

# Killing superalgebras and their uses

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Based on joint works with J. Figueroa-O'Farrill and P. de Medeiros

## Plan of the talk:

- $d = 11$  Supergravity
- Killing superalgebras
- Filtered deformations
- Spencer cohomology and Killing spinor equations
- Maximal supersymmetry
- High supersymmetry
- Spencer cohomology and Killing spinors in  $d = 4$  and  $d = 6$

## Prelude

to *“Killing superalgebras and their uses”*

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Let  $(V, \eta)$  be a real  $d$ -dimensional Lorentzian vector space. Today  $d = 11$  unless otherwise specified. The double-cover  $\text{Spin}(V)$  of the connected component  $\text{SO}^o(V) \subset \text{SO}(V)$  is identifiable with a group of invertible elements in the so-called Clifford algebra  $\mathcal{Cl}(V)$ . Such algebra is the vector space  $\mathcal{Cl}(V) = \Lambda^\bullet V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^d V$  with a special product.

$\mathcal{Cl}(V) \simeq \mathbb{R}(32) \oplus \mathbb{R}(32)$  always admits a representation as a matrix algebra. Since  $\text{Spin}(V) \subset \mathcal{Cl}(V)$ , we have a representation of  $\text{Spin}(V)$  (and hence of  $\mathfrak{so}(V) = \text{Lie}(\text{Spin}(V))$ ) by means of matrices acting on  $S = \mathbb{R}^{32}$ , called *spinor representation*.

Since  $\mathcal{Cl}(V) = \Lambda^\bullet V$ , we also have that polyvectors and forms on  $V$  correspond to matrices acting on  $S$ . The actions of such matrices are called *Clifford products between elements of  $\Lambda^\bullet V$  and elements of  $S$* .

## Prelude

to “Killing superalgebras and their uses”

**Def.** A *Lie superalgebra* is a real vector space of the form

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

endowed with a bilinear map  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$  such that

- $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ ,  $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$ ,  $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$ ;
- for any homogeneous  $X, Y$  (i.e. with  $X \in \mathfrak{g}_{\bar{i}}$ ,  $Y \in \mathfrak{g}_{\bar{j}}$ )

$$[X, Y] = -(-1)^{|X||Y|}[Y, X] \quad \left( |X| = \text{parity of } X = \begin{cases} 0 \\ 1 \end{cases} \right)$$

- for any homogeneous  $X, Y, Z$

$$(-1)^{|Z||X|}[X, [Y, Z]] + (-1)^{|Y||Z|}[Z, [X, Y]] + (-1)^{|X||Y|}[Y, [Z, X]] = 0$$

## Eleven-dimensional supergravity

Let  $(M, g, F)$  be Lorentzian mnfd  $(M, g)$ ,  $\dim M = 11$ , with closed  $F \in \Omega^4(M)$  and endowed with spin bundle  $S(M) \rightarrow M$  (the fiber  $S(M)_x \simeq S = \mathbb{R}^{32}$ ). The bosonic **eqs of supergravity** are two coupled PDE [Cremmer-Julia-Scherk '78]:

$$\left. \begin{aligned} \text{Ric}(X, Y) &= \frac{1}{2}g(i_X F, i_Y F) - \frac{1}{6}g(X, Y)\|F\|^2 \\ d * F &= \frac{1}{2}F \wedge F \end{aligned} \right\} (*)$$

Supersym. transf.  $\delta_\epsilon \Psi = D\epsilon + O(\Psi)$  of gravitino  $\Psi$  gives **superconnection** on  $S(M)$ :

$$D_X \epsilon = \nabla_X \epsilon - \frac{1}{24}[X \cdot F - 3F \cdot X] \cdot \epsilon$$

where  $X \in \mathfrak{X}(M)$  and  $\epsilon \in \Gamma(S(M))$ .

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**Def.** A **symmetry** of a solution of  $(*)$  is pair  $(\xi, \epsilon)$  given by

- (i) a *Killing vector field* for  $(g, F)$ , i.e., a v.f.  $\xi$  s.t.  $\mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$ ;
- (ii) a *Killing spinor*, i.e., a section  $\epsilon$  of  $S(M)$  s.t.  $D\epsilon = 0$ .

## Killing superalgebras

**Thm**[Figueroa-O'Farrill, Meessen, Philip '05] The v.s.  $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$  of symmetries of  $(M, g, F)$  has natural structure of Lie superalgebra, called **Killing superalgebra**.

**The Flat Model.**  $(M, g)$  Minkowski,  $F = 0$ . In this case  $D = \nabla$ ,  $\mathfrak{k}_1 \simeq S$ ,  $\mathfrak{k}_0 \simeq \mathfrak{so}(V) \oplus V$  and  $\mathfrak{k}$  is the Poincaré superalgebra  $\mathfrak{p} = (\mathfrak{so}(V) \oplus V) \oplus S$ .

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The Killing superalgebra is useful invariant of a supergravity bkgd:

- late '90s: first general check of AdS/CFT correspondence;
- early 2000s: it contracts under Penrose limit;
- mid 2000s: **homogeneity conjecture** by Meessen, i.e., if

$$\dim(\mathfrak{k}_1) > \frac{1}{2} \dim S = 16$$

then bkgd is locally homogeneous;

- it is useful to constructing bkgds with prescribed automorphism group.

## Supergravity solutions

- Local expressions for metric and 4-form of **low supersymmetric bkgds** have been derived solving the Killing spinor eqs: the  $G$ -structure method [Gauntlett, Gutowski, Pakis '03] and the spinorial geometry method [Gillard, Gran, Papadopoulos '05].
- There are  $(M, g)$  with parallel spinors that are not Ricci flat [Bryant '00].
- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].

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- There are  $(M, g)$  with parallel spinors that are not Ricci flat [Bryant '00].
- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].
- Classification of **highly supersymmetric bkgds** is largely open. Maximally supersymmetric bkgds are classified [Figueroa-O'Farrill, Papadopoulos '03] and there are non-existence results for 31 and 30 Killing spinors [Gran, Gutowski, Papadopoulos, Roest '07 & '10].
- There is one bkgd with 26 Killing spinors [Michelson '02] and also bkgds with 24, 22, 20, 18 [Gauntlett, Hull '02].

## Structural results for highly supersymmetric solutions

**The homogeneity thm**[Figuroa-O'Farrill, Hustler '12] If  $\dim(\mathfrak{k}_{\bar{1}}) > 16$  then bkgd is **locally homogeneous**.

On  $S$  there is  $\mathfrak{so}(V)$ -invariant symplectic form  $\langle -, - \rangle$  and transpose of Clifford multiplication  $V \otimes S \rightarrow S$  gives a way to square spinors: the **Dirac current**

$$\begin{aligned} k : \odot^2 S &\rightarrow V, \\ \eta(k(s, s), v) &= \langle s, v \cdot s \rangle \quad v \in V, s \in S. \end{aligned}$$

It turns out that  $k|_{\odot^2 S'} : \odot^2 S' \rightarrow V$  is surjective for all subspaces  $S' \subset S$  with  $\dim S' > 16$  so v.f.  $\xi \in \mathfrak{k}_{\bar{0}}$  already span  $T_x M$  at all  $x \in M$ .

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**Rem.** Supergravity eqs for homogeneous bkgds are algebraic and simpler than PDEs. However checking **supersymmetry is additional problem** — there exist many homog. bkgds which are not (highly) supersymmetric.

## Killing superalgebras are filtered deformations

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The nonzero Lie brackets of  $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} = (\mathfrak{so}(V) \oplus V) \oplus S$  are

$$[A, B] = AB - BA, \quad [A, s] = As, \quad [A, v] = Av, \quad [s, s] = k(s, s),$$

for all  $A, B \in \mathfrak{so}(V)$ ,  $s \in S$ ,  $v \in V$ . There exists a compatible  **$\mathbb{Z}$ -grading**

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$$

where  $\mathfrak{p}_{-2} = V$ ,  $\mathfrak{p}_{-1} = S$  and  $\mathfrak{p}_0 = \mathfrak{so}(V)$ . Compatibility means

- (i)  $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$  for all  $i, j \in \mathbb{Z}$ ;
- (ii)  $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$  and  $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$ .

## Killing superalgebras are filtered deformations

We shall be interested in graded subalgebras  $\mathfrak{a} \subset \mathfrak{p}$ , i.e.

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 = V' \oplus S' \oplus \mathfrak{h},$$

where  $V' \subset V$ ,  $S' \subset S$  and  $\mathfrak{h} \subset \mathfrak{so}(V)$ . If  $\dim S' > 16$  then  $V' = V$  (this is the algebraic fact underlying homogeneity thm). The Lie brackets of  $\mathfrak{a}$  are:

$$[A, B] = AB - BA$$

$$[A, v] = Av$$

$$[A, s] = As$$

$$[s, s] = \kappa(s, s)$$

$$[v, s] = 0$$

$$[v, w] = 0$$

$$A, B \in \mathfrak{h}, s \in S', v, w \in V'.$$

## Killing superalgebras are filtered deformations

There is natural filtration  $\mathfrak{a}^\bullet$  on  $\mathfrak{a}$ , i.e.

$$\mathfrak{a} = \mathfrak{a}^{-2} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{-1} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^0 = \mathfrak{a}_0 \supset \mathfrak{a}^1 = 0 .$$

**Def.** A **filtered deformation** of  $\mathfrak{a}$  is a Lie superalgebra  $\mathfrak{g}$  with same underlying vector space as  $\mathfrak{a}$  and a new Lie bracket  $[-, -]$  which satisfies:

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- (i)  $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j} \oplus \mathfrak{a}_{i+j+1} \oplus \cdots$ ,
- (ii) components of  $[-, -]$  of zero degree coincide with original bracket.

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$$[A, B] = AB - BA$$

$$[A, v] = Av + t\delta(A, v)$$

$$[A, s] = As$$

$$[s, s] = \kappa(s, s) + t\gamma(s, s)$$

$$[v, s] = t\beta(v, s)$$

$$[v, w] = t\alpha(v, w) + t^2\rho(v, w)$$

for some maps  $\delta : \mathfrak{h} \otimes V' \rightarrow \mathfrak{h}$ ,  $\gamma : \odot^2 S' \rightarrow \mathfrak{h}$ ,  $\beta : V' \otimes S' \rightarrow S'$ ,  $\alpha : \Lambda^2 V' \rightarrow V'$  and  $\rho : \Lambda^2 V' \rightarrow \mathfrak{h}$  subject to the Jacobi identities for all values of parameter  $t$ .

## Sketch of the proof

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The proof is simple. First let's start with a classical analogy [Kostant, '55]. Let  $(M, g)$  be a pseudo-Riemannian mnfd with Lie algebra  $\mathfrak{k}_0$  of Killing vectors. On the bundle  $\mathcal{E}_0 = TM \oplus \mathfrak{so}(TM)$  we have a connection  $D$  defined by

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A - R(X, \xi) \end{pmatrix}$$

for  $(\xi, A) \in \Gamma(\mathcal{E}_0)$ , where  $R : \Lambda^2 TM \rightarrow \mathfrak{so}(TM)$  is Riemann curvature. A section  $(\xi, A)$  is parallel precisely if  $\xi$  is Killing vector. This allows to localise at any  $x \in M$ . We introduce the notation  $(V, \eta) = (T_x M, g_x)$ ,  $\mathfrak{so}(V) = \mathfrak{so}(T_x M)$  so that the fiber of  $\mathcal{E}_0$  at  $x$  is  $\mathfrak{p}_0 = V \oplus \mathfrak{so}(V)$ . In particular  $\mathfrak{k}_0$  is identified with a (not yet  $\mathbb{Z}$ -graded!) subspace of  $\mathfrak{p}_0$  and for all  $(\xi, A_\xi), (\zeta, A_\zeta) \in \mathfrak{k}_0$  we have the bracket:

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + R(\xi, \zeta))$$

## Sketch of the proof

Let  $\text{ev}_x^{\bar{0}} : \mathfrak{k}_{\bar{0}} \longrightarrow V$  be the composition of evaluation at  $x$  with projection onto  $V = T_x M$  and set  $\text{Im}(\text{ev}_x^{\bar{0}}) = V' \subseteq V$ . Let also  $\mathfrak{h} = \text{Ker}(\text{ev}_x^{\bar{0}})$ : it consists of elements of  $\mathfrak{k}_{\bar{0}}$  of the form  $(0, A) \in V \oplus \mathfrak{so}(V)$  at  $x$  and defines a subspace of  $\mathfrak{so}(V)$ . There is a short exact sequence of v.s.

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k}_{\bar{0}} \xrightarrow{\text{ev}_x^{\bar{0}}} V' \longrightarrow 0$$

and hence a (non-canonical) v.s. isomorphism  $\mathfrak{k}_{\bar{0}} \cong \mathfrak{a}_{\bar{0}} := V' \oplus \mathfrak{h}$ . Geometrically this amounts to choosing a splitting i.e. for every  $v \in V'$  a Killing v.f.  $\xi \in \mathfrak{k}_{\bar{0}}$  with  $\xi(x) = v$ . Translating the Lie bracket of  $\mathfrak{k}_{\bar{0}}$  on  $\mathfrak{a}_{\bar{0}}$  shows that  $\mathfrak{k}_{\bar{0}}$  is a filt. def. of  $\mathfrak{a}_{\bar{0}} = V' \oplus \mathfrak{h} \subset \mathfrak{p}_{\bar{0}} = V \oplus \mathfrak{so}(V)$ . The Riemann curvature always contributes to components of the Lie bracket of positive filtration degree, independently of the choice of the splitting.

## Sketch of the proof

Let  $\text{ev}_x^{\bar{0}} : \mathfrak{k}_{\bar{0}} \longrightarrow V$  be the composition of evaluation at  $x$  with projection onto  $V = T_x M$  and set  $\text{Im}(\text{ev}_x^{\bar{0}}) = V' \subseteq V$ . Let also  $\mathfrak{h} = \text{Ker}(\text{ev}_x^{\bar{0}})$ : it consists of elements of  $\mathfrak{k}_{\bar{0}}$  of the form  $(0, A) \in V \oplus \mathfrak{so}(V)$  at  $x$  and defines a subspace of  $\mathfrak{so}(V)$ . There is a short exact sequence of v.s.

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We now superize this construction.

## Sketch of the proof

Let  $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$  be the vector bundle where  $\mathcal{E}_0 = TM \oplus \mathfrak{so}(TM)$  and  $\mathcal{E}_1 = S(M)$ . On  $\mathcal{E}$  we have a connection  $D$  defined on  $\mathcal{E}_0$  as before and by the superconnection on  $\mathcal{E}_1$ . Then the Killing superalgebra  $\mathfrak{k}$  is given by

$$\mathfrak{k}_1 = \{\varepsilon \in \Gamma(\mathcal{E}_1) | D\varepsilon = 0\} ,$$

$$\mathfrak{k}_0 = \{(\xi, A) \in \Gamma(\mathcal{E}_0) | D(\xi, A) = 0 \quad \text{and} \quad \nabla_\xi F + A \cdot F = 0\} ,$$

where  $A \cdot F$  is the natural action of  $\mathfrak{so}(TM)$  on 4-forms. Again an element of  $\mathfrak{k}$  is determined by the value at a point  $x \in M$  of the corresponding parallel section of  $\mathcal{E}$  and  $\mathfrak{k}$  defines a subspace of  $T_x M \oplus S(M)_x \oplus \mathfrak{so}(T_x M)$ . We introduce the notation

$$(V, \eta) = (T_x M, g_x) , \quad S = S(M)_x , \quad \mathfrak{so}(V) = \mathfrak{so}(T_x M) ,$$

and identify  $\mathfrak{k}$  with a (not yet  $\mathbb{Z}$ -graded) subspace of  $\mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$ . Again  $\mathfrak{k}$  is not, in general, a Lie subalgebra of the Poincaré superalgebra  $\mathfrak{p}$ .

## Sketch of the proof

The Lie brackets of  $\mathfrak{k}$  are given as follows. If  $(\xi, A_\xi), (\zeta, A_\zeta) \in \mathfrak{k}_0$  then (as before)

$$[(\xi, A_\xi), (\zeta, A_\zeta)] = (A_\xi \zeta - A_\zeta \xi, [A_\xi, A_\zeta] + \textcolor{red}{R}(\xi, \zeta)) . \quad (1)$$

The action of  $\mathfrak{k}_0$  on  $\mathfrak{k}_1$  is given by “spinorial Lie derivative” of Lichnerowicz-Kosmann

$$\begin{aligned} [(\xi, A_\xi), \varepsilon] &= \nabla_\xi \varepsilon + A_\xi \cdot \varepsilon \\ &= \frac{1}{24}(\xi \cdot F - 3F \cdot \xi) \cdot \varepsilon + A_\xi \cdot \varepsilon \end{aligned} \quad (2)$$

where  $A_\xi \cdot \varepsilon$  is the spinor representation of  $\mathfrak{so}(TM)$  on  $S(M)$  and we used  $D\varepsilon = 0$ . Lastly the square of a Killing spinor is its Dirac current, which belongs to  $\mathfrak{k}_0$ :

$$\begin{aligned} [\varepsilon, \varepsilon] &= (\kappa(\varepsilon, \varepsilon), -\nabla \kappa(\varepsilon, \varepsilon)) \\ &= (\kappa(\varepsilon, \varepsilon), \textcolor{red}{\gamma}^F(\varepsilon, \varepsilon)) \end{aligned} \quad (3)$$

where  $\gamma^F(\varepsilon, \varepsilon) \in \mathfrak{so}(TM)$  is given by  $\gamma^F(\varepsilon, \varepsilon)(X) = -\frac{1}{12}\kappa((X \cdot F - 3F \cdot X) \cdot \varepsilon, \varepsilon)$ . The red terms in (1), (2) and (3) are the terms which contribute to the components of the Lie bracket of positive filtration degree.

### Sketch of the proof

Let  $\text{ev}_x^{\bar{0}} : \mathfrak{k}_{\bar{0}} \longrightarrow V$  be as before and  $\text{ev}_x^{\bar{1}} : \mathfrak{k}_{\bar{1}} \longrightarrow S$  the evaluation at  $x$  in  $\mathfrak{k}_{\bar{1}}$ . Set  $\text{Im}(\text{ev}_x^{\bar{0}}) = V' \subseteq V$ ,  $\mathfrak{h} = \text{Ker}(\text{ev}_x^{\bar{0}})$  and  $\text{Im}(\text{ev}_x^{\bar{1}}) = S' \subseteq S$ . Choosing a splitting of

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{k}_{\bar{0}} \xrightarrow{\text{ev}_x^{\bar{0}}} V' \longrightarrow 0$$

we see that  $\mathfrak{k} \cong \mathfrak{a} := V' \oplus S' \oplus \mathfrak{h} \subset \mathfrak{p} = V \oplus S \oplus \mathfrak{so}(V)$  as a v.s. The result follows then from translating brackets (1), (2) and (3) of  $\mathfrak{k}$  on  $\mathfrak{a}$  under this identification.

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The proof is completed.

## Main Motivations and Questions

### Motivations.

- **Idea**: instead of studying directly bkgds, we set to study filt. def.
- The problem of classifying filt. def. of  $\mathbb{Z}$ -graded Lie (super)algebras is well-defined mathematically [Sternberg, Guillemin '70s] and subject of recent investigations [Kac, Cantarini, Cheng '00s] and [Kruglikov, The '14].
- According to Klein's Erlangen program, any geometry should be described by its transformation group. We shall set up a “supergravity Erlangen program”, systematising search for supergravity bkgds.

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### Questions.

- Is every filt. def. really a Killing superalgebra?
- Any 11-dimensional  $(M, g)$  with **closed**  $F \in \Omega^4(M)$  has associated Killing superalgebra. Should filt. def. be further constrained by eqs of supergravity?
- As we will see, filt. def. are useful also in relation to the problem of determining the geometries admitting **rigidly supersymmetric field theories**.

## Spencer cohomology

Filtered deformations are governed by **Spencer cohomology**, a bi-graded refinement of the usual Chevalley-Eilenberg cohomology of a Lie (super)algebra and its adjoint representation to the case of  $\mathbb{Z}$ -graded Lie (super)algebras. The **space of  $q$ -cochains** for *Poincaré superalgebra* is  $C^q(\mathfrak{p}_-, \mathfrak{p}) = \mathfrak{p} \otimes \Lambda^q \mathfrak{p}_-^*$ , where  $\mathfrak{p}_- = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} = V \oplus S$ . It decomposes  $C^q(\mathfrak{p}_-, \mathfrak{p}) = \bigoplus C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$  in components of different  $\deg = p$ .

|     |                    | $q$                                    |                                                                                                        |                                                                                                                            |                                                                                 |
|-----|--------------------|----------------------------------------|--------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| $p$ | 0                  | 1                                      | 2                                                                                                      | 3                                                                                                                          | 4                                                                               |
| 0   | $\mathfrak{so}(V)$ | $S \rightarrow S$<br>$V \rightarrow V$ | $\odot^2 S \rightarrow V$                                                                              |                                                                                                                            |                                                                                 |
| 2   |                    | $V \rightarrow \mathfrak{so}(V)$       | $\Lambda^2 V \rightarrow V$<br>$V \otimes S \rightarrow S$<br>$\odot^2 S \rightarrow \mathfrak{so}(V)$ | $\odot^3 S \rightarrow S$<br>$\odot^2 S \otimes V \rightarrow V$                                                           | $\odot^4 S \rightarrow V$                                                       |
| 4   |                    |                                        | $\Lambda^2 V \rightarrow \mathfrak{so}(V)$                                                             | $\odot^2 S \otimes V \rightarrow \mathfrak{so}(V)$<br>$\Lambda^2 V \otimes S \rightarrow S$<br>$\Lambda^3 V \rightarrow V$ | $\odot^4 S \rightarrow \mathfrak{so}(V)$<br>$\odot^3 S \otimes V \rightarrow S$ |

## Spencer cohomology and Killing spinors

**Thm**[Figueroa-O'Farrill, A.S.]  $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$ ,  $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^4 V$ .

Spencer cohomology recovers the 4-form of supergravity! But there is more: the  $\beta$ -component (remember  $\beta : V \otimes S \longrightarrow S$ ) of the Spencer cocycle is exactly

$$\beta(v, s) = v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s ,$$

where  $\varphi \in \Lambda^4 V$ . In other words, it indicates what relevant Killing spinor eqs are.

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**Rem.** The Killing spinor eqs in 11-dimensional supergravity encode *all* the information about bosonic bkgds. Indeed the **Clifford trace**

$$\sum_i e^i \cdot \mathcal{R}(e_i, -) : TM \longrightarrow \text{End}(S(M))$$

of curvature  $\mathcal{R} : \Lambda^2 TM \longrightarrow \text{End}(S(M))$  of  $D$  vanishes iff  $dF = 0$  and the field eqs are satisfied [Gauntlett, Pakis '03].

## Maximal supersymmetry

We classified maximally supersymmetric filt. def., i.e., filt. def.  $\mathfrak{g}$  of subalgebras  $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$  of  $\mathfrak{p}$  with  $\mathfrak{a}_{-2} = V$ ,  $\mathfrak{a}_{-1} = S$  and  $\mathfrak{a}_0 = \mathfrak{h}$  subalgebra of  $\mathfrak{so}(V)$ . The fact that  $S' = S$  means we have maximal supersymmetry, whereas  $V' = V$  (which is forced) means we are describing (locally) homogeneous geometries. We bootstrapped the computation of  $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$  from  $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$  and obtained:

**Thm**[Figueroa-O'Farrill, A.S.] The maximally supersymmetric filt. def. are exactly the Killing superalgebras of **maximally supersymmetric bkgds** and nothing else:

- (i)  $\mathfrak{p}$  itself for Minkowski spacetime;
- (ii)  $\mathfrak{osp}(8|4)$  for  $AdS_4 \times S^7$  [Freund, Rubin '80];
- (iii)  $\mathfrak{osp}(6, 2|4)$  for  $S^4 \times AdS_7$  [Pilch, van Nieuwenhuizen, Townsend '84];
- (iv) the Killing superalgebra of max. susy pp-wave [Kowalski-Glikman '84].

In all cases  $\mathfrak{h} = \mathfrak{so}(V) \cap \text{stab}(\varphi)$  where  $\varphi \in \Lambda^4 V$ .

## PDEs on Spinor Bilinears

For any section  $\varepsilon$  of  $S(M)$  we may define **differential forms** on  $M$  as follows:

**1**  $\omega^{(1)} \in \Omega^1(M)$ , where

$$\omega^{(1)}(X) = \langle \varepsilon, X \cdot \varepsilon \rangle$$

**2**  $\omega^{(2)} \in \Omega^2(M)$ , where

$$\omega^{(2)}(X_1, X_2) = \langle \varepsilon, (X_1 \wedge X_2) \cdot \varepsilon \rangle$$

**3**  $\omega^{(5)} \in \Omega^5(M)$ , where

$$\omega^{(5)}(X_1, \dots, X_5) = \langle \varepsilon, (X_1 \wedge \dots \wedge X_5) \cdot \varepsilon \rangle$$

The 1-form  $\omega^{(1)}$  is the metric dual of Dirac current  $\kappa = \kappa(\varepsilon, \varepsilon) \in \mathfrak{X}(M)$  of  $\varepsilon$ .

**Prop.** If  $D\varepsilon = 0$  then [Gauntlett-Pakis, '03]:

$$d\omega^{(2)} = -\iota_\kappa F \tag{1}$$

$$d\omega^{(5)} = \iota_\kappa \star F - \omega^{(2)} \wedge F. \tag{2}$$

In particular  $\iota_\kappa(d \star F - \frac{1}{2} F \wedge F) = 0$  and supergravity **Maxwell eqs are satisfied** if the space  $\mathfrak{k}_1$  of Killing spinors has  **$\dim \mathfrak{k}_1 > 16$** .

## High supersymmetry

**Thm**[Figueroa-O'Farrill, A.S.] Let  $(M, g)$  be 11-dimensional Lorentzian mnfd with **closed**  $F \in \Omega^4(M)$ . If space  $\mathfrak{k}_{\bar{1}}$  of Killing spinors has  **$\dim \mathfrak{k}_{\bar{1}} > 16$** , then  $(M, g, F)$  satisfies Einstein and Maxwell eqs of supergravity. This result is sharp.

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**Sketch of the proof.** The Jacobi identity in  $[S' S' V]$  gives

$$\frac{1}{2}R(v, \kappa(s, s))w = \kappa((X_v \beta)(w, s), s) - \kappa(\beta_v(s), \beta_w(s)) - \kappa(\beta_w \beta_v(s), s) \quad (1)$$

for all  $s \in S'$  and  $v, w \in V$ , for some map  $X : V \rightarrow \mathfrak{so}(V)$ . As  $\kappa(S', S') = V$ , this fully determines the curvature  $R$  and, by a further contraction, the Ricci tensor:

$$\begin{aligned} \text{Ric}(v, \kappa(s, s)) &= \frac{1}{2}F_{ab}^2 v^a \left\langle \Gamma^b s, s \right\rangle - \frac{1}{6}\|F\|^2 \langle v \cdot s, s \rangle \\ &\quad + \frac{1}{6} \langle (v \wedge F \wedge F + 2\iota_v \delta F - v \wedge dF) \cdot s, s \rangle. \end{aligned} \quad (2)$$

We then showed that the terms which depend on forms of different degree in

$$\odot^2 S' \subset \odot^2 S = \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V \quad (3)$$

satisfy the eqs separately (not immediate: embedding (3) is in general diagonal) 

## High supersymmetry

Furthermore if  $\dim \mathfrak{k}_{\bar{1}} > 16$  then  $(M, g, F)$  is the flat model if and only if  $F = 0$ . The theorem allows to establish a reconstruction result for highly supersymmetric bkgds and to map their classification to an algebraic problem.

**Def.** A filtered subdeformation  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\bar{1}}$  of  $\mathfrak{p}$  with  $\dim \mathfrak{g}_{\bar{1}} > 16$  is **aligned** if it is constructed out of a closed 4-form  $\varphi \in \Lambda^4 V$ .

**Thm**[Figueroa-O'Farrill, A.S.] The Killing superalgebra of any highly susy bkgd is aligned. Conversely any aligned filtered subdeformation  $\mathfrak{g}$  is (a subalgebra of) the Killing superalgebra  $\mathfrak{k}$  of a highly susy bkgd  $(M, g, F)$ . In particular highly susy bkgds, up to local equivalence, are in a **one-to-one correspondence** with maximal aligned filtered subdeformations of  $\mathfrak{p}$ , up to isomorphism of filtered subdeformations.

## Spencer cohomology and Killing spinors in $d = 4$

One can play the same game in any dimension and signature and look for new Killing spinor eqs. These eqs are well-suited to construct Killing superalgebras and in general they are related to geometries admitting rigid supersymmetric field theories.

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One can play the same game in any dimension and signature and look for new Killing spinor eqs. These eqs are well-suited to construct Killing superalgebras and in general they are related to geometries admitting rigid supersymmetric field theories.

**Thm**[de Medeiros, Figueroa-O'Farrill, A.S.] If  $d = 4$  then  $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$  and

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V .$$

The group  $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$  corresponds exactly to the scalar, vector and pseudoscalar auxiliary fields in “old” minimal off-shell formulation of  $N = 1$   $d = 4$  supergravity. The Spencer cocycle again indicates the relevant Killing spinor eqs:

$$\beta(v, s) = v \cdot (a + b \text{vol}) \cdot s - \frac{1}{2}(v \cdot \varphi + 3\varphi \cdot v) \cdot \text{vol} \cdot s ,$$

where  $v \in V$ ,  $s \in S$  and  $(a, b, \varphi) \in \Lambda^0 V \oplus \Lambda^0 V \oplus \Lambda^1 V$ .

## Maximal supersymmetry in $d = 4$

**Thm**[de Medeiros, Figueroa-O'Farrill, A.S.] If  $d = 4$  there are **three families** of filtered deformations  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  of subalgebras  $\mathfrak{a} = V \oplus S \oplus \mathfrak{h}$  of Poincaré superalgebra  $\mathfrak{p}$ :

- (i)  $\mathfrak{p}$  itself for Minkowski spacetime;
- (ii) In this case  $\mathfrak{h} = \mathfrak{so}(V)$ , there exist  $a, b \in \mathbb{R}$  with  $a^2 + b^2 > 0$  and the associated homogeneous Lorentzian mnfd  $(M = G/H, g)$ ,  $\text{Lie}(G) = \mathfrak{g}_{\bar{0}}$ ,  $\text{Lie}(H) = \mathfrak{h}$  is locally isometric to  $\text{AdS}_4$ ;
- (iii) There is nonzero  $\varphi \in V$ ,  $\mathfrak{h}$  is any subalgebra of  $\mathfrak{so}(V) \cap \text{stab}(\varphi)$  and the geometry is that of a Lie group with bi-invariant metric:
  - If  $\varphi$  is spacelike then  $(M, g)$  is locally isometric to  $\text{AdS}_3 \times \mathbb{R}$ ;
  - If  $\varphi$  is timelike then  $(M, g)$  is locally isometric to  $\mathbb{R} \times S^3$ ;
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**Rem.** Only (i) is **max. susy bkgd of supergravity** but (i)-(iii) are the Lorentzian  $d = 4$  geometries admitting  $N = 1$  rigid susy field theories [Festuccia-Seiberg '11], see also [Butter-Inverso-Lodato '15] for  $NW_4$ .

## Spencer cohomology and Killing spinors in $d = 6$

**Thm**[de Medeiros, Figueroa-O'Farrill, A.S.] If  $d = 6$  then  $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$  and

$$H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^3 V \oplus (V \otimes \mathfrak{sp}(1)) .$$

The Spencer cocycle indicates new Killing spinor equations on a Lorentzian 6-dimensional spin manifold  $(M, g)$ :

$$D_X \epsilon = \nabla_X \epsilon - \iota_X H \cdot \epsilon + 3\varphi(X) \cdot \epsilon - X \wedge \varphi \cdot \epsilon = 0 ,$$

where  $H \in \Omega^3(M)$ ,  $\varphi \in \Omega^1(M; \mathfrak{sp}(1))$ . The maximally supersymmetric bkgds are given by **three families**:

- (i) Minkowski spacetime with  $H = \varphi = 0$ ;
- (ii) If  $\varphi \neq 0$  and  $H = 0$  we have  $\text{AdS}_5 \times \mathbb{R}$ ,  $\mathbb{R} \times S^5$  and a conformally flat Lorentzian symmetric plane wave;
- (iii) If  $H \neq 0$  and  $\varphi = 0$  then  $H$  is parallel Cartan 3-form of a 6-dimensional Lie group with bi-invariant Lorentzian metric. If  $H$  is in addition **self-dual**, these are the maximally susy bkgds of  $d = 6$   $N = (1, 0)$  supergravity.

## References

- J. Figueroa-O'Farrill, A. S., *Spencer cohomology and 11-dimensional supergravity*, Comm. Math. Phys. **349** (2017), 627–660.
- J. Figueroa-O'Farrill, A. S., *Eleven-dimensional supergravity from filtered subdeformations of the Poincaré superalgebra*, J. Phys A: Math. Theor. **49** (2016), 7pp.
- J. Figueroa-O'Farrill, A. S., *On the algebraic structure of Killing superalgebras*, Adv. Theor. Math. Phys. **21** (2017), 1115–1160.

## References

- J. Figueroa-O’Farrill, A. S., *Spencer cohomology and 11-dimensional supergravity*, Comm. Math. Phys. **349** (2017), 627–660.
- J. Figueroa-O’Farrill, A. S., *Eleven-dimensional supergravity from filtered subdeformations of the Poincaré superalgebra*, J. Phys A: Math. Theor. **49** (2016), 7pp.
- J. Figueroa-O’Farrill, A. S., *On the algebraic structure of Killing superalgebras*, Adv. Theor. Math. Phys. **21** (2017), 1115–1160.
- P. de Medeiros, J. Figueroa-O’Farrill, A. S., *Killing superalgebras for Lorentzian four-manifolds*, J. High Energy Phys. **6** (2016), 50 pp.
- P. de Medeiros, J. Figueroa-O’Farrill, A. S., *Killing superalgebras for Lorentzian six-manifolds*, J. Geom. Phys. **132** (2018), 13–44.

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