spinor field s):

$$\frac{1}{4} R_{\mu\nu\sigma\tau} \Gamma^{\sigma\tau} \delta_B^A = 2\beta_{[\mu}^A \beta_{\nu]}^C \beta_{\tau]B} - 2\nabla_{[\mu} \beta_{\nu]}^A \beta_{\tau]B}. \quad (5.15)$$

The coefficients of $\mathbb{1}$, Γ^σ and $\Gamma^{\sigma\tau}$ in Equation (5.15) on each side of the equation must each match respectively. We will thus group the terms of $\beta_{[\mu} \beta_{\nu]}$ and $\nabla_{[\mu} \beta_{\nu]}$ appropriately. It will also prove fruitful to separate terms by their symmetry in $\otimes^2 \Delta$. To this end, recalling that $\phi \in \odot^2 \Delta$, we compute

$$(\phi_\mu \phi_\nu)_{AB} = \phi_{\mu AC} \phi_{\nu B}^C = \phi_{\nu B}^C \phi_{\mu CA} = -\phi_{\nu CB} \phi_{\mu A}^C = -\phi_{\nu BC} \phi_{\mu A}^C = -(\phi_\nu \phi_\mu)_{BA}, \quad (5.16)$$

so $[\phi_\mu, \phi_\nu] \in \odot^2 \Delta$ and $\{\phi_\mu, \phi_\nu\} \in \wedge^2 \Delta$. It will also be useful to introduce the notation $\langle \cdot \rangle$ to denote the trace in $\text{End } \Delta$. Since ϵ spans $\wedge^2 \Delta$, we can deduce that

$$\{\phi_\mu, \phi_\nu\}_{AB} = \frac{1}{2} \langle \{\phi_\mu, \phi_\nu\} \rangle \epsilon_{AB} \quad \text{and} \quad (\phi^\alpha \phi_\alpha)_{AB} = \frac{1}{2} \langle \phi^\alpha \phi_\alpha \rangle \epsilon_{AB}. \quad (5.17)$$

²The calculations in this section contain errors which lead to the ultimate conclusion Theorem (5.2.1) being incorrect, and also affect the results of later sections. This will not be corrected in this document, but corrected versions of these calculations and results will be published in an upcoming research article.

Turning to the first term on the right hand side of Equation (5.15), we have

$$\begin{aligned}
& \beta_{\mu}^A \beta_{\nu}^C \beta_B^C \\
&= \left(\frac{1}{4} \phi^{\alpha A} \Gamma_{\mu\alpha} - \frac{1}{2} \phi_{\mu}^A \Gamma^A + \frac{1}{8} \epsilon_{\mu\alpha\beta\sigma\tau} C^{\alpha\beta} \Gamma^{\sigma\tau} \delta_C^A + C_{\mu\alpha} \Gamma^{\alpha} \delta_C^A \right) \\
&\quad \left(\frac{1}{4} \phi^{\gamma C} \Gamma_{\nu\gamma} - \frac{1}{2} \phi_{\nu}^C \Gamma^C + \frac{1}{8} \epsilon_{\nu\gamma\delta\lambda\rho} C^{\gamma\delta} \Gamma^{\lambda\rho} \delta_B^C + C_{\nu\gamma} \Gamma^{\gamma} \delta_B^C \right) \\
&= \frac{1}{16} \phi^{\alpha A} \phi^{\gamma C} \Gamma_{\mu\alpha} \Gamma_{\nu\gamma} - \frac{1}{8} \phi^{\alpha A} \phi_{\nu}^C \Gamma_{\mu\alpha} \\
&\quad + \frac{1}{32} \epsilon_{\nu\gamma\delta\lambda\rho} \phi^{\alpha A} \Gamma_{\mu\alpha} \Gamma^{\lambda\rho} + \frac{1}{4} \phi^{\alpha A} C_{\nu\gamma} \Gamma_{\mu\alpha} \Gamma^{\gamma} \\
&\quad - \frac{1}{8} \phi_{\mu}^A \phi^{\gamma C} \Gamma_{\nu\gamma} + \frac{1}{4} \phi_{\mu}^A \phi_{\nu}^C \Gamma^C \\
&\quad - \frac{1}{16} \epsilon_{\nu\gamma\delta\lambda\rho} \phi_{\mu}^A \Gamma^{\lambda\rho} - \frac{1}{2} \phi_{\mu}^A C_{\nu\gamma} \Gamma^{\gamma} \\
&\quad + \frac{1}{32} \epsilon_{\mu\alpha\beta\sigma\tau} \phi^{\gamma A} \Gamma_{\nu\gamma} C^{\alpha\beta} \Gamma^{\sigma\tau} - \frac{1}{16} \epsilon_{\mu\alpha\beta\sigma\tau} \phi_{\nu}^A \Gamma_{\mu\alpha} C^{\alpha\beta} \Gamma^{\sigma\tau} \\
&\quad + \frac{1}{64} \epsilon_{\mu\alpha\beta\sigma\tau} \epsilon_{\nu\gamma\delta\lambda\rho} C^{\alpha\beta} C^{\gamma\delta} \Gamma^{\lambda\rho} \delta_B^A + \frac{1}{8} \epsilon_{\mu\alpha\beta\sigma\tau} C^{\alpha\beta} C_{\nu\gamma} \Gamma^{\sigma\tau} \Gamma^{\gamma} \delta_B^A \\
&\quad + \frac{1}{4} C_{\mu\alpha} \phi^{\gamma A} \Gamma_{\nu\gamma} \Gamma^{\alpha} - \frac{1}{2} C_{\mu\alpha} \phi_{\nu}^A \Gamma^{\alpha} \\
&\quad + \frac{1}{8} \epsilon_{\nu\gamma\delta\lambda\rho} C_{\mu\alpha} C^{\gamma\delta} \Gamma^{\lambda\rho} \delta_B^A + C_{\mu\alpha} C_{\nu\gamma} \Gamma^{\alpha} \Gamma^{\gamma} \delta_B^A \\
&= \frac{1}{16} \phi^{\alpha A} \phi^{\beta C} \Gamma_{\mu\alpha} \Gamma_{\nu\beta} + \frac{1}{4} \phi_{\mu}^A \phi_{\nu}^C \Gamma^C - \frac{1}{8} (\phi^{\alpha A} \phi_{\nu}^C \Gamma_{\mu\alpha} + \phi_{\mu}^A \phi^{\alpha C} \Gamma_{\nu\alpha}) \\
&\quad + \frac{1}{32} C^{\alpha\beta} \phi^{\gamma A} (\epsilon_{\nu\alpha\beta\sigma\tau} \Gamma_{\mu\gamma} \Gamma^{\sigma\tau} + \epsilon_{\mu\alpha\beta\sigma\tau} \Gamma^{\sigma\tau} \Gamma_{\nu\gamma}) \\
&\quad + \frac{1}{4} C^{\alpha\beta} \phi^{\gamma A} (g_{\alpha\nu} \Gamma_{\mu\gamma} \Gamma_{\beta} + g_{\alpha\mu} \Gamma_{\beta} \Gamma_{\nu\gamma}) \\
&\quad - \frac{1}{16} C^{\alpha\beta} (\epsilon_{\nu\alpha\beta\sigma\tau} \phi_{\mu}^A \Gamma^{\sigma\tau} + \epsilon_{\mu\alpha\beta\sigma\tau} \phi_{\nu}^A \Gamma^{\sigma\tau}) - \frac{1}{2} (C_{\nu\alpha} \phi_{\mu}^A + C_{\mu\alpha} \phi_{\nu}^A) \Gamma^{\alpha} \\
&\quad + \frac{1}{64} \epsilon_{\mu\alpha\beta\sigma\tau} \epsilon_{\nu\gamma\delta\lambda\rho} C^{\alpha\beta} C^{\gamma\delta} \Gamma^{\lambda\rho} \delta_B^A + C_{\mu\alpha} C_{\nu\beta} \Gamma^{\alpha} \Gamma^{\beta} \delta_B^A \\
&\quad + \frac{1}{8} \epsilon^{\rho\alpha\beta\sigma\tau} C_{\alpha\beta} C^{\gamma\delta} (g_{\rho\mu} g_{\gamma\nu} \Gamma_{\sigma\tau} \Gamma_{\delta} + g_{\rho\nu} g_{\gamma\mu} \Gamma_{\delta} \Gamma_{\sigma\tau}) \delta_B^A.
\end{aligned} \tag{5.18}$$

Then

$$\begin{aligned}
\beta_{[\mu}^A \beta_{\nu]}^C \beta_B^C &= \frac{1}{16} (\phi^{\alpha} \phi^{\beta})^A \Gamma_{[\mu|\alpha]} \Gamma_{\nu]\beta} + \frac{1}{8} [\phi_{\mu}, \phi_{\nu}]^A \Gamma_B + \frac{1}{8} [\phi^{\alpha}, \phi_{[\mu}]^A \Gamma_{\nu]\alpha} \\
&\quad + \frac{1}{32} C^{\alpha\beta} \phi^{\gamma A} \epsilon_{\alpha\beta\sigma\tau} [\Gamma^{\sigma\tau}, \Gamma_{\nu\gamma}] + \frac{1}{4} \phi^{\gamma A} C_{\beta[\mu} [\Gamma_{\nu]\gamma}, \Gamma^{\beta}] \\
&\quad + \frac{1}{64} \epsilon_{\alpha\beta\sigma\tau} \epsilon_{\mu\nu\gamma\delta\lambda\rho} C^{\alpha\beta} C^{\gamma\delta} \Gamma^{\lambda\rho} \delta_B^A \\
&\quad + C_{[\mu|\alpha]} C_{\nu]\beta} \Gamma^{\alpha} \Gamma^{\beta} \delta_B^A + \frac{1}{8} \epsilon_{\alpha\beta\sigma\tau} C^{\alpha\beta} C_{\nu]}^{\delta} [\Gamma^{\sigma\tau}, \Gamma_{\delta}] \delta_B^A.
\end{aligned} \tag{5.19}$$

We tackle this one term at a time (suppressing Δ indices). It's also useful to note that

$$C_{[\mu}^{\alpha} C_{\nu]\alpha} = 0 \text{ and } C_{[\mu|\sigma|} C_{\nu]\tau} = C_{\mu[\sigma} C_{\nu|\tau]}.$$

$$\begin{aligned} & (\phi^{\alpha} \phi^{\beta}) \Gamma_{[\mu|\alpha|} \Gamma_{\nu]\beta} \\ &= (\phi^{\alpha} \phi_{\beta}) \delta_{[\mu}^{\sigma} g_{\nu]\tau} \Gamma_{\sigma\alpha} \Gamma^{\tau\beta} \\ &= (\phi^{\alpha} \phi_{\beta}) \delta_{[\mu}^{\sigma} g_{\nu]\tau} \left[\Gamma_{\sigma\alpha}^{\tau\beta} + 4\delta_{[\alpha}^{[\tau} \Gamma_{\sigma]}^{\beta]} + 2\delta_{[\alpha}^{[\tau} \delta_{\sigma]}^{\beta]} \right] \\ &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta\sigma} [\phi^{\alpha}, \phi^{\beta}] \Gamma^{\sigma} - \{\phi^{\alpha}, \phi_{[\mu}\} \Gamma_{\nu]\alpha} - (\phi^{\alpha} \phi_{\alpha}) \Gamma_{\mu\nu} + \frac{1}{2} [\phi_{\mu}, \phi_{\nu}], \end{aligned} \quad (5.20)$$

$$\begin{aligned} & C^{\alpha\beta} \phi^{\gamma} \epsilon_{\alpha\beta\sigma\tau} [\Gamma^{\sigma\tau}, \Gamma_{\nu]\gamma}] \\ &= C^{\alpha\beta} \phi^{\gamma} \epsilon_{\alpha\beta\sigma\tau} [\delta_{\nu]}^{\theta} \left(8\delta_{[\theta}^{[\tau} \Gamma_{\gamma]}^{\sigma]} \right) = 4C^{\alpha\beta} \phi^{\gamma} (\epsilon_{\alpha\beta\gamma\sigma} \Gamma_{\nu]}^{\sigma} - \epsilon_{\mu\nu\alpha\beta\sigma} \Gamma_{\gamma}^{\sigma}), \end{aligned} \quad (5.21)$$

$$\begin{aligned} & \phi^{\gamma} C_{\beta[\mu} [\Gamma_{\nu]\gamma}, \Gamma^{\beta}] \\ &= \phi^{\gamma} C_{\beta[\mu} \delta_{\nu]}^{\theta} \left(4\delta_{[\gamma}^{\beta} \Gamma_{\theta]} \right) = 2(C_{\alpha[\mu} \phi^{\alpha} \Gamma_{\nu]} + C_{\mu\nu} \phi^{\alpha} \Gamma_{\alpha}), \end{aligned} \quad (5.22)$$

$$\begin{aligned} & \epsilon_{\alpha\beta\sigma\tau} [\mu \epsilon_{\nu]} \gamma \delta \lambda \rho C^{\alpha\beta} C^{\gamma\delta} \Gamma^{\sigma\tau} \Gamma^{\lambda\rho} \\ &= \epsilon_{\alpha\beta\sigma\tau} [\mu \epsilon_{\nu]} \gamma \delta^{\lambda\rho} C^{\alpha\beta} C^{\gamma\delta} \left[\epsilon^{\sigma\tau}{}_{\lambda\rho\theta} \Gamma^{\theta} + 4\delta_{[\lambda}^{[\tau} \Gamma_{\rho]}^{\sigma]} + 2\delta_{[\lambda}^{[\tau} \delta_{\rho]}^{\sigma]} \right] \\ &= C^{\alpha\beta} C_{\gamma\delta} \left[\epsilon_{\alpha\beta\sigma\tau} [\mu \epsilon_{\nu]}^{\gamma\delta\lambda\rho} \epsilon^{\sigma\tau}{}_{\lambda\rho\theta} \Gamma^{\theta} - 4\epsilon_{\alpha\beta\sigma\tau} [\mu \epsilon_{\nu]}^{\gamma\delta\lambda\tau} \Gamma_{\lambda}^{\sigma} - 2\epsilon_{\alpha\beta\sigma\tau} [\mu \epsilon_{\nu]}^{\gamma\delta\sigma\tau} \right] \\ &= C^{\alpha\beta} C_{\gamma\delta} \delta_{[\mu}^{\phi} g_{\nu]\psi} \left[\epsilon^{\psi\gamma\delta}{}_{\lambda\rho} \epsilon_{\phi\alpha\beta\sigma\tau} \epsilon^{\theta\lambda\rho\sigma\tau} \Gamma_{\theta} - 4\epsilon_{\phi\alpha\beta\sigma\tau} \epsilon^{\psi\gamma\delta\lambda\tau} \Gamma_{\lambda}^{\sigma} - 2\epsilon_{\phi\alpha\beta\sigma\tau} \epsilon^{\psi\gamma\delta\sigma\tau} \right] \\ &= C^{\alpha\beta} C_{\gamma\delta} \delta_{[\mu}^{\phi} g_{\nu]\psi} \left[12\epsilon^{\psi\gamma\delta}{}_{[\alpha\beta} \Gamma_{\phi]} + 96\delta_{[\phi}^{\psi} \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} \Gamma_{\sigma]}^{\sigma} - 24\delta_{[\phi}^{\psi} \delta_{\alpha}^{\gamma} \delta_{\beta]}^{\delta} \right] \\ &= -4C^{\alpha\beta} C^{\gamma\delta} (\epsilon_{\alpha\beta\gamma\delta} [\mu \Gamma_{\nu]} + 2\epsilon_{\mu\nu\gamma\delta\alpha} \Gamma_{\beta}) \\ &\quad - 96C^{\alpha\beta} \delta_{[\mu}^{\phi} g_{\nu][\phi} C_{\alpha\beta} \Gamma_{\sigma]}^{\sigma} - 24C^{\alpha\beta} \delta_{[\mu}^{\phi} g_{\nu][\phi} C_{\alpha\beta}] \\ &= -4C^{\alpha\beta} C^{\gamma\delta} (\epsilon_{\alpha\beta\gamma\delta} [\mu \Gamma_{\nu]} + 2\epsilon_{\mu\nu\gamma\delta\alpha} \Gamma_{\beta}) \\ &\quad - 16(2C^{\alpha\tau} C_{\alpha[\mu} \Gamma_{\nu]\tau} - C_{\sigma[\mu} C_{\nu]\tau} \Gamma^{\sigma\tau} + 3C^{\alpha\beta} C_{\alpha\beta} \Gamma_{\mu\nu}), \end{aligned} \quad (5.23)$$

$$\begin{aligned} & C_{[\mu|\alpha|} C_{\nu]\beta} \Gamma^{\alpha} \Gamma^{\beta} \\ &= (C_{[\mu|\alpha|} C_{\nu]\beta} \Gamma^{\alpha\beta} + C_{[\mu}^{\alpha} C_{\nu]\alpha}) = C_{[\mu|\alpha|} C_{\nu]\beta} \Gamma^{\alpha\beta}, \end{aligned} \quad (5.24)$$

$$\begin{aligned} & \epsilon_{\alpha\beta\sigma\tau} [\mu C^{\alpha\beta} C_{\nu]}^{\delta} [\Gamma^{\sigma\tau}, \Gamma_{\delta}] \\ &= \epsilon_{\alpha\beta\sigma\tau} [\mu C^{\alpha\beta} C_{\nu]}^{\delta} \left(4\delta_{\delta}^{[\tau} \Gamma^{\sigma]} \right) = 4\epsilon_{\alpha\beta\sigma\tau} [\mu C^{\alpha\beta} C_{\nu]}^{\tau} \Gamma^{\sigma}. \end{aligned} \quad (5.25)$$

Equation (5.15) is then

$$\begin{aligned}
& R_{\mu\nu\sigma\tau}\Gamma^{\sigma\tau}\delta_B^A \\
&= 8(\beta_{[\mu}\beta_{\nu]})^A{}_B - 8\nabla_{[\mu}\beta_{\nu]}^A{}_B \\
&= -\frac{1}{4}\epsilon_{\mu\nu\alpha\beta\sigma}[\phi^\alpha, \phi^\beta]^A{}_B\Gamma^\sigma - \frac{1}{2}\{\phi^\alpha, \phi_{[\mu}\}^A{}_B\Gamma_{\nu]\alpha} - \frac{1}{2}(\phi^\alpha\phi_\alpha)^A{}_B\Gamma_{\mu\nu} \\
&\quad + \frac{1}{4}[\phi_\mu, \phi_\nu]^A{}_B + [\phi^\alpha, \phi_{[\mu}]^A{}_B\Gamma_{\nu]\alpha} + [\phi_\mu, \phi_\nu]^A{}_B \\
&\quad + C^{\alpha\beta}\phi^{\gamma A}{}_B(\epsilon_{\alpha\beta\gamma\sigma}[\mu\Gamma^\sigma{}_{\nu]} - \epsilon_{\mu\nu\alpha\beta\sigma}\Gamma^\sigma{}_\gamma) \\
&\quad + 4(C_{\alpha[\mu}\phi^{\alpha A}{}_B\Gamma_{\nu]} + C_{\mu\nu}\phi^{\alpha A}{}_B\Gamma_\alpha) \\
&\quad - \frac{1}{2}C^{\alpha\beta}C^{\gamma\delta}(\epsilon_{\alpha\beta\gamma\delta}[\mu\Gamma_{\nu]} + 2\epsilon_{\mu\nu\gamma\delta\alpha}\Gamma_\beta)\delta_B^A \\
&\quad - 2(2C^{\alpha\tau}C_{\alpha[\mu}\Gamma_{\nu]\tau} - C_{\sigma[\mu}C_{\nu]\tau}\Gamma^{\sigma\tau} + 3C^{\alpha\beta}C_{\alpha\beta}\Gamma_{\mu\nu})\delta_B^A \\
&\quad + 8C_{[\mu|\alpha}C_{\nu]\beta}\Gamma^{\alpha\beta}\delta_B^A + 4\epsilon_{\alpha\beta\sigma\tau}[\mu C^{\alpha\beta}C_{\nu]}{}^\tau\Gamma^\sigma\delta_B^A \\
&\quad - [2g_{\sigma[\nu}\nabla_{\mu]}\phi_\tau^A{}_B\Gamma^{\sigma\tau} - 4\nabla_{[\mu}\phi_{\nu]}^A{}_B + \epsilon_{\alpha\beta\sigma\tau}[\nu\nabla_{\mu]}C^{\alpha\beta}\Gamma^{\sigma\tau}\delta_B^A + 8\nabla_{[\mu}C_{\nu]\sigma}\Gamma^\sigma\delta_B^A] \\
&= \left[6C_{[\mu|\sigma}C_{\nu]\tau} - 4C^\alpha{}_\tau C_{\alpha[\mu}g_{\nu]\sigma} - 6C^{\alpha\beta}C_{\alpha\beta}g_{\mu\sigma}g_{\tau\nu} \right. \\
&\quad \left. - \frac{1}{4}\langle\{\phi_\tau, \phi_{[\mu}\}\rangle g_{\nu]\sigma} - \frac{1}{4}\langle\phi^\alpha\phi_\alpha\rangle g_{\sigma[\mu}g_{\nu]\tau} + \epsilon_{\alpha\beta\sigma\tau}[\mu\nabla_{\nu]}C^{\alpha\beta}] \Gamma^{\sigma\tau}\delta_B^A \right. \\
&\quad + \left[-\frac{1}{2}C^{\alpha\beta}C^{\gamma\delta}(\epsilon_{\alpha\beta\gamma\delta}[\mu g_{\nu]\sigma} + 2\epsilon_{\mu\nu\gamma\delta\alpha}g_{\beta\sigma}) + 4\epsilon_{\alpha\beta\sigma\tau}[\mu C^{\alpha\beta}C_{\nu]}{}^\tau - 8\nabla_{[\mu}C_{\nu]\sigma}] \Gamma^\sigma\delta_B^A \right. \\
&\quad + \left[[\phi_\tau, \phi_{[\mu}]^A{}_B g_{\nu]\sigma} + C^{\alpha\beta}\phi^{\gamma A}{}_B(\epsilon_{\alpha\beta\gamma\sigma}[\mu g_{\nu]\tau} - \epsilon_{\mu\nu\alpha\beta\sigma}g_{\gamma\tau}) + 2g_{\sigma[\mu}\nabla_{\nu]}\phi_\tau^A{}_B] \Gamma^{\sigma\tau} \right. \\
&\quad + \left[4(C_{\alpha[\mu}\phi^{\alpha A}{}_B g_{\nu]\sigma} + C_{\mu\nu}\phi_\sigma^A{}_B) - \frac{1}{4}\epsilon_{\mu\nu\alpha\beta\sigma}[\phi^\alpha, \phi^\beta]^A{}_B \right] \Gamma^\sigma \\
&\quad \left. + \left[\frac{5}{4}[\phi_\mu, \phi_\nu]^A{}_B + 4\nabla_{[\mu}\phi_{\nu]}^A{}_B \right] \right]
\end{aligned} \tag{5.26}$$

5.2.2 Imposing the symmetries of the Riemann tensor

The $\Lambda^2\Delta$ component of the $\Gamma^{\sigma\tau}$ coefficient gives the Riemann tensor:

$$\begin{aligned}
R_{\mu\nu\sigma\tau} &= 6C_{\mu[\sigma}C_{\nu]\tau} - 4C_{[\mu}{}^\alpha g_{\nu][\sigma}C_{\tau]\alpha} - 6C^{\alpha\beta}C_{\alpha\beta}g_{\mu[\sigma}g_{\tau]\nu} \\
&\quad + \frac{1}{4}\langle\{\phi_{[\sigma}, \phi_{\mu]}\}\rangle g_{\nu][\tau]} - \frac{1}{4}\langle\phi^\alpha\phi_\alpha\rangle g_{\sigma[\mu}g_{\nu]\tau} + \epsilon_{\alpha\beta\sigma\tau}[\mu\nabla_{\nu]}C^{\alpha\beta}.
\end{aligned} \tag{5.27}$$

Since $C_{[\mu|\sigma}C_{\nu]\tau} = C_{\mu[\sigma}C_{\nu]\tau}$ and the Riemann tensor $R \in \odot^2(\Lambda^2 V)$, its $\Lambda^2(\Lambda^2 V)$ component vanishes:

$$\epsilon_{\alpha\beta\sigma\tau}[\mu\nabla_{\nu]}C^{\alpha\beta} - \epsilon_{\alpha\beta\mu\nu}[\sigma\nabla_{\tau]}C^{\alpha\beta} = 0. \tag{5.28}$$

Contracting with $\epsilon^{\mu\nu\theta\phi\psi}$,

$$\begin{aligned}
0 &= \epsilon^{\mu\nu\theta\phi\psi} (\epsilon_{\alpha\beta\sigma\tau\mu} \nabla_\nu C^{\alpha\beta} - \epsilon_{\alpha\beta\mu\nu[\sigma} \nabla_{\tau]} C^{\alpha\beta}) \\
&= 24\delta_\alpha^{[\nu} \delta_\beta^\theta \delta_\sigma^\phi \delta_\tau^{\psi]} \nabla_\nu C^{\alpha\beta} - 12\delta_\alpha^{[\theta} \delta_\beta^\phi \delta_{[\sigma}^{\psi]} \nabla_{\tau]} C^{\alpha\beta} \\
&= 24\delta_\sigma^{[\phi} \delta_\tau^{\psi]} \nabla_\nu C^{\nu\theta]} - 12\delta_{[\sigma}^{[\psi} \nabla_{\tau]} C^{\theta\phi]} \\
&= 6(2\delta_\sigma^{[\phi} \delta_\tau^{\psi]} \nabla_\nu C^{|\nu|\theta]} - \delta_\sigma^{[\phi} \nabla_\tau C^{\psi\theta]} - \delta_\tau^{[\psi} \nabla_\sigma C^{\phi\theta]}) - 12\delta_{[\sigma}^{[\psi} \nabla_{\tau]} C^{\theta\phi]} \\
&= -12\left(\delta_\sigma^{[\theta} \delta_\tau^{\phi]} C^{\psi]} + 2\delta_{[\sigma}^{[\theta} \nabla_{\tau]} C^{\phi\psi]}\right),
\end{aligned} \tag{5.29}$$

then contracting σ and θ in this equation gives us

$$\nabla_\tau C^{\phi\psi} + \delta_\tau^{[\phi} \nabla_\sigma C^{\psi]\sigma} = 0, \tag{5.30}$$

and a final contraction of τ and ψ yields $\nabla_\tau C^{\tau\phi} = 0$, which substituted back into the above equation gives us

$$\nabla C = 0. \tag{5.31}$$

Now, recalling that $C^\alpha_{[\mu} C_{\nu]\alpha} = 0$ and $C_{[\mu|\sigma|} C_{\nu]\tau} = C_{\mu[\sigma} C_{|\nu|\tau]}$, applying the algebraic Bianchi identity $R_{[\mu\nu\sigma]\tau} = 0$ to Equation (5.27) gives us

$$C_{[\mu\nu} C_{\sigma]\tau} = 0 \quad \text{or} \quad C \wedge C = 0. \tag{5.32}$$

Equation (5.27) now explicitly obeys the algebraic symmetries of the Riemann tensor, while the $\bigwedge^2 \Delta$ component of the coefficient of Γ^σ in Equation (5.26) vanishes identically.

5.2.3 Solving the remaining equations

Turning now to the $\odot^2 \Delta$ component of Equation (5.26), again the coefficients of 1 , Γ^σ and $\Gamma^{\sigma\tau}$ must vanish separately:

$$[\phi_{[\tau}, \phi_{|\mu]} g_{\nu]|\sigma}] + C^{\alpha\beta} \phi^\gamma (\epsilon_{\alpha\beta\gamma[\sigma|\mu} g_{\nu]|\tau]} - \epsilon_{\mu\nu\alpha\beta[\sigma} g_{\tau]\gamma}) + 2g_{[\sigma|\mu} \nabla_{\nu]} \phi_{\tau]} = 0, \tag{5.33}$$

$$4(C_{\alpha[\mu} \phi^\alpha g_{\nu]\sigma} + C_{\mu\nu} \phi_\sigma) - \frac{1}{4} \epsilon_{\mu\nu\alpha\beta\sigma} [\phi^\alpha, \phi^\beta] = 0, \tag{5.34}$$

$$\frac{5}{4} [\phi_\mu, \phi_\nu] + 4\nabla_{[\mu} \phi_{\nu]} = 0. \tag{5.35}$$

Contracting ν and σ in Equation (5.34) gives $C_{\mu\alpha} \phi^\alpha = 0$, and thus we have

$$C_{\mu\nu} \phi_\sigma = \frac{1}{16} \epsilon_{\mu\nu\sigma\alpha\beta} [\phi^\alpha, \phi^\beta] \quad \text{or} \quad [\phi_\mu, \phi_\nu] = \frac{4}{3} \epsilon_{\mu\nu\sigma\alpha\beta} C^{\alpha\beta} \phi^\sigma. \tag{5.36}$$

On the other hand, contracting the same indices in Equation (5.33) gives

$$\frac{1}{2} [\phi_\tau, \phi_\mu] - \frac{1}{4} C^{\alpha\beta} \phi^\gamma \epsilon_{\alpha\beta\gamma\tau\mu} - \frac{3}{2} \nabla_\mu \phi_\tau - \frac{1}{2} g_{\mu\tau} \nabla_\sigma \phi^\sigma = 0, \tag{5.37}$$

which upon substitution of Equation (5.36) yields

$$\frac{1}{6} [\phi_\tau, \phi_\mu] - \frac{3}{2} \nabla_\mu \phi_\tau - \frac{1}{2} g_{\mu\tau} \nabla_\sigma \phi^\sigma = 0. \tag{5.38}$$

The symmetric part of this equation is

$$3\nabla_{(\tau}\phi_{\mu)} + g_{\mu\tau}\nabla_{\sigma}\phi^{\sigma} = 0. \quad (5.39)$$

Contracting the indices, we see that $\nabla_{\sigma}\phi^{\sigma}=0$, so also $\nabla_{(\tau}\phi_{\mu)} = 0$. The skew-symmetric part of Equation (5.38) is

$$[\phi_{\tau}, \phi_{\mu}] + 9\nabla_{[\tau}\phi_{\mu]} = 0, \quad (5.40)$$

thus comparing with Equation (5.35) and then with Equation (5.38), we have

$$[\phi_{\mu}, \phi_{\nu}] = 0, \quad (5.41)$$

$$\nabla\phi = 0. \quad (5.42)$$

Together, these imply that $\phi = \varphi \otimes r$, for a parallel vector field φ and a constant $r \in \odot^2 \Delta$. Finally, since the commutator vanishes, Equation (5.36) gives

$$C \otimes \phi = 0 \quad (5.43)$$

so that either $C = 0$ or $\phi = 0$. Note that since $\nabla C = 0$ and $\nabla\phi = 0$, $\nabla R = 0$; in particular the differential Bianchi identity is trivial.

In summary, in this section we have proven the following theorem.

Theorem 5.2.1. *Let (M, g, C, ϕ) be a 5-dimensional background with superconnection given by Equation (5.8). If it is maximally supersymmetric then $\nabla R = 0$ and there are two nontrivial cases:*

1. $C = 0, \phi \neq 0$:

- $\phi = \varphi \otimes r$ for some parallel one-form φ and some $r \in \odot^2 \Delta$,
- the Riemann, Ricci and scalar curvatures are given by

$$R_{\mu\nu\sigma\tau}^{\phi} = \frac{1}{4} [2\varphi_{[\sigma}\varphi_{|\mu}g_{\nu|\tau]} - \varphi^{\alpha}\varphi_{\alpha}g_{\mu[\sigma}g_{\tau]\nu}] \langle r^2 \rangle, \quad (5.44)$$

2. $\phi = 0, C \neq 0$:

- C is a parallel 2-form such that $C \wedge C = 0$,
- the Riemann, Ricci and scalar curvatures are given by

$$R_{\mu\nu\sigma\tau}^C = 6C_{\mu[\sigma}C_{|\nu|\tau]} - 4C_{[\mu}^{\alpha}g_{\nu][\sigma}C_{\tau]\alpha} - 6C^{\alpha\beta}C_{\alpha\beta}g_{\mu[\sigma}g_{\tau]\nu}, \quad (5.45)$$

5.3 The Killing superalgebra

We now define the Killing superalgebra of a supersymmetric background and demonstrate sufficient conditions for it to be a Lie superalgebra.

5.3.1 Dirac fields

First, we extend the Dirac bilinears of Definition 3.2.9 to sections of the appropriate bundles on our spin manifold. All of the bilinears were defined in terms of the quaternionic structure on the Clifford algebra and spin module, so given a Clifford bundle $\mathcal{CL}(M)$ and associated spin bundles $\mathbb{S}$ and $\mathbb{S}$ we can simply lift those definitions and define the following fibre-wise. By symmetry, we only need the values on the diagonal.

Definition 5.3.1. On a background (M, g, C, ϕ) , for each spinor field s , we define

1. The *Dirac scalar*, given by

$$\mu_p(s, s) = \epsilon_{AB} \bar{s}_p^A s_p^B \quad (5.46)$$

for $p \in M$,

2. The *Dirac 1-form*

$$\kappa(s, s)^\flat(X) = \mu(s, X \cdot s) \quad (5.47)$$

for all $X \in \mathfrak{X}(M)$,

3. The *Dirac current* $\kappa(s, s)$, which is the dual vector field to the Dirac 1-form; that is, κ is given by

$$g(\kappa(s_1, s_2), X) = \mu(s_1, X \cdot s_2) \quad (5.48)$$

for all $X \in \mathfrak{X}(M)$,

4. the *Dirac 2-form* with values in $\odot^2 \Delta$

$$\omega(s, s)^{AB}(X.Y) = \bar{s}^A(X \wedge Y) \cdot s^B \quad (5.49)$$

for all $X, Y \in \mathfrak{X}(M)$.

In components,

$$\kappa_\mu(s, s) = \epsilon_{AB} \bar{s}^A \Gamma_\mu s^B, \quad (5.50)$$

$$\omega_{\mu\nu}^{AB}(s, s) = \epsilon_{AB} \bar{s}^A \Gamma_{\mu\nu} s^B. \quad (5.51)$$

5.3.2 The Killing superalgebra

Definition 5.3.2. The *Killing superalgebra* of the background (M, g, C, ϕ) is $\mathfrak{K} = \mathfrak{K}_0 \oplus \mathfrak{K}_1$ where

$$\mathfrak{K}_0 = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X g = \mathcal{L}_X C = \mathcal{L}_X \phi = 0\}, \quad (5.52)$$

is the space of *Killing vector fields* (note that this is more restrictive than the usual definition of Killing vectors) and

$$\mathfrak{K}_1 = \{s \in \Gamma(\mathbb{S}) \mid \mathcal{D}_X s = 0\} \quad (5.53)$$

is the space of *Killing spinor fields*. There is a superbracket $[\cdot, \cdot]$ on $\mathfrak{K}$ which is defined as:

- the vector field commutator on $\mathfrak{K}_0 \otimes \mathfrak{K}_0$, $[X, Y] = X \circ Y - Y \circ X$,
- the Dirac current on $\mathfrak{K}_1 \otimes \mathfrak{K}_1$, $[s, t] = \kappa(s, t)$,
- the spinorial Lie derivative on $\mathfrak{K}_0 \otimes \mathfrak{K}_1$, $[X, s] = \mathcal{L}_X s$ (and $[s, X] = -\mathcal{L}_X s$ on $\mathfrak{K}_1 \otimes \mathfrak{K}_0$).

Theorem 5.3.3. *If (M, g, C, ϕ) is a background such that*

- *either $C = 0$ or $\phi = 0$,*
- *$\nabla C = 0$ and $C \wedge C = 0$,*
- *$\phi = \varphi \otimes r$ for a one-form φ such that $\nabla \varphi = 0$ and $r \in \odot^2 \Delta$,*

(in particular if the background is maximally supersymmetric) then the Killing superalgebra as described above is a Lie superalgebra.

The remainder of this section is dedicated to proving this statement. We begin with a couple of lemmas which will be required later on.

Lemma 5.3.4. *The Dirac current is equivariant under the action of $\mathcal{L}$;*

$$\mathcal{L}_X \kappa(s, s) = 2\kappa(\mathcal{L}_X s, s) \quad (5.54)$$

for all spinor fields s and Killing vector fields X .

Proof. Let X be a Killing vector field, Y a vector field and s a spinor field. Then

$$\begin{aligned} Y \cdot ((\nabla X) \cdot s) &= -\frac{1}{4} Y^\lambda \nabla_\mu X_\nu \Gamma_\lambda \Gamma^{\mu\nu} s \\ &= -\frac{1}{4} Y_\lambda \nabla_\mu X_\nu \Gamma^{\lambda\mu\nu} s - \frac{1}{2} Y^\mu \nabla_\mu X_\nu \Gamma^\nu s \\ &= -\frac{1}{8} \epsilon^{\lambda\mu\nu\sigma\tau} Y_\lambda \nabla_\mu X_\nu \Gamma_{\sigma\tau} s - \frac{1}{2} (\nabla_Y X) \cdot s \end{aligned} \quad (5.55)$$

and so, since $\mu(s, \Gamma_{\sigma\tau} s) = 0$,

$$\mu(s, Y \cdot (\nabla X) \cdot s) = -\frac{1}{2} \mu(s, (\nabla_Y X) \cdot s). \quad (5.56)$$

Using this and the definition of the Dirac current, we have

$$\begin{aligned} g(\mathcal{L}_X \kappa(s, s), Y) &= Xg(\kappa(s, s), Y) - g(\kappa(s, s), \mathcal{L}_X Y) \\ &= g(\nabla_X \kappa(s, s), Y) + g(\kappa(s, s), \nabla_X Y) - g(\kappa(s, s), [X, Y]) \\ &= 2g(\kappa(s, \nabla_X s), Y) + g(\kappa(s, s), \nabla_Y X) \\ &= \mu(s, 2Y \cdot (\nabla_X s) + (\nabla_Y X) \cdot s) \\ &= \mu(s, 2Y \cdot [\nabla_X s - (\nabla X) \cdot s]) \\ &= 2\mu(s, Y \cdot \mathcal{L}_X s) \\ &= 2g(\kappa(s, \mathcal{L}_X s), Y), \end{aligned} \quad (5.57)$$

hence the result, since Y is arbitrary. $\square$

Lemma 5.3.5. *If s is a Killing spinor field, the gradient of its Dirac current is*

$$\nabla \kappa(s, s) = -\gamma(s, s) \quad \text{or} \quad \nabla_\mu \kappa_\nu = \gamma_{\mu\nu}. \quad (5.58)$$

Proof. Recall that κ is symmetric and $\nabla_X s = \beta_X s$, so

$$\nabla_X \kappa(s, s) = 2\kappa(s, \nabla_X s) = 2\kappa(s, \beta_X s) = -\gamma(s, s)X, \quad (5.59)$$

where we have used the cocycle condition Equation (4.68). $\square$

Lemma 5.3.6. *The spinorial Lie derivative is compatible with the superconnection:*

$$[\mathcal{L}_X, \mathcal{D}_Y] = \mathcal{D}_{[X, Y]} \quad \text{or} \quad \mathcal{L}_X \mathcal{D}_Y s = \mathcal{D}_{[X, Y]} s + \mathcal{D}_Y \mathcal{L}_X s \quad (5.60)$$

for all $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathbb{S}_{\mathbb{C}})$.

Proof. Using Lemma 5.1.3 and the definition of $\mathcal{D}$,

$$\begin{aligned} [\mathcal{L}_X, \mathcal{D}_Y]s &= \mathcal{L}_X \mathcal{D}_Y s - \mathcal{D}_Y \mathcal{L}_X s \\ &= \mathcal{L}_X \nabla_Y s - \mathcal{L}_X \beta_Y s - \nabla_Y \mathcal{L}_X s + \beta_Y \mathcal{L}_X s \\ &= [\mathcal{L}_X, \nabla_Y]s - \beta_Y \mathcal{L}_X s - \beta_{(\mathcal{L}_X Y)} s + \beta_Y \mathcal{L}_X s \\ &= \nabla_{[X, Y]} s - \beta_{[X, Y]} s \\ &= \mathcal{D}_{[X, Y]} s. \end{aligned} \quad (5.61)$$

$\square$

Proof of Theorem 5.3.3. Throughout, let s be a Killing spinor field and let μ, κ and ω be the norm, current and 2-form associated to s (suppressing (s, s) in the notation). First, note that the super skew-symmetry of the bracket follows directly from the definition. We will next show that $\mathfrak{K}$ is closed. We begin by recalling the beginning of the usual derivation of the Killing equation, as we can also use the same techniques on the Lie derivative of C and ϕ . For all vector fields X, Y ,

$$\begin{aligned} (\mathcal{L}_\kappa g)(X, Y) &= \mathcal{L}_\kappa(g(X, Y)) - g(\mathcal{L}_\kappa X, Y) - g(X, \mathcal{L}_\kappa Y) \\ &= (\nabla_\kappa g)(X, Y) + g(\nabla_\kappa X, Y) + g(X, \nabla_\kappa Y) - g([\kappa, X], Y) - g(X, [\kappa, Y]) \\ &= g(\nabla_X \kappa, Y) + g(X, \nabla_Y \kappa) \\ &= -g(\gamma(s, s)X, Y) - g(X, \gamma(s, s)Y) \\ &= 0, \end{aligned} \quad (5.62)$$

where we have used the defining properties of the Levi-Civita connection, Lemma 5.3.5, and the fact that γ takes values in $\mathfrak{so}(V)$. Note that this means that the Dirac current of a Killing spinor field being a Killing vector field is equivalent to the cocycle condition Equation (4.68).

Now recall that

$$\gamma_{\mu\nu}(s, s) = \frac{1}{4} \epsilon_{\mu\nu\alpha\sigma\tau} \phi^\alpha_{AB} \omega^{\sigma\tau AB} + 2C_{\mu\nu} \mu + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta\sigma} C^{\alpha\beta} \kappa^\sigma, \quad (5.63)$$

We have

$$(\mathcal{L}_\kappa C)(X, Y) = (\nabla_\kappa)(X, Y) - C(\gamma(s, s)X, Y) - C(X, \gamma(s, s)Y), \quad (5.64)$$

or in components,

$$\begin{aligned} (\mathcal{L}_\kappa C)_{\mu\nu} &= \kappa^\lambda \nabla_\lambda C_{\mu\nu} + 2C_{\lambda[\mu} \gamma(s, s)^{\lambda}_{\nu]} \\ &= \kappa^\lambda \nabla_\lambda C_{\mu\nu} + \frac{1}{2} C_{\lambda[\mu} \epsilon^{\lambda}_{\nu]\alpha\sigma\tau} \phi^\alpha_{AB} \omega^{\sigma\tau AB} \\ &\quad + 4C_{\lambda[\mu} C^{\lambda}_{\nu]} \mu + C_{\lambda[\mu} \epsilon^{\lambda}_{\nu]\alpha\beta\sigma} C^{\alpha\beta} \kappa^\sigma \\ &= \kappa^\lambda \nabla_\lambda C_{\mu\nu} + \frac{1}{2} C_{\lambda[\mu} \epsilon^{\lambda}_{\nu]\alpha\sigma\tau} \phi^\alpha_{AB} \omega^{\sigma\tau AB} + C_{\lambda[\mu} \epsilon^{\lambda}_{\nu]\alpha\beta\sigma} C^{\alpha\beta} \kappa^\sigma. \end{aligned} \quad (5.65)$$

For $\phi = \varphi \otimes r$, we actually just need $\mathcal{L}_\kappa \varphi$ to vanish. Indeed,

$$\begin{aligned} (\mathcal{L}_\kappa \varphi)(X) &= \mathcal{L}_\kappa(\varphi(X)) - \varphi(\mathcal{L}_\kappa X) \\ &= (\nabla_\kappa \varphi)(X) + \varphi(\nabla_\kappa X) - \varphi(\nabla_\kappa X - \nabla_X \kappa) \\ &= (\nabla_\kappa \varphi)(X) - \varphi(\gamma(s, s)X), \end{aligned} \quad (5.66)$$

where we have again used Lemma 5.3.5; hence in components,

$$\begin{aligned} (\mathcal{L}_\kappa \varphi)_\mu &= \kappa^\lambda \nabla_\lambda \varphi_\mu + \gamma_{\mu\nu} \varphi^\nu \\ &= \kappa^\lambda \nabla_\lambda \varphi_\mu + \frac{1}{4} \epsilon_{\mu\nu\alpha\sigma\tau} \varphi^\nu \varphi^\alpha \omega^{\sigma\tau AB} r_{AB} + 2C_{\mu\nu} \varphi^\nu \mu + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta\sigma} \varphi^\nu C^{\alpha\beta} \kappa^\sigma \\ &= \kappa^\lambda \nabla_\lambda \varphi_\mu + 2C_{\mu\nu} \varphi^\nu \mu + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta\sigma} \varphi^\nu C^{\alpha\beta} \kappa^\sigma. \end{aligned} \quad (5.67)$$

Thus $\mathcal{L}_\kappa C = \mathcal{L}_\kappa \phi = 0$ if the conditions in the statement of the theorem hold³.

We have thus shown that $[\mathfrak{K}_1, \mathfrak{K}_1] \subseteq \mathfrak{K}_0$. That $[\mathfrak{K}_0, \mathfrak{K}_0] \subseteq \mathfrak{K}_0$ follows from the fact that $\mathcal{L}$ is a Lie algebra representation of vector fields on the space of tensor fields. It remains to be shown that $[\mathfrak{K}_1, \mathfrak{K}_0] + [\mathfrak{K}_0, \mathfrak{K}_1] \subseteq \mathfrak{K}_1$; that is, $\mathcal{L}_K s$ is Killing for all Killing vector fields K and Killing spinor fields s . Indeed, for any vector field X , using Lemma 5.3.6,

$$\mathcal{D}_X \mathcal{L}_K s = \mathcal{D}_{[X, K]} s + \mathcal{L}_K \mathcal{D}_X s = 0. \quad (5.68)$$

It remains to be shown only that the Jacobi identity is satisfied. For three Killing vector fields this is clear because the bracket is simply the commutator. For two Killing vector fields K, L and a Killing spinor field s ,

$$[s, [K, L]] + [K, [L, s]] - [L, [s, K]] = -\mathcal{L}_{[K, L]} s + \mathcal{L}_K \mathcal{L}_L s - \mathcal{L}_L \mathcal{L}_K s, \quad (5.69)$$

so the Jacobi identity is equivalent to $\mathcal{L}$ being a Lie algebra representation of vector fields on spinor fields, which follows from Lemma 5.1.3. By symmetry, for the one Killing

³Note, however, that if the conditions of the theorem *do not* hold, the vanishing of these derivatives produces a system of first-order nonlinear differential equations. For any solution (C, φ) to these equations, the Killing superalgebra still closes. This would produce a more general class of (non-maximally) supersymmetric backgrounds.

field and two Killing spinor field case, we need only consider the identity where both spinor fields are the same:

$$[K, [s, s]] + [s, [s, K]] - [s, [K, s]] = [K, \kappa(s, s)] - \kappa(s, \mathcal{L}_K s) - \kappa(s, \mathcal{L}_K s), \quad (5.70)$$

so this Jacobi identity is equivalent to κ being equivariant under the representation $\mathcal{L}$ of Killing vector fields on spinor and vector fields, which is simply Lemma 5.3.4. Finally, for three Killing spinor fields, again we need only consider the case where they are all the same; this reduces to the vanishing of

$$\begin{aligned} [[s, s], s] &= \mathcal{L}_\kappa s = \nabla_\kappa s + \frac{1}{4} \nabla_\mu \kappa_\nu \Gamma^{\mu\nu} s \\ &= \beta_\kappa s + \frac{1}{4} \gamma_{\mu\nu}(s, s) \Gamma^{\mu\nu} s \\ &= \beta_\kappa s + \gamma(s, s) \cdot s, \end{aligned} \quad (5.71)$$

which is equivalent to the cocycle condition Equation (4.69). $\square$

5.4 Maximally supersymmetric geometries

We now seek to classify the maximally supersymmetric backgrounds (M, g, C, ϕ) . Our starting point is Theorem 5.2.1. In all cases, the Riemannian curvature is covariantly constant ($\nabla R = 0$) so (M, g) is a so-called *locally symmetric space*.

We will argue that it is possible to assume, up to local isometry, that (M, g) is in fact a *symmetric space*, meaning that for all $x \in M$, there exists an isometry θ_x which is involutive ($\theta_x^2 = \text{Id}_M$) and for which x is an isolated fixed point. This definition is given only for completeness and will not be used directly; we will use without proof certain properties of symmetric spaces and defer full discussion to the literature [37, 38].

The Riemannian symmetric spaces were classified by Cartan [39], while the Lorentzian case was elucidated by Cahen and Wallach [40]. The general pseudo-Riemannian case remains an open problem [37, 38].

5.4.1 Geometric toolbox

Here follows a collection of geometric definitions and results which we will use to classify our symmetric backgrounds. It will allow us to first simplify the problem at hand by only considering the local geometry, then to break down the spaces into simpler components, and finally to identify the components.

Definition 5.4.1. A smooth map $f : M \rightarrow N$ is *local diffeomorphism* if for each $x \in M$, there exists an open neighbourhood U_x such that $f(U_x)$ is open and $f|_{U_x} : U_x \rightarrow f(U_x)$ is a diffeomorphism.

Such a local diffeomorphism of pseudo-Riemannian manifolds (M, g_M) and (N, g_N) which preserves the metric: $f_* g_N = g_M$; that is, $g_M(X, Y) = g_N(df(X), df(Y))$, is called a *local isometry*.

Global isometry between two spaces is the natural notion of equivalence for pseudo-Riemannian manifolds, while local isometry is a slightly weaker notion of equivalence. We simplify our task by classifying supersymmetric backgrounds only up to local isometry.

Recall that a *universal cover* $(\tilde{X}, \pi)$ of a connected topological space X is a simply-connected topological space $\tilde{X}$ and a covering map $\pi : \tilde{X} \rightarrow X$. If X satisfies certain topological conditions (which are satisfied if it is a connected manifold), the universal cover exists and is unique. Furthermore, any smooth structure or metric on X pull back along the covering map to the universal cover $\tilde{X}$, meaning that it is a pseudo-Riemannian manifold whenever X is, and π is then a local isometry. It can be shown that (M, g) is locally symmetric if and only if its universal cover is a symmetric space, so since we only seek to classify (M, g) up to local isometry, we assume without loss of generality that (M, g) is a simply-connected Lorentzian symmetric space.

Symmetric spaces are in particular homogeneous, meaning that the group of isometries G of M acts transitively on M (that is, for all pairs of points $x, y \in M$, there exists some $g \in G$ such that $y = gx$), so there is a G -equivariant diffeomorphism $M \cong G/H$ where H is the stabiliser of some arbitrarily chosen point $o \in M$. The actual stabiliser subgroup H depends on the choice of o , but the manifold G/H is always diffeomorphic to M . This observation allows us to effectively localise at a point $o \in M$.

Definition 5.4.2. A pseudo-Riemannian manifold (M, g) is called *decomposable* if it is locally isometric to a product $(M_1, g_1) \times (M_2, g_2)$ for some pseudo-Riemannian manifolds⁴ (M_1, g_1) , (M_2, g_2) , otherwise it is called *indecomposable*.

We will now give a result which will assist us in decomposing spaces into products, the de Rham–Wu Decomposition Theorem. For this we’ll need the notion of *holonomy*. We give now a general definition of holonomy, but we will actually only need it for the Levi-Civita connection on the tangent bundle. A modern treatment of holonomy on general pseudo-Riemannian spaces is given in [41].

Definition 5.4.3. Let $E \rightarrow B$ be a smooth vector bundle equipped with a connection ∇ . Then given two points $x, y \in B$, a path γ from x to y defines an invertible linear map $E_x \rightarrow E_y$ by parallel transportation.

The *holonomy group* of ∇ at a point $x \in B$ is the subgroup $\text{Hol}_x(B, \nabla)$ of $GL(E_x)$ consisting of linear transformations which arise as parallel transports around a loop based at x .

Because the Levi-Civita connection ∇ of a pseudo-Riemannian manifold (M, g) preserves the metric, parallel transportation preserves the inner product on tangent spaces; in particular at any point $x \in M$ the holonomy group $\text{Hol}_x(M, \nabla)$ preserves the inner product on $T_x M$ and so is a subgroup of $O(T_x M)$. In fact, it is a Lie group, and its identity component lies in $SO(T_x M)$.

We can now state a version of the de Rham–Wu theorem. This uses the language of *smooth distributions* of the tangent bundle and their *maximal integral submanifolds*. We

⁴Note that if the signature of g is (p, q) and the signature of g_i is (p_i, q_i) , we must have $p = p_1 + p_2$, $q = q_1 + q_2$. In particular, if (M, g) is Riemannian then so are its factors, and if it is Lorentzian then one factor must be Lorentzian and the other negative-Riemannian (its metric is negative-definite).

won't fully define these concepts here, but intuitively we consider a smooth distribution D of the tangent bundle TM to be a choice of a subspace D_x of the tangent space $T_x M$ at each $x \in M$ which smoothly varies as we vary x ; an integral submanifold of such a distribution is a smooth manifold N with an immersion $i : N \hookrightarrow M$ such that at each point $x \in N$ the image of $di_x : T_x N \hookrightarrow T_x M$ is D_x . An integral submanifold is maximal if it is not a proper submanifold of any other integral manifold of D ⁵

Theorem 5.4.4 (de Rham–Wu [42, 43]). *Let (M, g) be a simply connected pseudo-Riemannian manifold and ∇ its Levi-Civita connection. Suppose that at some point, the action of $\text{Hol}_x(M, \nabla)$ on $T_x M$ preserves some proper, nontrivial, nondegenerate subspace $T_x^1 M$. Then*

1. *The action of $\text{Hol}_x(M, \nabla)$ also preserves the orthogonal subspace $T_x^2 M = (T_x^1 M)^\perp$, so there is a decomposition*

$$T_x M = T_x^1 M \oplus T_x^2 M, \quad (5.72)$$

preserved by $\text{Hol}_x(M, \nabla)$,

2. *This decomposition extends to a decomposition of the tangent bundle (given by parallel transportation of vectors at x)*

$$TM = T^1 M \oplus T^2 M, \quad (5.73)$$

where $T^1 M$ and $T^2 M$ are distributions on the tangent bundle with fibres $T_x^1 M$ and $T_x^2 M$ at x , which are orthogonal and preserved by holonomy in the sense that at any point $y \in M$, $T_y^2 M = (T_y^1 M)^\perp$ and $\text{Hol}_y(M, \nabla)$ preserves $T_y^1 M$ and $T_y^2 M$,

3. *Both of these distributions are integrable, with maximal integral submanifolds M_1, M_2 . These submanifolds are simply-connected and can be given the metrics g_1 and g_2 which are pullbacks of g along the immersion maps,*
4. *The original manifold (M, g) is decomposable as a product of its integral submanifolds; there is an isometry*

$$(M, g) \cong (M_1, g_1) \times (M_2, g_2), \quad (5.74)$$

5. *If M is symmetric, so are its integral submanifolds.*

This gives us a practical tool which we can use to decompose our Lorentzian symmetric space into products of Lorentzian and negative-Riemannian symmetric spaces. The next step will be to identify the factors. The following theorem of Cahen and Wallach, as stated in [44] (but with our mostly-minus convention for the metric signature) will help:

Theorem 5.4.5 (Cahen–Wallach[40, 44]). *Let (M, g) be a simply-connected Lorentzian symmetric space. Then it is isometric to a product $(L, g_L) \times (R, -g_R)$ where (L, g_L) is either*

⁵A distribution is not guaranteed to have any integral submanifolds; if it does it is called *integrable* and its maximal submanifold is unique up to diffeomorphism. Frobenius' Theorem gives a necessary and sufficient condition for a distribution to be integrable.

- $(\mathbb{R}, dt^2)$, the real line with the standard metric,
- UdS , the universal cover of de Sitter space dS ,
- $UAdS$, the universal cover of anti-de Sitter space AdS ,
- $CW_d(A)$, for $d \geq 3$, a Cahen-Wallach pp-wave space as described below,

and (R, g_R) is a simply-connected Riemannian symmetric space.

The Cahen-Wallach space is a type of *Brinkmann space* or *plane-fronted parallel-ray wave (pp-wave)*. Pp-waves can be defined as Lorentzian manifolds which support a nonzero parallel null vector field [45, 46]; they are a model for spacetime through which radiation is propagating. A Cahen-Wallach space (fully constructed in eg. [40, 44]) $CW_d(A)$ is a particular family of pp-waves with coordinates $(x^+, x^-, x^1, \dots, x^{d-2})$ and the metric

$$g = 2dx^+dx^- - \left(\sum_{i,j=1}^{d-2} A_{ij}x^i x^j \right) (dx^-)^2 - \sum_{i=1}^{d-2} (dx^i)^2, \quad (5.75)$$

where $A \in \mathbb{R}(d-2)$ with $A_{ij} = A_{ji}$ (indeed, in the full construction A appears as a symmetric bilinear form). The parallel null vector field in such a space is ∂_+ . It is useful to make some brief comments about these spaces:

- The Latin indices $i, j, k \dots$ take values in $\{1, \dots, d-2\}$ and we will use the Einstein summation convention on these indices: $a_i b^i := \sum_i a_i b^i$.
- Minkowski space can be seen as a trivial CW pp-wave, $CW_d(0) = \mathbb{R}^{1,(d-1)6}$.
- Two such spaces $CW_d(A)$ and $CW_d(B)$ are isometric if and only if $A_{ij} = c O^k_i O^l_j B_{kl}$ for some $c \in \mathbb{R}$ and some orthogonal matrix O .
- The only nonzero components of the Riemann and Ricci tensors and the scalar curvature are

$$R_{i-j-} = A_{ij}, \quad R_{--} = \eta^{ij} A_{ij}, \quad R = 0. \quad (5.76)$$

- For $d \geq 4$, the Weyl tensor vanishes (equivalently $CW_d(A)$ is conformally flat) if and only if A is a scalar matrix: $A_{ij} = a \eta_{ij}$ where $a = \frac{1}{d-2} \eta^{ij} A_{ij}$.
- $CW_d(A)$ is indecomposable if and only if A is nondegenerate.

5.4.2 $C = 0, \phi = 0$

We are now ready to put this geometric toolbox to work. There are three main cases to check. The first case, $C = 0, \phi = 0$, is essentially trivial; here the Riemann curvature vanishes, so (M, g) is Minkowski space by the Cahen-Wallach theorem. In the remainder of this section, we deal with the other cases.

⁶Setting $A = 0$ in Equation (5.75) recovers the Minkowski metric in lightcone coordinates, $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1)$, $x^- = \frac{1}{\sqrt{2}}(x^0 - x^1)$.

5.4.3 $C = 0$, $\phi \neq 0$

This case is virtually identical to the $H = 0$ case in 6 dimensions (See Proposition 26 and the material following it in [4]); we reproduce the argument here, although note that [4] uses a mostly-plus signature so some modifications are required.

By Theorem 5.2.1, $\phi = \varphi \otimes r$ for a parallel one-form φ and $r \in \bigodot^2 \Delta$. It will be useful in the following to regard φ as a vector field rather than as a one-form (that is, we identify it with its dual vector field). Without loss of generality we can choose the trace $\langle r^2 \rangle = -4$, and then the Riemann, Ricci and scalar curvatures are

$$R_{\mu\nu\sigma\tau} = \varphi^2 g_{\mu[\sigma} g_{\tau]\nu} - 2\varphi_{[\sigma} \varphi_{|\mu} g_{\nu]|\tau]}, \quad R_{\mu\nu} = \frac{3}{2}(\varphi^2 g_{\mu\nu} - \varphi_\mu \varphi_\nu), \quad R = 6\varphi^2. \quad (5.77)$$

First we note that the Weyl tensor vanishes:

$$\begin{aligned} C_{\mu\nu\sigma\tau} &= R_{\mu\nu\sigma\tau} - \frac{4}{D-2} g_{[\mu[\sigma} R_{\tau]|\nu]} + \frac{2}{(D-1)(D-2)} R g_{\mu[\sigma} g_{\tau]\nu} \\ &= \varphi^2 g_{\mu[\sigma} g_{\tau]\nu} + 2\varphi_{[\mu} g_{\nu][\sigma} \varphi_{\tau]} - 2(\varphi^2 g_{\mu[\sigma} g_{\tau]\nu} + \varphi_{[\mu} g_{\nu][\sigma} \varphi_{\tau]}) + \varphi^2 g_{\mu[\sigma} g_{\tau]\nu} \\ &= 0 \end{aligned} \quad (5.78)$$

which, since $D \geq 4$, is equivalent to conformal flatness of the metric.

Since $\nabla\varphi = 0$ and M is connected, it follows from the fact that parallel transport is invertible that φ is nowhere vanishing. Furthermore, φ^2 is constant, hence so is the causal type (timelike $\varphi^2 > 0$, spacelike $\varphi^2 < 0$ or null $\varphi^2 = 0$). There are two main branches distinguished by whether or not φ is null.

φ spacelike or timelike

Since φ is nowhere vanishing at each point $x \in M$ it defines a nonzero vector φ_x . Since it is parallel, it is preserved by parallel transportation and in particular φ_x is preserved by $\text{Hol}_x(M, \nabla)$, which means that the holonomy group actually preserves the subspaces $\text{span}(\varphi_x)$ and $\text{span}(\varphi_x)^\perp$ of $T_x M$, on both of which g is nondegenerate. By the de Rham–Wu decomposition theorem, this means that (M, g) decomposes as a product of a one-dimensional manifold and a 4-dimensional manifold; in particular

- $(N, g_N) \times (\mathbb{R}, -dx^2)$ where (N, g_N) is 4-dimensional Lorentzian, if φ is spacelike,
- $(\mathbb{R}, dt^2) \times (N, -g_N)$ where (N, g_N) is 4-dimensional Riemannian, if φ is timelike.

We define the symmetric tensor

$$h_{\mu\nu} = g_{\mu\nu} - \frac{\varphi_\mu \varphi_\nu}{\varphi^2}, \quad (5.79)$$

the pullback to N of which coincides with $\pm g_N$ (with sign depending on the causal type of φ). The curvature tensors are

$$R_{\mu\nu\sigma\tau} = \varphi^2 h_{\mu[\sigma} h_{\tau]\nu}, \quad R_{\mu\nu} = \frac{3}{2} \varphi^2 h_{\mu\nu}, \quad R = 6\varphi^2. \quad (5.80)$$

The sectional curvature K perpendicular to φ (and therefore parallel to N) is given by

$$K(X, Y) = g(R(X, Y)Y, X) / (X^2 Y^2 - g(X, Y)^2) = \frac{1}{2} \varphi^2 = \frac{1}{12} R. \quad (5.81)$$

where X, Y are vector fields perpendicular to φ . This is also the sectional curvature of (N, g_N) , so (N, g_N) is a symmetric, conformally flat Einstein manifold with constant sectional and scalar curvature depending on φ^2 , hence we can identify it:

- If φ is timelike, (N, g_N) is a symmetric Riemannian manifold with positive scalar and sectional curvature, so must be S^4 ,
- If φ is spacelike, (N, g_N) is a symmetric Lorentzian manifold with negative scalar and sectional curvature and so must be the covering space of AdS_4 .

φ null

In this case, the de Rham–Wu theorem does not apply as above since at any point g is degenerate on the span of φ . We begin instead with the observation that (M, g) is a Lorentzian manifold with a parallel, null vector field φ ; it is thus a pp-wave. The curvature tensors are

$$R_{\mu\nu\sigma\tau} = 2\varphi_{[\mu}g_{\nu][\sigma}\varphi_{\tau]}, \quad R_{\mu\nu} = -\frac{3}{2}\varphi_\mu\varphi_\nu, \quad R = 0. \quad (5.82)$$

We now recall Theorem 5.4.5 and use the notation there. The geometry is scalar-flat, so the Lorentzian part (L, g_L) cannot be UdS or $UAdS$ or a product involving them since they both have constant nonzero curvature. The Riemann curvature orthogonal to φ is zero:

$$R(X, Y)_{\mu\nu} = (\varphi_{[\mu}g_{\nu]\sigma}\varphi_\tau - \varphi_{[\mu}g_{\nu]\tau}\varphi_\sigma)X^\sigma Y^\tau = 0 \quad (5.83)$$

where X, Y are vector fields which are (at least locally) orthogonal to φ . In particular, the (negative-)Riemannian part $(R, -g_r)$ must be flat. Since the Riemann curvature is not zero, the (L, g_L) cannot also be flat, in particular it is not $(\mathbb{R}, dt^2)$, so it must be a (possibly decomposable) Cahen–Wallach pp-wave.

Since we can see from the metric in Equation (5.75) that $CW_5(A)$ can include a flat Riemannian part if A is allowed to be degenerate, we can assume without loss of generality now that $(M, g) = (L, g_L) = CW_5(A)$ with $\phi = \partial_+$ being the distinguished parallel null vector. We already demonstrated conformal flatness, so from the comments below Equation (5.75), $A_{ij} = a\eta_{ij}$ for some $a \in \mathbb{R}$. Furthermore, comparing the Riemann tensors in Equations (5.76) and (5.82) yields $a = -\frac{1}{2}$.

5.4.4 $C \neq 0, \phi = 0$

By Theorem 5.2.1, $\nabla C = 0$ and $C \wedge C = 0$. Localising at a point $o \in M$, which we will call the origin, we have $C_o \wedge C_o = 0$ in $\bigwedge^4 T_o^*M$. This is a kind of *Plücker relation* which implies that $C_o = \alpha_o \wedge \beta_o$ for some nonzero, linearly independent $\alpha_o, \beta_o \in \bigwedge^2 T_o^*M$.

It follows that $C = \alpha \wedge \beta$ on a neighbourhood of o for some linearly independent one-forms α, β . As we did with φ above, we will identify these one-forms with their dual vector fields.

Note that $C_{\mu\nu} = \alpha_\mu \beta_\nu - \alpha_\nu \beta_\mu$, so

$$C_{\mu\nu} \alpha^\nu = (\alpha \cdot \beta) \alpha_\mu - \alpha^2 \beta_\mu, \quad (5.84)$$

$$C_{\mu\nu} \beta^\nu = \beta^2 \alpha_\mu - (\alpha \cdot \beta) \beta_\mu, \quad (5.85)$$

$$C_\mu{}^\lambda C_{\nu\lambda} = \alpha^2 \beta_\mu \beta_\nu + \beta^2 \alpha_\mu \alpha_\nu - (\alpha \cdot \beta)(\alpha_\mu \beta_\nu + \alpha_\nu \beta_\mu) \quad (5.86)$$

$$|C|^2 := C_{\mu\nu} C^{\mu\nu} = 2[\alpha^2 \beta^2 - (\alpha \cdot \beta)^2] \quad (5.87)$$

The Riemann tensor (from Theorem 5.2.1) is

$$\begin{aligned} R_{\mu\nu\sigma\tau} = & 3(C_{\mu\sigma} C_{\nu\tau} - C_{\mu\tau} C_{\nu\sigma}) \\ & - (C_\mu{}^\lambda g_{\nu\sigma} C_{\tau\lambda} - C_\mu{}^\lambda g_{\nu\tau} C_{\sigma\lambda} - C_\nu{}^\lambda g_{\mu\sigma} C_{\tau\lambda} + C_\nu{}^\lambda g_{\mu\tau} C_{\sigma\lambda}) \\ & - 3|C|^2 (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}). \end{aligned} \quad (5.88)$$

Let X, Y be vector fields perpendicular everywhere to the $\alpha \wedge \beta$ -distribution in TM ; that is, $X \cdot \alpha = X \cdot \beta = Y \cdot \alpha = Y \cdot \beta = 0$. Then, at least locally near o , $C_{\mu\nu} X^\nu = C_{\mu\nu} Y^\nu = 0$. We thus have

$$R_{\mu\nu\sigma\tau} X^\sigma Y^\tau = -3|C|^2 (X_\mu Y_\nu - X_\nu Y_\mu) \quad (5.89)$$

and, with some working,⁷

$$R_{\mu\nu\sigma\tau} \alpha^\sigma \beta^\tau = |C|^2 (\alpha_\mu \beta_\nu - \alpha_\nu \beta_\mu) = |C|^2 C_{\mu\nu}. \quad (5.90)$$

We can identify three cases distinguished by the nature of the plane spanned by α_o and β_o ; since it is a subspace of a Lorentzian vector space, it is either Euclidean, Lorentzian or degenerate. We will also use each of these three terms to refer to C if the plane spanned by $\alpha_o \wedge \beta_o$ has that property.

C Euclidean

The action of the holonomy group leaves C invariant; it must therefore also leave $\alpha_o \wedge \beta_o$, and thus the plane it defines, invariant. The de Rham–Wu Theorem then gives a decomposition $(M, g) = (L, g_L) \times (R, -g_R)$ where (L, g_L) is a 3-dimensional Lorentzian manifold, on which we will use upper-case Latin indices, and (R, g_R) is a 2-dimensional Riemannian manifold, on which we use lower-case Latin indices.

The curvature of (L, g_L) is given by Equation (5.89):

$$R_{MNST} = -3|C|^2 (g_{MS} g_{NT} - g_{MT} g_{NS}), \quad R_{MN} = -6|C|^2 g_{MN}, \quad R = -18|C|^2, \quad (5.91)$$

and its tangent bundle is the $\alpha \wedge \beta$ -distribution.

⁷Shortly before submission, it was realised that there were inconsistencies with this part of the Riemann curvature, which in turn revealed that there were likely mistakes in the calculation of the Riemann curvature formula Equation (5.2.1).

Since it is two-dimensional, the Riemann tensor of $(R, -g_R)$ has only a single degree of freedom which can be deduced from Equation (5.90):

$$R_{mnst} = |C|^2(g_{ms}g_{nt} - g_{mt}g_{ns}), \quad R_{mn} = |C|^2 g_{mn}, \quad R = 2|C|^2. \quad (5.92)$$

By the Cauchy-Schwarz inequality, $|C|^2 > 0$. We can thus deduce that $(L, g_L) = UAdS_3$ and $(R, g_R) = S^2$.

C Lorentzian

This case is very similar to the Euclidean case. We have a decomposition $(M, g) = (L, g_L) \times (R, -g_R)$ where this time the Lorentzian factor (L, g_L) is 2-dimensional and has the $\alpha \wedge \beta$ -distribution as its tangent bundle, and the Riemannian (R, g_R) is 3-dimensional; we still use upper-case Latin indices on L and lower-case on R . The curvature of (L, g_L) is

$$R_{MNST} = |C|^2(g_{MS}g_{NT} - g_{MT}g_{NS}), \quad R_{MN} = |C|^2 g_{MN}, \quad R = 2|C|^2, \quad (5.93)$$

and the curvature of $(R, -g_R)$ is

$$R_{mnst} = -3|C|^2(g_{ms}g_{nt} - g_{mt}g_{ns}), \quad R_{mn} = -6|C|^2 g_{mn}, \quad R = -18|C|^2. \quad (5.94)$$

In this case the sign of $|C|^2$ is indefinite. We obtain:

- $(L, g_L) = UAdS_2$ and $(R, g_R) = S^3$ for $C < 0$,
- $(L, g_L) = \mathbb{R}^{1,1}$ and $(R, g_R) = \mathbb{R}^3$ for $C = 0$,
- $(L, g_L) = UoS_2$ and $(R, g_R) = H^3$ for $C > 0$.

C degenerate

Since the $\alpha \wedge \beta$ -plane is degenerate here, de Rham–Wu decomposition doesn't work. We do however have $|C|^2 = 0$, so by inspecting Equations (5.89) and (5.90), we see that the curvature either strictly parallel or strictly orthogonal to the plane is zero. The Riemannian tensor in this case is simply

$$R_{\mu\nu\sigma\tau} = 3(C_{\mu\sigma}C_{\nu\tau} - C_{\mu\tau}C_{\nu\sigma}) - (C_{\mu}^{\lambda}g_{\nu\sigma}C_{\tau\lambda} - C_{\mu}^{\lambda}g_{\nu\tau}C_{\sigma\lambda} - C_{\nu}^{\lambda}g_{\mu\sigma}C_{\tau\lambda} + C_{\nu}^{\lambda}g_{\mu\tau}C_{\sigma\lambda}), \quad (5.95)$$

so there are no UoS or $UAdS$ factors and we must have a Cahen-Wallach pp-wave $CW_5(A)$. Thus all components of the Riemann tensor must vanish apart from $R_{i-j-} = A_{ij}$. Imposing this condition eventually forces all components of C to 0 apart from C_{i-} and also yields

$$A_{ij} = 3C_{i-}C_{j-} + \eta_{ij}C_{-}^k C_{k-}. \quad (5.96)$$

5.4.5 Summary

The following theorem summarises our results on the maximally supersymmetric backgrounds.

Theorem 5.4.6. *If (M, g, C, ϕ) is a maximally supersymmetric 5-dimensional background then (M, g) is a locally symmetric space which falls into one of three families:*

1. $\phi = 0, C = 0$: (M, g) is locally isometric to Minkowski space $\mathbb{R}^{1,4}$.
2. $\phi \neq 0, C = 0$: $\phi = \varphi \otimes r$ for a parallel vector field φ and $r \in \odot^2 \Delta$ and (M, g) is locally isometric to one of the following, depending on the causal type of φ :
 - $AdS_4 \times \mathbb{R}^{0,1}$ with $R_{AdS_4} = 6\phi^2$ for φ spacelike,
 - $\mathbb{R}^{1,0} \times S^4$ with $R_{S^4} = 6\phi^2$ for φ timelike,
 - $CW_d(A)$ with $A_{ij} = -\frac{1}{2}\eta_{ij}$ for φ null.
3. $\phi = 0, C \neq 0$: $\nabla C = 0, C \wedge C = 0$ and (M, g) is locally isometric to one of the following, depending on the causal type of C , where $|C|^2 = C^{\mu\nu}C_{\mu\nu}$:
 - $AdS_3 \times S^2$ with $R_{AdS_3} = -18|C|^2$ and $R_{S^2} = 2|C|^2$ for C Euclidean,
 - Either
 - $AdS_2 \times S^3$ with $R_{AdS_2} = 2|C|^2$ and $R_{S^3} = -18|C|^2$ if $|C|^2 < 0$,
 - $\mathbb{R}^{1,4}$ if $|C|^2 = 0$,
 - $dS_2 \times H^3$ with $R_{dS_2} = 2|C|^2$ and $R_{H^3} = -18|C|^2$ if $|C|^2 > 0$,
 for C Lorentzian,
 - $CW_d(A)$ with $A_{ij} = 3C_{i-}C_{j-} + \eta_{ij}C^k{}_-C_{k-}$ for C degenerate.

Chapter 6

Results and Analysis

6.1 Algebraic results

We found in Chapter 4 (in particular Theorem 4.3.1) that the nonzero part of the second generalised Spencer cohomology of the Poincaré superalgebra $\mathfrak{s}$ in 5 dimensions is

$$H^{2,2}(\mathfrak{s}_-, \mathfrak{s}) \cong \bigwedge^2 V \oplus (V \otimes \odot^2 \Delta), \quad (6.1)$$

with unique cocycle representatives of the cohomology classes being given explicitly by Proposition 4.3.5. The filtered deformations of $\mathfrak{s}$ are parametrised, up to isomorphism, by the $\mathfrak{so}(V)$ -invariant cohomology classes in this space.

6.2 Geometric results

In Chapter 5, cocycles of Proposition 4.3.5 are used to define the superconnection for a 5-dimensional “background”, a Lorentzian spin manifold equipped with a 2-form C and a $\odot^2 \Delta$ -valued 1-form ϕ (either or both of which may be zero). This superconnection is given by

$$\mathcal{D}_X s = \nabla_X s - \frac{1}{4}(X \cdot C - 3C \cdot X) \cdot s + \frac{1}{8}(X \cdot \phi + 3\phi \cdot X) \cdot s \quad (6.2)$$

for a vector field X and a spinor field s . Killing spinor fields are defined to be those spinor fields which are parallel with respect to this connection. Setting $\phi = 0$ recovers the definition of Killing spinors in minimal 5-dimensional supergravity [21].

Backgrounds for which the dimension of the space of Killing spinors is maximal are referred to as maximally supersymmetric, and Theorem 5.2.1 gives necessary and sufficient conditions on C , ϕ and the curvature of the background for maximal supersymmetry. The conditions on C and ϕ are sufficient (without the condition on the Riemann tensor) to have a well-defined Killing superalgebra, the algebra which encodes the supersymmetry of the space.

Theorem 5.4.6 classifies the maximally supersymmetric backgrounds up to local isometry. In the $\phi = 0$ case, apart from the flat model we recover the maximally supersymmetric

backgrounds $AdS_3 \times S^2$, $AdS_2 \times S^3$ and pp-wave of 5-dimensional minimal supergravity [21, 47]. We also find $dS_2 \times H^3$ as a $\phi = 0$ background, and for $C = 0$ there are backgrounds $AdS_4 \times \mathbb{R}^{0,1}$, $\mathbb{R}^{1,0} \times S^4$ and a different pp-wave; this case is virtually identical to the analogous result in 6 dimensions [4].

Since with $\phi = 0$, $C \neq 0$ our definition of Killing spinors coincides with that of supergravity, the maximally symmetric supergravity solutions should all appear in our calculations. However, in [21] there are also maximally supersymmetric backgrounds which did not arise here, specifically the generalised Gödel model, the near-horizon geometry of a BMPV black hole (as well as three other potential solutions that Gauntlett *et al.* suggested may be isometric to BMPV or to one of the others). We do not have an explanation for this at this stage, but we note that the BMPV solutions are a one-parameter family which interpolate between the $AdS_2 \times S^3$ and $AdS_3 \times S^2$. It is possible, therefore, that our application of the de Rham decomposition theorem in the case of the nondegenerate C -plane was invalid. Indeed, it is conceivable that the causal type of the C -plane may change as the base point is varied, even if C is parallel, and that the extremal $AdS_2 \times S^3$ and $AdS_3 \times S^2$ backgrounds are obtained only when the causal type remains constant. Our CW pp-wave solution is a limit of the Gödel solution, so again in the degenerate C -plane case it seems that we have inserted some assumption which restricts us to only an extremal solution of a parametrised family. As previously observed in a footnote, it is also likely that our determination of the formula for the Riemann curvature was incorrect.

Chapter 7

Conclusions

7.1 Summary of results

We explored the cohomology of the minimal Poincaré superalgebra in 5 spacetime dimensions, determining those cohomology groups relevant to its filtered deformations. We showed how this algebraic calculation can be used to define a connection on 5-dimensional Lorentzian spin manifold in terms of a 1-form ϕ and a 2-form C , providing a formalism for supersymmetry in a curved 5-dimensional spacetime, allowing us to define the notion of a supersymmetric background. By considering the condition for the curvature of this connection to vanish, we determined a set of conditions equivalent to maximal supersymmetry of a background. We showed that these conditions are sufficient for the supersymmetry algebra, the Killing superalgebra, to be well-defined. Finally, we made a partially successful attempt at classifying the maximally supersymmetric backgrounds, but not without some problems that will hopefully be resolved in future work.

Our notion of a supersymmetric background agrees with that of 5-dimensional minimal supergravity when $\phi = 0$, and we recovered some of the maximally supersymmetric supergravity backgrounds, but not all of them.

7.2 Future work

Our results are closely reminiscent and those obtained by similar methods in 6 spacetime dimensions, as we mentioned previously, but it is worth spelling these similarities out in more detail. In 6 dimensions, the relevant Spencer cohomology (and therefore the filtered deformations and supersymmetric backgrounds) are given by a (self-dual, in the case of no R-symmetry) 3-form and a 1-form [4]. In 5 dimensions, the analogous result is a 2-form, which is dual to a 3-form, and a 1-form. The maximally supersymmetric backgrounds corresponding to the 1-forms in either dimension are directly analogous. Under Kaluza-Klein reduction, a self-dual 3-form in 6 dimensions reduces to a 2-form and its dual 3-form.

It is a natural question, therefore, whether 5-dimensional results could be derived from the 6-dimensional results via Kaluza-Klein dimensional reduction, without the long ex-

plicit derivations given here. During this project, we explored the possibility that this could be implemented at the level of the cohomology calculation by finding the relative cohomology $C^{d,p}(\mathfrak{s}_-, L, \mathfrak{s})$ with respect to a spacelike line L in V in 6 dimensions and relating it to the $C^{d,p}(\mathfrak{s}, \mathfrak{s})$ cohomology via a central extension of $\mathfrak{s}$ by L . This was ultimately unsuccessful, as taking the relative cohomology loses the information of the $\odot^2 \Delta$ -valued 1-form. Despite the failure of this condition, the similarity of the two cases is still tantalising, and we intend to further explore the connection between the two cases. A more immediately tractable problem would be to perform dimensional reduction on the 6-dimensional maximally supersymmetric geometries.

Indeed, the $D = 5$ case is especially fertile ground for exploring dimensional reduction, situated as it is between the 4- and 6-dimensional cases which were already covered in [3, 4] (however, due to the dimensionality of the spinor representations, we expect the dimensional reduction of the $\mathcal{N} = 1$, $D = 5$ case to be related to $\mathcal{N} = 2$, extended supersymmetry in $D = 4$; [3] covers only $\mathcal{N} = 1$). If dimensional reduction can be understood here, it may provide a formalism for more efficient calculation of the Spencer cohomology and filtered deformations of the Poincaré superalgebra in other dimensions and with extended supersymmetry, as well as the maximally symmetric background, Killing superalgebras, etcetera.

Further work which could be done in $D = 5$ but which we did not have time to complete in this work includes

- Resolving the concerns over the maximally supersymmetric backgrounds with a 2-form field previously discussed,
- Finding the filtered (sub-)deformations of the Poincaré superalgebra,
- Finding the Killing superalgebras for the maximally supersymmetric backgrounds and showing that they are filtered deformations of the Poincaré superalgebra,
- Finding non-maximally supersymmetry backgrounds
- Considering subalgebras of the Poincaré superalgebra more closely,
- Considering modifications to the Poincaré superalgebra, including extended supersymmetry and R-symmetry.

Indeed, the work of this project is expected to continue as part of a PhD project, and the questions above are more than ample as a starting point for further research.

Appendix A

List of symbols

D	Spacetime dimension
$\mathcal{N}$	Number of supersymmetries of a theory
η	Minkowski metric with mostly-minus signature
e_μ	Elements of an orthonormal basis for η
g	Pseudo-Riemannian metric
k	Field or skew-field
$k(n)$	k -algebra of $n \times n$ matrices over k
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{H}$	Skew-field of quaternions
$\mathfrak{g}$	Lie algebra or superalgebra
$\mathfrak{Z}(\mathfrak{g})$	Centre of the Lie algebra or superalgebra $\mathfrak{g}$
$\mathfrak{p}$	Poincaré algebra
$\mathfrak{s}$	Poincaré superalgebra
$\mathfrak{t}$	Subalgebra of the Poincaré superalgebra
$\mathfrak{k}$	Killing superalgebra
$\otimes, \otimes$	Tensor power, algebra, product
$\odot, \odot$	Symmetric power, algebra, product
$\bigwedge, \wedge$	Exterior power, algebra, product
∂	Cochain differential; partial derivative
$\Omega(M)$	Space, algebra, de Rham complex of differential forms on M
d	Exterior derivative
$\text{Gr } V$	Associated graded space to filtered space V
$\mathcal{CL}(V)$	Clifford algebra of a pseudo-inner product space V
Σ	8-dimensional irreducible Clifford module with quaternionic structure in $D = 5$
Δ	4-dimensional trivial Clifford module with quaternionic structure in $D = 5$
S	8-dimensional Clifford module of symplectic Majorana spinors in $D = 5$
$\Gamma_{\mu_1 \dots \mu_p}$	Gamma-endomorphism of S

∇	Connection, usually Levi-Civita or its spin lift
$\mathcal{D}$	Superconnection
$\mathcal{L}$	(Spinorial) Lie derivative
$\$$	Spinor bundle with fibre Σ
$\mathcal{H}$	Trivial bundle with fibre Δ
$\mathbb{S}_{\mathbb{C}}$	Tensor bundle $\$ \otimes \mathcal{H}$ with fibre $S \otimes C = \Sigma \otimes_{\mathbb{C}} \Delta$
$\mathbb{S}$	Symplectic Majorana spinor bundle with fibre S
$\Gamma(B)$	Space of sections of a bundle $B \rightarrow M$
$\mathfrak{X}(M)$	Space of vector fields, $\Gamma(TM)$

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