

Lie superalgebras of symmetries in 11-dimensional supergravity

Andrea Santi

University of Rome “Tor Vergata” (Italy)

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Based on joint works with J. Figueroa-O’Farrill and P. de Medeiros

First part:

- Introduction to $d = 11$ supergravity
- Killing superalgebras and filtered deformations
- Spencer cohomology and Killing spinor equations
- Classification results: maximal supersymmetry

Second part:

- High supersymmetry
- Reconstruction theorem
- Outlook

First Part

Eleven-dimensional supergravity

Let (M, g, F) be Lorentzian mnfd (M, g) , $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with spinor bundle $S(M) \longrightarrow M$ (fiber is spinor module $S = \mathbb{R}^{32}$). The **eqs of supergravity** are two coupled PDE [Cremmer-Julia-Scherk '78]:

$$\left. \begin{aligned} \text{Ric}(X, Y) &= \frac{1}{2}g(i_X F, i_Y F) - \frac{1}{6}g(X, Y)\|F\|^2 \\ d \star F &= \frac{1}{2}F \wedge F \end{aligned} \right\} (*)$$

Supersymmetry transformation of the so-called gravitino gives **connection** on $S(M)$:

$$D_X \epsilon = \nabla_X \epsilon - \frac{1}{24}[X \cdot F - 3F \cdot X] \cdot \epsilon$$

where $X \in \mathfrak{X}(M)$ and $\epsilon \in \Gamma(S(M))$.

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Def.

- (i) **Even symmetry** of a solution of $(*)$ is vector field ξ s.t. $\mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$;
- (ii) **Odd symmetry** of a solution of $(*)$ is a section ϵ of $S(M)$ s.t. $D\epsilon = 0$.

Killing superalgebras

Thm[Figueroa-O'Farrill, Meessen, Philip '05] The v.s. $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of symmetries of a supergravity background (M, g, F) has natural structure of Lie superalgebra, called **Killing superalgebra**.

The Flat Model: (M, g) Minkowski spacetime, $F = 0$. In this case $D = \nabla$, $\mathfrak{k}_{\bar{1}} \simeq S$, $\mathfrak{k}_{\bar{0}} \simeq \mathfrak{so}(V) \oplus V$ and $\mathfrak{k}$ is the Poincaré superalgebra $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} = (\mathfrak{so}(V) \oplus V) \oplus S$.

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mid 2000s: **homogeneity conjecture** by Meessen, i.e., if

$$\dim(\mathfrak{k}_{\bar{1}}) > \frac{1}{2} \dim S = 16$$

then bkgd is locally homogeneous.

Supergravity backgrounds

- Local expressions for metric g and 4-form F of **low supersymmetric bkgds** have been derived solving Killing spinor eqs: G -structure method [Gauntlett, Gutowski, Pakis '03], spinorial geometry method [Gillard, Gran, Papadopoulos '05].
- There are (M, g) with parallel spinors that are not Ricci flat [Bryant '00].

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- There are (M, g) with parallel spinors that are not Ricci flat [Bryant '00].
- Classification of **highly supersymmetric bkgds** is largely open. Maximally supersymmetric bkgds are classified [Figueroa-O'Farrill, Papadopoulos '03], and $\nexists$ bkgds with 31 or 30 Killing spinors [Gran, Gutowski, Papadopoulos, Roest '10].
- There is one known bkgd with 26 Killing spinors [Michelson '02] and bkgds with 24, 22, 20, 18 Killing spinors [Gauntlett, Hull '02].

Structural results for highly supersymmetric backgrounds I

The homogeneity thm[Figueroa-O'Farrill, Hustler '12] If $\dim(\mathfrak{k}_{\bar{1}}) > 16$ then bkgd is **locally homogeneous**.

On S there is $\mathfrak{so}(V)$ -invariant symplectic form $\langle -, - \rangle$ and transpose of Clifford multiplication $V \otimes S \rightarrow S$ gives a way to square spinors: the **Dirac current**

$$\begin{aligned} k : \odot^2 S &\longrightarrow V, \\ \eta(k(s, s), v) &= \langle s, v \cdot s \rangle \quad v \in V, \ s \in S. \end{aligned}$$

It turns out that $k|_{\odot^2 S'} : \odot^2 S' \rightarrow V$ is surjective for any subspace $S' \subset S$ with $\dim S' > 16 \implies \mathfrak{k}_{\bar{0}}$ spans $T_x M$ at all $x \in M$.

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Warning. Supergravity eqs for homogeneous bkgds are algebraic and not PDEs. However **checking supersymmetry is a difficult problem**: $\exists$ homogeneous bkgds which are not (highly) supersymmetric.

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The nonzero Lie brackets of $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} = (\mathfrak{so}(V) \oplus V) \oplus S$ are

$$[A, B] = AB - BA, \quad [A, s] = As, \quad [A, v] = Av, \quad [s, s] = k(s, s),$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$, $v \in V$. There exists a compatible **$\mathbb{Z}$ -grading**

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$$

where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. Compatibility means

- (i) $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$ for all $i, j \in \mathbb{Z}$;
- (ii) $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$ and $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$.

Structural results for highly supersymmetric backgrounds II

We are interested in **graded subalgebras** $\mathfrak{a} \subset \mathfrak{p}$, i.e.

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 = V' \oplus S' \oplus \mathfrak{h} ,$$

where $V' \subset V$, $S' \subset S$ and $\mathfrak{h} \subset \mathfrak{so}(V)$. There is natural filtration $\mathfrak{a}^\bullet$ on $\mathfrak{a}$, i.e.

$$\mathfrak{a} = \mathfrak{a}^{-2} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{-1} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^0 = \mathfrak{a}_0 \supset \mathfrak{a}^1 = 0 .$$

Def. A **filtered deformation** of $\mathfrak{a}$ is a Lie superalgebra $\mathfrak{g}$ with same underlying vector space as $\mathfrak{a}$ and a new Lie bracket $[-, -]$ which satisfies:

- (i) $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j} \oplus \mathfrak{a}_{i+j+1} \oplus \cdots$,
- (ii) components of $[-, -]$ of zero degree coincide with original bracket.

Structural results for highly supersymmetric backgrounds II

Explicitly

$$[A, B] = AB - BA$$

$$[A, v] = Av + t\delta(A, v)$$

$$[A, s] = As$$

$$[s, s] = \kappa(s, s) + t\gamma(s, s)$$

$$[v, s] = t\beta(v, s)$$

$$[v, w] = t\alpha(v, w) + t^2\rho(v, w)$$

for all $A, B \in \mathfrak{h}$, $s \in S'$, $v, w \in V'$. Here $\alpha : \Lambda^2 V' \rightarrow V'$, $\beta : V' \otimes S' \rightarrow S'$, $\gamma : \odot^2 S' \rightarrow \mathfrak{h}$, $\delta : \mathfrak{h} \otimes V' \rightarrow \mathfrak{h}$, $\rho : \Lambda^2 V' \rightarrow \mathfrak{h}$ are subject to Jacobi identities for all values of the parameter t .

Motivations and first questions

Motivations.

- **State of the art.** Two important classes of bkgds:
 - (i) (M, g, F) with small $\dim \mathfrak{k}_1$ are well-understood, via various special Ansätze either on M , g or F ;
 - (ii) Much less is known on (M, g, F) with $\dim \mathfrak{k}_1 > 16$.
- **Idea:** instead of bkgds, we set to study filtered subdeformations of $\mathfrak{p}$.
- Filtered deformations are ubiquitous: homogeneous spaces, G -structures, parabolic geometries [Kruglikov, The '14], classification of primitive infinite dimensional Lie superalgebras [Sternberg, Guillemin '60s, Kac, Cantarini, Cheng '00s].
- **Upshot:** reformulation in terms of deformation theory $\leadsto$ structural results. Long term goal is to build up an “Erlangen program” for supergravity.

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First questions.

- Is every filtered subdeformation of $\mathfrak{p}$ really a Killing superalgebra?
- Any 11-dimensional (M, g) with **closed** $F \in \Omega^4(M)$ has associated Killing superalgebra. Should filt. def. be further constrained by eqs of supergravity?

Spencer cohomology

Infinitesimal filtered deformations are governed by **Spencer cohomology**, a bi-graded refinement of Chevalley-Eilenberg cohomology of a Lie (super)algebra and its adjoint representation to the case of $\mathbb{Z}$ -graded Lie (super)algebras. The **space of q -cochains** for Poincaré superalgebra is $C^q(\mathfrak{p}_-, \mathfrak{p}) = \mathfrak{p} \otimes \Lambda^q \mathfrak{p}_-^*$, where $\mathfrak{p}_- = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} = V \oplus S$. It decomposes $C^q(\mathfrak{p}_-, \mathfrak{p}) = \bigoplus C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ in components of different $\deg = p$.

		q			
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \rightarrow S$ $V \rightarrow V$	$\odot^2 S \rightarrow V$		
2		$V \rightarrow \mathfrak{so}(V)$	$\Lambda^2 V \rightarrow V$ $V \otimes S \rightarrow S$ $\odot^2 S \rightarrow \mathfrak{so}(V)$	$\odot^3 S \rightarrow S$ $\odot^2 S \otimes V \rightarrow V$	$\odot^4 S \rightarrow V$
4			$\Lambda^2 V \rightarrow \mathfrak{so}(V)$	$\odot^2 S \otimes V \rightarrow \mathfrak{so}(V)$ $\Lambda^2 V \otimes S \rightarrow S$ $\Lambda^3 V \rightarrow V$	$\odot^4 S \rightarrow \mathfrak{so}(V)$ $\odot^3 S \otimes V \rightarrow S$

Spencer cohomology and Killing spinors

Thm[Figueroa'O-Farrill, A. S. '17]

- (i) $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^4 V$ and $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$;
- (ii) The component $\beta : V \otimes S \longrightarrow S$ of the Spencer cocycle coincides with $\beta^\varphi(v, s) := (v \cdot \varphi - 3\varphi \cdot v) \cdot s$, where $\varphi \in \Lambda^4 V$;
- (iii) The brackets of a **filtered deformation** $\mathfrak{g}$ of $\mathfrak{a} = V \oplus S' \oplus \mathfrak{h}$, $\dim S' > 16$, are:

$$\begin{aligned} [A, B] &= AB - BA & [A, s] &= As \\ [A, v] &= Av + \delta(A, v) & [v, s] &= \beta^\varphi(v, s) + X_v s \\ [v, w] &= \alpha(v, w) + \rho(v, w), & [s, s] &= \kappa(s, s) + \gamma^\varphi(s, s) - X_{\kappa(s, s)} \end{aligned}$$

for all $A, B \in \mathfrak{h}$, $v, w \in V$, $s \in S'$, where

$$\begin{array}{ll} \alpha(v, w) = X_v w - X_w v & X : V \longrightarrow \mathfrak{so}(V) \\ \delta(A, v) = [A, X_v] - X_{Av} & \beta^\varphi(v, s) = (v \cdot \varphi - 3\varphi \cdot v) \cdot s \\ \rho(v, w) = [X_v, X_w] - X_{\alpha(v, w)} + R(v, w) & \gamma^\varphi(s, s)(v) = -2\kappa(\beta^\varphi(v, s), s) \end{array}$$

Classification result

Thm[Figueroa'O-Farrill, A. S. '17]

We classified the filtered deformations $\mathfrak{g}$ with $\dim S' = 32$ (i.e., $S' = S$)
– exactly one of the following *four cases* occurs:

- 1 $\mathfrak{g} = \mathfrak{p}$;
- 2 $\mathfrak{g} = \mathfrak{osp}(8|4)$;
- 3 $\mathfrak{g} = \mathfrak{osp}(6, 2|4)$;
- 4 $\mathfrak{g}$ = a particular solvable Lie superalgebra.

In the first case $\varphi = 0$, whereas in all the remaining cases $\varphi \in \Lambda^4 V$ is precisely in one of the three real orbits underlying the unique (complex) minimal orbit of $\mathrm{SO}(V^{\mathbb{C}})$ on $\mathbb{P}(\Lambda^4 V^{\mathbb{C}})$. Moreover $\mathfrak{h} = \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$.

A brief detour from $d = 11$

General strategy: one can play same game in any dimension and signature, looking for Killing spinor eqs and Einstein/Maxwell eqs that arise from Spencer cohomology.

Thm[de Medeiros, Figueroa-O'Farrill, A.S. '18] If $d = 6$ then:

- (i) $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^3 V \oplus (V \otimes \mathfrak{sp}(1))$ and $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$;
- (ii) The Spencer cocycle indicates new Killing spinor equations for Lorentzian 6-dimensional spin manifolds:

$$D_X \epsilon = \nabla_X \epsilon - \iota_X H \cdot \epsilon + 3\Phi(X) \cdot \epsilon - X \wedge \Phi \cdot \epsilon = 0 ,$$

where $H \in \Omega^3(M)$, $\Phi \in \Omega^1(M; \mathfrak{sp}(1))$;

- (iii) We classified the maximally supersymmetric filtered subdeformations of $\mathfrak{p}$, in particular they split into three families:

- 1 $H = \Phi = 0$;
- 2 $\Phi \neq 0$, $H = 0$,
- 3 $H \neq 0$, $\Phi = 0$ (if H is **self-dual** $\leadsto$ Killing superalgebras of maximally supersymmetric bkgds of $d = 6$ $N = (1, 0)$ supergravity).

Second Part

PDEs on Spinor Bilinears

For any section ε of $S(M)$ we may define **differential forms** on M as follows:

1 $\omega^{(1)} \in \Omega^1(M)$, where

$$\omega^{(1)}(X) = \langle \varepsilon, X \cdot \varepsilon \rangle$$

2 $\omega^{(2)} \in \Omega^2(M)$, where

$$\omega^{(2)}(X_1, X_2) = \langle \varepsilon, (X_1 \wedge X_2) \cdot \varepsilon \rangle$$

3 $\omega^{(5)} \in \Omega^5(M)$, where

$$\omega^{(5)}(X_1, \dots, X_5) = \langle \varepsilon, (X_1 \wedge \dots \wedge X_5) \cdot \varepsilon \rangle$$

The 1-form $\omega^{(1)}$ is the metric dual of the Dirac current $\kappa = \kappa(\varepsilon, \varepsilon) \in \mathfrak{X}(M)$ of ε .

Prop. If $D\varepsilon = 0$ then:

$$d\omega^{(2)} = -\iota_\kappa F \tag{1}$$

$$d\omega^{(5)} = \iota_\kappa \star F - \omega^{(2)} \wedge F. \tag{2}$$

In particular $\iota_\kappa(d \star F - \frac{1}{2}F \wedge F) = 0$ if $dF = 0$, and the supergravity **Maxwell eqs** are satisfied if $\dim \mathfrak{k}_\Gamma > 16$.

Structural results for highly supersymmetric backgrounds III

It had long been conjectured $\exists$ critical threshold which forces the Einstein and Maxwell eqs of supergravity. In 2017, we gave the following positive answer:

Thm[Figueroa-O'Farrill, A.S. '17] Let (M, g) be 11-dimensional Lorentzian mnfd with **closed** $F \in \Omega^4(M)$. If **$\dim \mathfrak{k}_{\bar{1}} > 16$** , then:

- 1 (M, g, F) automatically satisfies Einstein and Maxwell eqs of supergravity.
(The bound on $\dim \mathfrak{k}_{\bar{1}}$ is sharp);
- 2 $F = 0$ iff (M, g, F) is the flat model.

Structural results for highly supersymmetric backgrounds III

Idea of the proof. The Jacobi identity in $[S'S'V]$ gives

$$\frac{1}{2}R(v, \kappa(s, s))w = \kappa((X_v \beta^\varphi)(w, s), s) - \kappa(\beta_v^\varphi(s), \beta_w^\varphi(s)) - \kappa(\beta_w^\varphi \beta_v^\varphi(s), s)$$

for all $s \in S'$ and $v, w \in V$. As $\kappa(S', S') = V$, this determines the curvature R and, by a further contraction, the Ricci tensor:

$$\begin{aligned} \text{Ric}(v, \kappa(s, s)) &= \frac{1}{2}F_{ab}^2 v^a \langle s, \Gamma^b s \rangle - \frac{1}{6}\|F\|^2 \langle s, v \cdot s \rangle \\ &\quad - \frac{1}{6} \langle (v \wedge F \wedge F + 2\iota_v d^* F - v \wedge dF) \cdot s, s \rangle. \end{aligned}$$

We then showed that the terms which depend on forms of different degree in

$$\odot^2 S' \subset \odot^2 S = \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V$$

satisfy the eqs separately (not immediate: the embedding is diagonal!) ■

Structural results for highly supersymmetric backgrounds III

Cor. [Reconstruction Thm] The bkgds (M, g, F) with $\dim(\mathfrak{k}_1) > 16$ are in a *1:1 correspondence* with (maximal, odd-generated) filtered subdeformations $\mathfrak{g}$ of $\mathfrak{p}$ with $\dim S' > 16$ and closed associated infinitesimal deformation $\varphi \in \Lambda^4 V$.

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.....time to delve into the rabbit hole.....

The rabbit hole

The usage of filtered subdeformations to construct bkgds has been hindered by the amount of algebraic data that is required to reconstruct the deformation. We are now going to significantly simplify this data: in the proposed method $\mathfrak{g}$ is recovered from certain pairs (S', φ) and $SO(V)$ -orbits of 4-vectors $\varphi \in \Lambda^4 V$ may be taken as a starting point.

Dirac Kernels and Lie pairs

Let $\mathfrak{g}$ be a filtered subdeformation of $\mathfrak{p}$ as in the Reconstruction Theorem, with the graded Lie algebra $\mathfrak{a} = V \oplus S' \oplus \mathfrak{h}$ and infinitesimal deformation $\varphi \in \Lambda^4 V$. Since $\dim S' > 16$, restricting the Dirac current to $\odot^2 S'$ gives rise to a short exact sequence

$$0 \longrightarrow \mathfrak{D} \longrightarrow \odot^2 S' \xrightarrow{\kappa} V \longrightarrow 0 ,$$

where

$$\mathfrak{D} = \odot^2 S' \cap (\Lambda^2 V \oplus \Lambda^5 V) = \left\{ \omega \in \odot^2 S' \mid \omega^{(1)} = 0 \right\}$$

is called the **Dirac kernel** of S' . A splitting of the above short exact sequence — that is, a linear map $\Sigma : V \rightarrow \odot^2 S'$ such that $\Sigma(v)^{(1)} = v$ for all $v \in V$ — is called a **section** associated to S' .

Dirac Kernels and Lie pairs

Def. Let $\mathfrak{h}_{(\varphi, S')}$ be the subspace of $\mathfrak{so}(V)$ given by

$$\mathfrak{h}_{(\varphi, S')} = \gamma^\varphi(\mathfrak{D}) = \left\{ \gamma^\varphi(\omega) \mid \omega \in \odot^2 S' \text{ with } \omega^{(1)} = 0 \right\}.$$

The pair (φ, S') is called a **Lie pair** if

$$\mathfrak{h}_{(\varphi, S')} \subset \mathfrak{stab}(\varphi) \cap \mathfrak{stab}(S').$$

This is a system of coupled algebraic equations, quadratic on φ and cubic on S' .

Prop. Let $\mathfrak{g}$ and $\mathfrak{a} = V \oplus S' \oplus \mathfrak{h}$ be as above. Then (φ, S') is a Lie pair and

- (i) the isotropy algebra $\mathfrak{h} = \mathfrak{h}_{(\varphi, S')}$;
- (ii) the map $X : V \rightarrow \mathfrak{so}(V)$ is determined by the identity $X = \gamma^\varphi \circ \Sigma$.

In particular $\mathfrak{g}$ is determined, up to isomorphism, by the associated (φ, S') .

Cor. Any highly supersymmetric supergravity bkgd (M, g, F) is determined, up to local isometry, by $(\varphi, S') = (F|_o, \mathfrak{k}_{\bar{1}}|_o)$.

The reconstruction problem for supergravity backgrounds

It is not true that every Lie pair (φ, S') has a corresponding $\mathfrak{g}$, in other words, it is associated to a supergravity bkgd. First of all, it bears reminding that the right-hand sides of the Lie brackets take values in $\mathfrak{a} = V \oplus S' \oplus \mathfrak{h}$, but the individual terms may not. Explicitly,

$$\beta^\varphi(v, s) + X_v s \in S' \quad (1)$$

$$[A, X_v] - X_{Av} \in \mathfrak{h} \quad (2)$$

$$\gamma^\varphi(s, s) - X_{\kappa(s, s)} \in \mathfrak{h} \quad (3)$$

$$[X_v, X_w] - X_{X_v w - X_w v} + R(v, w) \in \mathfrak{h} \quad (4)$$

for all $A \in \mathfrak{h}$, $v, w \in V$, $s \in S'$. Moreover the Lie brackets are subject to the Jacobi identities. There are ten components of them, and five are simply satisfied by equivariance and the fact that $\mathfrak{h} = \mathfrak{h}_{(\varphi, S')} \subset \mathfrak{stab}(\varphi) \cap \mathfrak{stab}(S')$. The Jacobi identity with three odd elements is one Spencer cocycle condition.

The reconstruction problem for supergravity backgrounds

The remaining four components of the Jacobi identities are more involved:

$$R : \Lambda^2 V \rightarrow \mathfrak{so}(V) \quad \text{is } \mathfrak{h}\text{-equivariant} \quad (5)$$

$$\frac{1}{2}R(v, \kappa(s, s))w = \kappa((X_v \beta^\varphi)(w, s), s) - \kappa(\beta_v^\varphi(s), \beta_w^\varphi(s)) - \kappa(\beta_w^\varphi \beta_v^\varphi(s), s) \quad (6)$$

$$R(v, w)s = (X_v \beta^\varphi)(w, s) - (X_w \beta^\varphi)(v, s) + [\beta_v^\varphi, \beta_w^\varphi](s) \quad (7)$$

$$R(u, v)w + R(v, w)u + R(w, u)v = 0 \quad (8)$$

$$(X_u R)(v, w) + (X_v R)(w, u) + (X_w R)(u, v) = 0 \quad (9)$$

for all $u, v, w \in V, s \in S'$.

In summary, if we aim to construct bkgds via filtered subdeformations, a long task awaits us: find a Lie pair (φ, S') , compute the isotropy $\mathfrak{h} = \mathfrak{h}_{(\varphi, S')}$ and X via the Dirac kernel, check that equations (1)-(9) are satisfied, for a putative curvature tensor R to be spelled out, and finally don't forget to check $d\varphi = 0$.

The reconstruction problem for supergravity backgrounds

The following result drastically simplifies the situation:

Thm[A.S. '22] Let (φ, S') be a Lie pair satisfying (1) and assume there is $R : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ so that (6)-(7) hold. Then (2)-(5) and (8)-(9) hold too.

Cor.[Reconstruction Thm - Strong Version] Highly supersymmetric bgkds are in a **1:1 correspondence** with Lie pairs (φ, S') s.t.

$$\beta^\varphi(v, s) + X_v s \in S' \quad (10)$$

$$d\varphi(v_0, \dots, v_4) = \sum_{i < j} (-1)^{i+j} \varphi(\alpha(v_i, v_j), v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_4) = 0 \quad (11)$$

$$\frac{1}{2} R(v, \kappa(s, s))w = \kappa((X_v \beta^\varphi)(w, s), s) - \kappa(\beta_v^\varphi(s), \beta_w^\varphi(s)) - \kappa(\beta_w^\varphi \beta_v^\varphi(s), s) \quad (12)$$

$$R(v, w)s = (X_v \beta^\varphi)(w, s) - (X_w \beta^\varphi)(v, s) + [\beta_v^\varphi, \beta_w^\varphi](s) \quad (13)$$

for some $R : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ and all $v, w, v_0, \dots, v_4 \in V, s \in S'$.

A general strategy

The construction of supergravity bkgds breaks then into three steps:

- 1 Consider a $\text{Spin}(V)$ -orbit of Lie pairs (φ, S') (and therefore the associated graded subalgebra $\mathfrak{a} = V \oplus S' \oplus \mathfrak{h}_{(\varphi, S')}$ of the Poincaré superalgebra $\mathfrak{p}$);
- 2 Check (10) and (11) (roughly speaking, this gives the Killing superalgebra of the bkgd at the level of infinitesimal deformation);
- 3 See if there exists $R : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ satisfying (13) and then check (12) (here we fully reconstructed the Killing superalgebra as a deformation).

The idea has been carried out successfully in a few cases, but an algorithmic implementation of the proposed strategy with computers could be beneficial. The Reconstruction Thm has also been used to give an alternative and simpler proof of the fact that $\nexists$ bkgds with $\dim(\mathfrak{k}_{\bar{1}}) = 31$ and to set the following:

Night Thm[A.S. '25] Let (M, g, F) be a bkgd such that $F|_o = \varphi \in \Lambda^4 V$ has $\text{rank}(\varphi) \leq 6$ and an Euclidean support. If $\dim(\mathfrak{k}_{\bar{1}}) > 26$, then $\dim(\mathfrak{k}_{\bar{1}}) = 32$.

Outlook

1 A number of different bkgds have been proposed in the literature:

- (i) Maximally supersymmetric bkgds,
- (ii) pp-waves (i.e., Brinkmann spaces with flat transverse geometry),
- (iii) Gödel bkgds (i.e., admitting closed timelike curves).

pp-waves are known with any even $18 \leq \dim(\mathfrak{k}_1) \leq 26$, and $\dim(\mathfrak{k}_1) = 20$ for Gödel bkgds. **The classification with $16 < \dim(\mathfrak{k}_1) \leq 32$ is widely open.**

2 All known bkgds with $\dim(\mathfrak{k}_1) \geq 24$ are Lorentzian symmetric spaces. $\exists$ a **critical threshold which forces the homogeneous mnfd to be symmetric?**

3 $\nexists$ known highly supersymmetric bkgd s.t. $\text{rank}(\varphi) = 11$, and all known bkgds with $\text{rank}(\varphi) = 10$ have $\dim(\mathfrak{k}_1) \leq 22$. On the other hand, those with $\dim(\mathfrak{k}_1) \geq 24$ have $\text{rank}(\varphi) \leq 8 \rightsquigarrow$ **rank(φ) as a useful organizing principle** + usage of real Vinberg's θ -groups to classify orbits of 4-vectors.

4 **Supersymmetry gap is still unknown:** $\exists$ bkgds s.t. $27 \leq \dim(\mathfrak{k}_1) \leq 29$?

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