BRIEF NOTES ON THE CALCULUS OF VARIATIONS

JOSÉ FIGUEROA-O'FARRILL

ABSTRACT. These are some brief notes on the calculus of variations aimed at undergraduate students in Mathematics and Physics. The only prerequisites are several variable calculus and the rudiments of linear algebra and differential equations. These are usually taken by second-year students in the University of Edinburgh, for whom these notes were written in the first place.

CONTENTS

1. INTRODUCTION

The calculus of variations gives us precise analytical techniques to answer questions of the following type:

- Find the shortest path (i.e., **geodesic**) between two given points on a surface.
- Find the curve between two given points in the plane that yields a surface of revolution of minimum area when revolved around a given axis.
- Find the curve along which a bead will slide (under the effect of gravity) in the shortest time.

It also underpins much of modern mathematical physics, via Hamilton's principle of least action. It can be used both to generate interesting differential equations, and also to prove the existence of solutions, even when these cannot be found analytically, as in the recently discovered solution to the three-body problem [1]

The calculus of variations is concerned with the problem of extremising "functionals." This problem is a generalisation of the problem of finding extrema of functions of several variables. In a sense to be made precise below, it is the problem of finding extrema of functions of an *infinite* number of variables. In fact, these variables will themselves be functions and we will be finding extrema of "functions of functions" or *functionals*.

This generalisation is actually quite straight-forward, provided we understand the finite-dimensional case. Let us start by reviewing this.

2. Finding extrema of functions of several variables

We start by introducing some notation. Let $x \in \mathbb{R}^n$ be an arbitrary point. We shall denote by $\mathbb{R}^n_x$ the space of vectors based at the point x. The space $\mathbb{R}^n_x$ is called the **tangent space to $\mathbb{R}^n$ at the point** x.

Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}$ be a differentiable function. Recall that a point $x \in U$ is a **critical point** of the function f if $Df(x) = 0$, where $Df(x) \in (\mathbb{R}^n_x)^*$ is the derivative matrix of f at x.

☞   Here $(\mathbb{R}^n_x)^*$ is the **dual space** to $\mathbb{R}^n_x$; that is, the space of linear functions $\mathbb{R}^n_x \to \mathbb{R}$. It is again a vector space and is called the **cotangent space** to $\mathbb{R}^n$ at x.

This condition is equivalent to $Df(x)\varepsilon = 0$ for all tangent vectors ε at x; that is, for all $\varepsilon \in \mathbb{R}^n_x$. In turn this condition is equivalent to

$$\boxed{\left. \frac{d}{ds}f(x + s\varepsilon)\right|_{s=0} = 0 \qquad \forall \varepsilon \in \mathbb{R}^n_x} . \tag{1}$$

There are three main ingredients in this equation: the point $x \in U \subset \mathbb{R}^n$, a function f defined on U and the tangent space $\mathbb{R}^n_x$ at x. We will now generalise this to functionals.

3. A motivating example: geodesics

As a motivating example, let us consider the problem of finding the shortest path between two points in the plane: P and Q, say. It is well-known that the answer is the straight line joining these two points, but let us derive this.

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[1]See, for example, the following article in the Notices of the AMS:

`http://www.ams.org/notices/200105/fea-montgomery.pdf`

By a path between P and Q we mean a twice continuously differentiable curve (a C^2 curve for short)

$$x : [0, 1] \to \mathbb{R}^2 \qquad t \mapsto (x^1(t), x^2(t))$$

with the condition that $x(0) = P$ and $x(1) = Q$. The arclength of such a path is obtained by integrating the norm of the velocity vector

$$S[x] = \int_0^1 \|\dot{x}(t)\| dt \; ,$$

where

$$\|\dot{x}(t)\| = \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2} \; .$$

☞   Notice that $\dot{x}(t) \in \mathbb{R}^2_{x(t)}$. In fact, the tangent space at a point is the space of velocities of curves passing through that point.

Finding the shortest path between P and Q means minimising the arclength over the space of all paths between P and Q. To use equation (1) we need to identify its ingredients in the present problem. The rôle of $U \subset \mathbb{R}^n$ is played here by the (infinite-dimensional) space of paths in $\mathbb{R}^2$ from P to Q, and the function to be minimised is the arclength S. The final ingredient needed in order to mimic (1) is the analogue of the tangent space $\mathbb{R}^n_x$. These are the vectors based at x, hence they can be understood as differences of points $y - x$ for $y, x \in \mathbb{R}^n$. In our case, they are differences of C^2 curves $x(t)$ and $y(t)$ from P to Q. Let $\varepsilon(t) = y(t) - x(t)$ be one such difference of curves. Then $\varepsilon : [0, 1] \to \mathbb{R}^2$ is itself a C^2 function with the condition that $\varepsilon(0) = \varepsilon(1) = 0 \in \mathbb{R}^2$. Such a ε is called an (endpoint-fixed) variation, hence the name of the theory.

☞   Strictly speaking, for every fixed t, $\varepsilon(t) \in \mathbb{R}^2_{x(t)}$; that is, it is a tangent vector at $x(t)$. Moreover the endpoint conditions are $\varepsilon(0) = 0 \in \mathbb{R}^2_P$ and $\varepsilon(1) = 0 \in \mathbb{R}^2_Q$. However, we can (and will) identify all the tangent spaces with $\mathbb{R}^2$ by translating them to the origin in $\mathbb{R}^2$ and this is why we have written ε as a map $\varepsilon : [0, 1] \to \mathbb{R}^2$ and $\varepsilon(0) = \varepsilon(1) = 0$.

The condition for a path x being a critical point of the arclength functional S is now given by a formula analogous to (1):

$$\left. \frac{d}{ds} S[x + s\varepsilon] \right|_{s=0} = 0 \qquad \text{for all endpoint-fixed variations } \varepsilon.$$

As we now show, this condition translates into a differential equation for the path x. Notice that

$$S[x + s\varepsilon] = \int_0^1 \|\dot{x}(t) + s\dot{\varepsilon}(t)\| dt$$

$$= \int_0^1 \langle \dot{x}(t) + s\dot{\varepsilon}(t), \dot{x}(t) + s\dot{\varepsilon}(t) \rangle^{1/2} dt \; ,$$

whence

$$\frac{d}{ds} S[x + s\varepsilon] = \int_0^1 \frac{d}{ds} \langle \dot{x} + s\dot{\varepsilon}, \dot{x} + s\dot{\varepsilon} \rangle^{1/2} dt$$

$$= \int_0^1 \frac{\langle \dot{x} + s\dot{\varepsilon}, \dot{\varepsilon} \rangle}{\|\dot{x} + s\dot{\varepsilon}\|} dt \; .$$

Evaluating at $s = 0$, we find

$$\frac{d}{ds} S[x + s\varepsilon] \bigg|_{s=0} = \int_0^1 \frac{\langle \dot{x}, \dot{\varepsilon} \rangle}{\|\dot{x}\|} dt$$

$$= \int_0^1 \left\langle \frac{\dot{x}}{\|\dot{x}\|}, \dot{\varepsilon} \right\rangle dt \ .$$

Integrating by parts and using that $\varepsilon(0) = \varepsilon(1) = 0$, we find that

$$\frac{d}{ds} S[x + s\varepsilon] \bigg|_{s=0} = - \int_0^1 \left\langle \frac{d}{dt} \left(\frac{\dot{x}}{\|\dot{x}\|} \right), \varepsilon \right\rangle dt \ .$$

Therefore a path x is a critical point of the arclength functional S if and only if

$$\int_0^1 \left\langle \frac{d}{dt} \left(\frac{\dot{x}}{\|\dot{x}\|} \right), \varepsilon \right\rangle dt = 0 \ . \tag{2}$$

We will prove in the next section that this actually implies that

$$\frac{d}{dt} \left(\frac{\dot{x}}{\|\dot{x}\|} \right) = 0 \ , \tag{3}$$

which says that the velocity vector $\dot{x}$ has constant direction; i.e., that it is a straight line. There is only one straight line joining P and Q and it is clear from the geometry that this path actually minimises arclength.

✎ **Exercise 1.** *Generalise the preceding discussion to paths in $\mathbb{R}^n$ between any two distinct points.*

4. The fundamental lemma of the calculus of variations

In this section we prove an easy result from analysis which was used above to go from equation (2) to equation (3). This result is fundamental to the calculus of variations.

Theorem 1 (Fundamental Lemma of the Calculus of Variations). *Let $f : [0, 1] \to \mathbb{R}^n$ be a continuous function which obeys*

$$\int_0^1 \langle f(t), h(t) \rangle \, dt = 0$$

for all C^2 functions $h : [0, 1] \to \mathbb{R}^n$ with $h(0) = h(1) = 0$. Then $f \equiv 0$.

We will prove the case $n = 1$ and leave the general case as an (easy) exercise.

Proof for $n = 1$. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which obeys

$$\int_0^1 f(t)h(t)dt = 0$$

for all C^2 functions $h : [0, 1] \to \mathbb{R}$ with $h(0) = h(1) = 0$. Then we will prove that $f \equiv 0$. Assume for a contradiction that there is a point $t_0 \in [0, 1]$ for which $f(t_0) \neq 0$. We will assume in addition that $f(t_0) > 0$, with a similar proof working in the case $f(t_0) < 0$. Because f is continuous, there is a neighbourhood U of t_0 in which $f(t) > c > 0$ for all $t \in U$.

☞ If $t_0 \neq 0, 1$, then we can take $U = (t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$. If $t_0 = 0$ or $t_0 = 1$ we take $U = [0, \varepsilon)$ or $U = (1 - \varepsilon, 1]$ respectively.

We will now construct a C^2 function $h : [0, 1] \rightarrow \mathbb{R}$ with the following properties:

(P1) $h(t) = 0$ for all t outside the neighbourhood U; and

(P2) $\int_0^1 h(t)dt = \int_U h(t)dt > 0$.

Postponing for a moment the construction of such a function, let us see how their existence allows us to prove the Lemma. Let us estimate the integral

$$
\begin{aligned}
\int_0^1 f(t)h(t)dt = \int_U f(t)h(t)dt && \text{using (P1)} \\
> c \int_U h(t)dt && \text{since } f > c \text{ on } U \\
> 0 && \text{using (P2).}
\end{aligned}
$$

This violates the hypothesis of the Lemma, hence we deduce that there is no point t_0 for which $f(t_0) \neq 0$. □

✎ **Exercise 2.** *Prove the Fundamental Lemma for general n.*

We now come to the construction of the function h in the above proof. Consider the function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$
\theta(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \ . \end{cases}
$$

This function is clearly smooth (i.e., infinitely differentiable) at every point except, perhaps, at $t = 0$. However it is an easy exercise to prove that θ is smooth there as well.

✎ **Exercise 3.** *Prove that θ so defined is a smooth function.*

Now define the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$
\varphi(t) = \theta(t)\theta(1 - t) \ .
$$

Being the product of two smooth functions, it is clearly smooth. Moreover, it vanishes outside the interval $(0, 1)$. By rescaling t, we can make a function $\varphi_{a,b}$ which vanishes outside any interval (a, b):

$$
\varphi_{a,b}(t) = \varphi\left(\frac{t - a}{b - a}\right) \ . \tag{4}
$$

Furthermore, it is easy to show that

$$
\int_a^b \varphi_{a,b}(t)dt > 0 \ .
$$

The function h in the proof above can be taken to be the restriction to $[0, 1]$ of one of the $\varphi_{a,b}$ for suitable a and b.

Remark 1. We have actually proven something stronger than stated. Because the function h constructed above is not just C^2 but in fact smooth (i.e., C^∞), it is enough to check that $\int fh\,dt = 0$ only for smooth h to deduce that f vanishes.

5. The Euler–Lagrange equation

Let $\mathcal{C}_{P,Q}$ be the space of C^2 curves $x : [0,1] \to \mathbb{R}^n$ with $x(0) = P$ and $x(1) = Q$. Let $L : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a sufficiently differentiable function (typically smooth in applications) and let us consider the functional $S : \mathcal{C}_{P,Q} \to \mathbb{R}$ defined by

$$S[x] = \int_0^1 L(x(t), \dot{x}(t), t)\, dt \ .$$

The function L is called the **lagrangian** and the functional S is called the **action**. Extremising S will yield a differential equation for x. Recall that a path x is a critical point for the action if, for all endpoint-fixed variations ε, we have

$$\left. \frac{d}{ds} S[x + s\varepsilon] \right|_{s=0} = 0 \ .$$

Differentiating under the integral sign, we find

$$\begin{aligned}
0 &= \int_0^1 \left. \frac{d}{ds} L(x + s\varepsilon, \dot{x} + s\dot{\varepsilon}, t) \right|_{s=0} dt \\
&= \int_0^1 \left(\sum_{i=1}^n \frac{\partial L}{\partial x^i} \varepsilon^i + \sum_{i=1}^n \frac{\partial L}{\partial \dot{x}^i} \dot{\varepsilon}^i \right) dt \\
&= \int_0^1 \sum_{i=1}^n \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} \right) \varepsilon^i dt \ ,
\end{aligned}$$

where we have integrated by parts and used that $\varepsilon(0) = \varepsilon(1) = 0$. Using the Fundamental Lemma, this is equivalent to

$$\boxed{\frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i}} \ , \tag{5}$$

for all $i = 1, 2, \ldots, n$. This is the **Euler–Lagrange equation**.

As an example, let us reconsider the lagrangian $L(x, \dot{x}, t) = \|\dot{x}\|$. Then

$$\frac{\partial L}{\partial x^i} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^i} = \frac{\dot{x}^i}{\|\dot{x}\|} \ ,$$

and the Euler–Lagrange equation simply says that $\frac{\dot{x}^i}{\|\dot{x}\|}$ is constant, as we saw above.

✍

Exercise 4. *Let $x : [0,1] \to \mathbb{R}$ be a C^3 function. Let the lagrangian L depend also on the second derivative $\ddot{x}$. Derive the Euler–Lagrange equation arising from extremising the action*

$$S[x] = \int_0^1 L(x, \dot{x}, \ddot{x}, t) dt \ .$$

Generalise this to lagrangians depending on the first k derivatives of x, which should now be a C^{k+1} function. Generalise this further to lagrangians depending on the first k derivatives of $x : [0,1] \to \mathbb{R}^n$.
(Hint: For lagrangians depending on the first k derivatives of x, the variations and their first $k-1$ derivatives must vanish at the endpoints.)

6. Hamilton's principle of least action

Consider a particle of mass m moving in $\mathbb{R}^3$ under the influence of a potential $V : \mathbb{R}^3 \to \mathbb{R}$. Let $x : \mathbb{R} \to \mathbb{R}^3$ denote the trajectory of this particle. Define the **kinetic energy** of the trajectory to be the function $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(\dot{x}) = \tfrac{1}{2} m \|\dot{x}\|^2 \ .$$

We define the lagrangian to be the difference between the kinetic and potential energies

$$L(x, \dot{x}) = T(\dot{x}) - V(x) \ .$$

The action of the trajectory from time t_0 to time t_1 is the integral

$$S[x] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t)) dt \ .$$

Hamilton's **Principle of Least Action** says that particles follow trajectories which minimise the action. Such trajectories are therefore called **physical trajectories**.

For the above lagrangian, we have

$$\frac{\partial L}{\partial x^i} = -\frac{\partial V}{\partial x^i} \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \ ,$$

and the Euler–Lagrange equation is nothing but Newton's second law:

$$m\ddot{x}^i = -\frac{\partial V}{\partial x^i} \ ,$$

where we recognise the right-hand side of this equation as the force due to the potential V.

More generally, for any lagrangian (not necessarily of the form $T - V$) one calls the quantity $\frac{\partial L}{\partial x^i}$ the **force**, the quantity $\frac{\partial L}{\partial \dot{x}^i}$ the **momentum**, and the quantity $\sum_{i=1}^{n} \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L$ the **energy**. For the above lagrangian $L = T - V$, the energy is $T + V$.

✍ **Exercise 5** (Conservation of energy). *Prove that if the lagrangian does not depend explicitly on t, then the energy is constant along physical trajectories.*

✍ **Exercise 6** (Conservation of momentum). *Prove that if the lagrangian does not depend explicitly on one of the coordinates, say x^1, then the corresponding momentum $\frac{\partial L}{\partial \dot{x}^1}$ is constant along physical trajectories.*

In Section 9 we will see that these and other conservation laws result from *symmetries* of the lagrangian.

7. Some further problems

We are now ready to solve some of the problems stated in the introduction. We will leave the solution as exercises. They are easy to solve applying the Euler–Lagrange equation to suitable actions.

7.1. Minimal surface of revolution.
Consider two points in the plane with coordinates (x_1, y_1) and (x_2, y_2) with $x_2 > x_1$. Let $f : [x_1, x_2] \to \mathbb{R}$ be a C^2 function with the property that $f(x_1) = y_1$ and $f(x_2) = y_2$. The graph of this function is a curve from (x_1, y_1) to (x_2, y_2). Now consider revolving this curve around the x-axis to yield a surface of revolution. The surface area of the resulting surface of revolution is given by the following integral

$$S[f] = 2\pi \int_{x_0}^{x_1} f(x) \sqrt{1 + f'(x)^2} \, dx \ ,$$

where $f'(x)$ is the derivative of $f(x)$ with respect to x.

✍ **Exercise 7.** *Derive a differential equation for the function f so that the curve it defines yields a surface of revolution of minimum area when revolved about the x-axis. Solve the equation and deduce the shape of the curve. Such curves are called* catenaries.

7.2. The brachistochrone.
Consider a bead of mass m which can slide down a wire frame under the influence of gravity but without any friction. Suppose that the bead is dropped from rest from a height h. Let τ denote the time it takes to slide down to the ground. This time will depend on the shape of the wire. The shape for which τ is minimal is called the *brachistochrone* (Greek for "shortest time").

We will assume that the wire has no torsion, so that the motion of the bead happens in one plane: the (x, z) plane with z the vertical displacement and x the horizontal displacement. We choose our axes in such a way that wire touches the ground at the origin of the plane: $(0, 0)$. The shape of the wire is given by a function $z = z(x)$, with $z(0) = 0$ and $z(h) = \ell$. Let s denote the length *along* the wire from the origin to the point (x, z) on the wire.

The kinetic energy of the bead at any time t after being dropped is given by

$$T = \tfrac{1}{2} m \left(\frac{ds}{dt} \right)^2 \ ,$$

whereas the potential energy is given by

$$V = -mg \left(h - z \right) \ .$$

☞ The m in T is called the inertial mass and the m in V is called the gravitational mass. They are denoted by the same symbol because they are the same: this is the celebrated equivalence principle. This was famously demonstrated by Galileo by showing that bodies fall at the same rate regardless of their masses.

Energy is conserved because there is no friction, whence $T + V$ is a constant. To compute it, we evaluate it at the moment the bead is dropped. Because it is dropped from rest, $ds/dt = 0$ and hence $T = 0$. Since the bead is dropped from a height h, the potential energy also vanishes, and we have that $T + V = 0$. From this identity we can solve for ds/dt:

$$\frac{ds}{dt} = -\sqrt{2g \left(h - z \right)} \ , \tag{6}$$

where we have chosen the negative sign for the square root, because as the bead falls, s decreases. Now, the length element along the wire is given by

$$ds = dx \sqrt{1 + z'(x)^2} \ . \tag{7}$$

Let us rewrite equation (6) as

$$dt = -\frac{1}{\sqrt{2g\,(h-z)}}\,ds \ .$$

and insert equation (7) in this equation, to obtain

$$dt = -\frac{\sqrt{1+z'(x)^2}}{\sqrt{2g\,(h-z(x))}}\,dx \ .$$

Integrating this expression, we obtain the time τ taken by the bead to fall from the point (ℓ, h) to the point $(0,0)$:

$$\tau = \frac{1}{\sqrt{2g}} \int_0^\ell \frac{\sqrt{1+z'(x)^2}}{\sqrt{(h-z(x))}}\,dx \ .$$

This formula defines a functional on functions $z : [0,\ell]$, $x \mapsto z(x)$, with $z(0) = 0$ and $z(h) = \ell$, given by

$$S[z] = \int_0^\ell \frac{\sqrt{1+z'(x)^2}}{\sqrt{(h-z(x))}}\,dx \ ,$$

where we have conveniently reabsorbed the constant $\sqrt{2g}$ into the functional.

Exercise 8. *Extremise the action $S[z]$ defined above to find an equation for the brachistochrone. Solve the equation and deduce that the brachistochrone is an arc of a cycloid. Show that the cycloid is also the* tautochrone *(Greek for "equal time"): that is, that no matter where along the cycloid you drop the bead from (provided you drop it from rest), it will take the same amount of time to reach the bottom.*
(Hint: The cycloid is the locus traced by a point on the rim of a wheel which rolls without slipping along a horizontal axis. It can be parametrised as follows:

$$(x(t), z(t)) = (a(t+\sin t), a(1-\cos t)) \ ,$$

where a is radius of the wheel.)

7.3. **Geodesics on the sphere.** Let P and Q be any two distinct points on the unit sphere S^2 in $\mathbb{R}^3$. Let $x : [0,1] \to S^2 \subset \mathbb{R}^3$ be a C^2 curve from P to Q. In spherical polar coordinates, we can write

$$x(t) = (\cos\theta(t)\sin\varphi(t), \sin\theta(t)\sin\varphi(t), \cos\varphi(t)) \ .$$

The arclength is computed by integrating $\|\dot{x}\|$. An easy calculation yields

$$\|\dot{x}\| = \sqrt{\dot{\varphi}^2 + (\sin\varphi)^2\dot{\theta}^2} \ ,$$

whence the arclength of the path defines a functional on functions θ and φ

$$S[\theta, \varphi] = \int_0^1 \sqrt{\dot{\varphi}^2 + (\sin\varphi)^2\dot{\theta}^2}\,dt \ .$$

The shortest path between P and Q can now be found by extremising the above functional. It is however technically easier to parametrise the path in terms of the angle φ itself, in such

a way that the path is given by specifying the function $\varphi \mapsto \theta(\varphi)$. In terms of this function, the arclength functional becomes

$$S[\theta] = \int_{\varphi_P}^{\varphi_Q} \sqrt{1 + (\sin\varphi)^2(\theta')^2} d\varphi \ ,$$

where θ' is now the derivative of θ with respect to φ.

✍ **Exercise 9** (Geodesics on the sphere). *Extremising the above functional $S[\theta]$, prove that the shortest path between any two points P and Q on the unit sphere lies on a great circle; that is, on the intersection of the sphere with a plane through the centre of the sphere.*

8. Second variation

Finding extrema of a function involves more than finding its critical points. One has to decide whether these points are indeed extrema (i.e., maxima or minima) or not. To determine the type of a critical point one needs to compute the hessian matrix of the function. Something similar happens in the calculus of variations. We start by reviewing the finite-dimensional situation.

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and let $x \in U$ be a critical point. As we have seen above, this means that equation (1) holds. The hessian matrix $\mathrm{Hess}_x f$ of f at x is a symmetric bilinear form on $\mathbb{R}^n_x$, defined by

$$(\mathrm{Hess}_x f)(\varepsilon, \eta) = \left.\frac{\partial^2}{\partial s \partial t} f(x + s\varepsilon + t\eta)\right|_{s=t=0} \ ,$$

for all $\varepsilon, \eta \in \mathbb{R}^n_x$. If the hessian is positive-definite (resp. negative-definite) at a critical point, then this point is a minimum (resp. maximum). If the hessian is positive-semidefinite (resp. negative-semidefinite) then we have a degenerate minimum (resp. maximum). If the hessian is indefinite then we have neither.

Let us apply this to lagrangians of the form $L(x, \dot{x})$ where $x : [0,1] \to \mathbb{R}^n$ is a C^2 function. We will define a bilinear form

$$H_x(\varepsilon, \eta) = \left.\frac{\partial^2 S[x + u\varepsilon + v\eta]}{\partial u \partial v}\right|_{u=v=0} \ .$$

We compute this as follows:

$$
\begin{aligned}
H_x(\varepsilon, \eta) &= \left[\frac{\partial}{\partial u}\left(\frac{\partial}{\partial v}S[x + u\varepsilon + v\eta]\right)_{v=0}\right]_{u=0} \\
&= \frac{\partial}{\partial u}\int_0^1 \sum_{i=1}^n \left(\frac{\partial L(x + u\varepsilon)}{\partial x^i} - \frac{d}{dt}\frac{\partial L(x + u\varepsilon)}{\partial \dot{x}^i}\right)\eta^i dt\bigg|_{u=0} \\
&= \int_0^1 \sum_{i,j=1}^n \left(\frac{\partial^2 L}{\partial x^j \partial x^i}\varepsilon^j + \frac{\partial^2 L}{\partial \dot{x}^j \partial x^i}\dot{\varepsilon}^j - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x^j \partial \dot{x}^i}\varepsilon^j + \frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^i}\dot{\varepsilon}^j\right)\right)\eta^i dt \ .
\end{aligned}
$$

We can rewrite this result as

$$H_x(\varepsilon, \eta) = \int_0^1 (J_{ij}\varepsilon^j)\eta^i dt \ ,$$

where J_{ij} is a matrix of differential operators called the *Jacobi operator*.

Determining whether this bilinear form is definite, semidefinite or indefinite is usually quite hard, but in some simple cases it can be settled using elementary means.

✍ **Exercise 10.** *Compute the second variation for the arclength in $\mathbb{R}^n$ and prove that it is positive-definite; that is, prove that for all variations ε,*

$$H_x(\varepsilon, \varepsilon) \geq 0 \qquad and \qquad H_x(\varepsilon, \varepsilon) = 0 \iff \varepsilon \equiv 0 \ .$$

By analogy with the finite-dimensional case, conclude that critical paths are indeed minimising in this case. (This is geometrically obvious, of course.)

(Hint: The answer is

$$H_x(\varepsilon, \eta) = \int_0^1 \sum_{i,j=1}^n \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \dot{\varepsilon}^i \dot{\eta}^j \, dt \ ,$$

where

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} = \frac{1}{\|\dot{x}\|^3} \left(\|\dot{x}\|^2 \delta_{ij} - \dot{x}^i \dot{x}^j \right) \ .$$

You may find the Cauchy–Schwarz inequality useful in proving positive-definiteness.)

9. NOETHER'S THEOREM AND CONSERVATION LAWS

Let $S[x] = \int_0^1 L(x, \dot{x}) dt$ be an action for C^2 curves $x : [0,1] \to \mathbb{R}^n$ with $x(0) = P$ and $x(1) = Q$, for some points $P, Q \in \mathbb{R}^n$.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. Then the composition $y = \varphi \circ x : [0,1] \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ from the point $\varphi(P)$ to the point $\varphi(Q)$. If x extremises S, there is no reason to believe that y should too. However there is one case when this is true.

We say that φ is a symmetry of the lagrangian L if

$$L(x, \dot{x}) = L(y, \dot{y}) \ .$$

Equivalently one says that L is invariant under φ.

☞ Notice that $\dot{y}$ can be computed using the chain rule:

$$\dot{y}(t) = \frac{d}{dt} \varphi(x(t)) = D\varphi(x(t))\dot{x}(t) \ ,$$

where $D\varphi$ is the derivative matrix of φ. In this case it is an $n \times n$ matrix.

Lemma 1. *Let $x : [0,1] \to \mathbb{R}^n$ solve the Euler–Lagrange equation for the lagrangian L. If L is invariant under $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, then $y = \varphi \circ x$ also solves the Euler–Lagrange equation for L. In other words, φ takes solutions to solutions.*

Proof. Since $L(x, \dot{x}) = L(y, \dot{y})$, it trivially follows that

$$\frac{\partial L}{\partial x^i} = \frac{\partial L}{\partial y^i} \qquad \frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial \dot{y}^i} \ ,$$

whence

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} \iff \frac{\partial L}{\partial y^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} \ .$$

□

🛑 The converse of this lemma is *not* true. There are transformations taking solutions to solutions which are not symmetries of the lagrangian.

Now consider not one function φ but a one-parameter family $\varphi_s : \mathbb{R}^n \to \mathbb{R}^n$ of smooth functions, defined for all $s \in \mathbb{R}$. Assume moreover that this family satisfies the following two properties:

(D1) $\varphi_0 = \mathrm{id}$, where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^n$; and

(D2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$, for all $s, t \in \mathbb{R}$.

✍ **Exercise 11.** *Show that these properties imply that φ_s is invertible, with inverse $\varphi_s^{-1} = \varphi_{-s}$. Show that the family $\{\varphi_s\}$ defines a group isomorphic to $(\mathbb{R}, +)$.*

The family $\{\varphi_s\}$ is called a **one-parameter group of diffeomorphisms** of $\mathbb{R}^n$.

The following theorem tells us what happens when a lagrangian is invariant under a one-parameter group of diffeomorphisms; that is, when it is invariant under φ_s for all s.

Theorem 2 (Noether's Theorem). *Let $S[x] = \int_0^1 L(x, \dot{x})dt$ be an action for curves $x : [0, 1] \to \mathbb{R}^n$, and let L be invariant under a one-parameter group of diffeomorphisms $\{\varphi_s\}$. Then the* **Noether charge** *I, defined by*

$$I(x, \dot{x}) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i} \frac{\partial \varphi_s^i(x)}{\partial s}\bigg|_{s=0} ,$$

is conserved; that is, $dI/dt = 0$ along physical trajectories.

Proof. Let $x(t)$ be a solution of the Euler–Lagrange equation for L. Then by the above lemma, so does $y(s, t) := \varphi_s(x(t))$ for every s; in other words,

$$\frac{\partial L}{\partial y^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} .$$

Because L is invariant under φ_s for every s, $L(y, \dot{y})$ does not depend on s. Taking the derivative with respect to s, we obtain

$$0 = \frac{\partial}{\partial s} L(y, \dot{y}) = \sum_{i=1}^{n} \frac{\partial L}{\partial y^i} \frac{\partial y^i}{\partial s} + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{y}^i} \frac{\partial \dot{y}^i}{\partial s} .$$

Using the Euler–Lagrange equation, we can rewrite this as

$$0 = \sum_{i=1}^{n} \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i} \frac{\partial y^i}{\partial s} + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{y}^i} \frac{\partial \dot{y}^i}{\partial s}$$

$$= \frac{d}{dt} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{y}^i} \frac{\partial y^i}{\partial s} .$$

Finally we evaluate at $s = 0$, using that $y(0, t) = x(t)$ by property (D1) above, to arrive at

$$\frac{d}{dt} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i} \frac{\partial y^i}{\partial s}\bigg|_{s=0} = 0 ,$$

which completes the proof. □

As an example, consider a lagrangian $L = L(\dot{x})$, where $x : [0,1] \to \mathbb{R}$, which does not depend explicitly on x. This means that L is invariant under the one-parameter group of diffeomorphisms $\varphi_s(x) = x + s$. According to Noether's theorem the momentum $\partial L / \partial \dot{x}$ is conserved.

✎ **Exercise 12** (Conservation of momentum revisited). *Redo Exercise 6 by exhibiting a one-parameter symmetry group of L and using Noether's theorem.*

✎ **Exercise 13** (Conservation of angular momentum). *Let $L = \frac{1}{2}m\|\dot{x}\|^2 - V(x)$ be a lagrangian for plane curves $x : [0,1] \to \mathbb{R}^2$. Assume that V only depends on $\|x\|$. Show that L is invariant under the one-parameter symmetry group $\varphi_s : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\varphi_s(x) = (x^1 \cos s - x^2 \sin s, x^1 \sin s + x^2 \cos s) .$$

Find the expression for the Noether charge associated to this symmetry.
(Answer: You should find $I = m(x^1 \dot{x}^2 - x^2 \dot{x}^1)$.)

10. ISOPERIMETRIC PROBLEMS

The original isoperimetric problem was posed by the ancient Greeks: find the closed plane curve of a given length that encloses the largest area. They even managed to convince themselves that the intuitive answer (the circle) was correct. The reason this problem is called isoperimetric is that one is maximising the area inside the curve while keeping the perimeter fixed. More generally, an isoperimetric problem is one where one is trying to extremise a functional subject to a (functional) constraint. In this section we will learn how to deal with such constrained extremisation in the context of the variational calculus. Let us start by setting up the classical isoperimetric problem in this context.

Let $x : [0,1] \to \mathbb{R}^2$ be a C^2 curve which is closed: $x(0) = x(1)$. The area enclosed by the curve is given by the following functional:

$$S[x] = \tfrac{1}{2} \int_0^1 (x^1 \dot{x}^2 - x^2 \dot{x}^1) dt ,$$

whereas the perimeter of the curve is given by the following functional:

$$A[x] = \int_0^1 \|\dot{x}\| dt .$$

The isoperimetric problem is the following: extremise $S[x]$ subject to $A[x] = \ell$.

Surely you recognise the finite-dimensional analogue to this problem. Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be functions of n variables. One can then extremise f subject to $g = 0$. As in SVC, one can use the method of Lagrange multipliers. We define a new function $F : U \times \mathbb{R} \to \mathbb{R}$ of $n+1$ variables (the new variable, typically denoted λ, is the Lagrange multiplier) by $F(x, \lambda) = f(x) - \lambda g(x)$ and one simply extremises F without any constraints. The resulting equations are

$$\frac{\partial F}{\partial \lambda} = 0 \implies g(x) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial x^i} = 0 \implies \frac{\partial f}{\partial x^i} = \lambda \frac{\partial g}{\partial x^i} .$$

The method of Lagrange multipliers extends to the calculus of variations. Suppose that we want to extremise the action

$$S[x] = \int_0^1 L(x, \dot{x}, t) dt$$

on functions $x : [0, 1] \to \mathbb{R}^n$, subject to the constraint

$$A[x] = \int_0^1 K(x, \dot{x}, t) dt = 0 \ .$$

🛑 Without loss of generality we have taken the constraint to be $A[x] = 0$ as opposed to $A[x] = c$ for some constant c. Clearly if $A[x] = c$, $A'[x] = A[x] - c = 0$.

The method of Lagrange multipliers says that we should construct a new functional depending in addition on one extra parameter λ (*not* a function, but a constant)

$$\tilde{S}[x, \lambda] = S[x] - \lambda A[x] \ ,$$

and extremise $\tilde{S}[x, \lambda]$ in the space of functions $x : [0, 1] \to \mathbb{R}^n$. Any solution of the resulting Euler–Lagrange equation will depend on $2n$ constants of integration *and* the parameter λ. These are then fixed by the $2n$ boundary conditions for $x(0)$ and $x(1)$ *and* the constraint $A[x] = 0$.

🛑 The only reason we have $2n$ constants of integration is because the lagrangian is first-order; that is, it depends only on x and $\dot{x}$. This means that the resulting Euler–Lagrange equation is a second-order ordinary differential equation for the n component functions of x and hence there are $2n$ constants of integration: 2 constants per component function. In general, if the lagrangian depends on x and its first k derivatives, we will have kn constants of integration and an equal number of boundary conditions.

Before doing an example, let us see why this works. Recall that a function $x : [0, 1] \to \mathbb{R}^n$ is a critical point of the functional $S[x]$ if for any variation ε,

$$\frac{d}{ds} S[x + s\varepsilon] \Big|_{s=0} = 0 \ .$$

In the presence of a constraint $A[x] = 0$, we would have to consider only those variations which preserve the constraint; that is, only those ε for which $A[x + s\varepsilon] = 0$ for all s. This condition is generally too strong and there may not be any nontrivial variations satisfying this. Instead we introduce a two-parameter family of variations: $S[x + s\varepsilon + r\eta]$ and we choose the parameters s and r in such a way that $A[x + s\varepsilon + r\eta] = 0$. At a fixed function x and for fixed variations ε and η, the condition $A[x + s\varepsilon + r\eta] = 0$ defines a curve in the (r, s) plane: $g(r, s) = 0$. Hence, for fixed x, ε, η, we want to extremise the function $f(r, s) = S[x + s\varepsilon + r\eta]$ subject to the condition $g(r, s) = 0$.

The method of Lagrange multipliers for functions of two variables (here s and r) says that we should extremise the function

$$F(r, s, \lambda) = f(r, s) - \lambda g(r, s) \ ,$$

which is nothing but

$$F(r, s, \lambda) = S[x + s\varepsilon + r\eta] - \lambda A[x + s\varepsilon + r\eta] \ .$$

This function has a critical point if the following conditions are satisfied:

$$\frac{\partial F}{\partial r} \Big|_{r=s=0} = \frac{\partial F}{\partial s} \Big|_{r=s=0} = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial \lambda} \Big|_{r=s=0} = 0 \ .$$

✍ **Exercise 14.** *Convince yourself that imposing these equations for all variations ε and η is the same as extremising the modified functional $\tilde{S}[x, \lambda]$ and imposing the constraint $A[x] = 0$.*

Let us now do an example. This is a variant of the original isoperimetric problem. Consider a C^2 function $f : [0, 1] \to \mathbb{R}$ with the property that $f(0) = f(1) = 0$ and $f(x) > 0$ everywhere else. Its graph $y = f(x)$ is a curve from the origin to the point $(1, 0)$ and lying in the upper half-plane. We would like to maximise the area under the curve provided that the arclength is fixed to some number $\ell \geq 1$. In other words, we want to maximise the area functional

$$S[f] = \int_0^1 f(x)\, dx$$

subject to the constraint

$$A[f] = \int_0^1 \sqrt{1 + f'(x)^2}\, dx = \ell \ .$$

According to the method of Lagrange multipliers, we must extremise the modified functional

$$\tilde{S}[f, \lambda] = \int_0^1 \left(f(x) - \lambda\sqrt{1 + f'(x)^2} \right)\, dx \ .$$

The Euler–Lagrange equation resulting from this functional is

$$1 + \frac{d}{dx}\left(\lambda\frac{f'(x)}{\sqrt{1 + f'(x)^2}} \right) = 0 \ .$$

Integrating once we find

$$x + \lambda\frac{f'(x)}{\sqrt{1 + f'(x)^2}} = c_1 \ ,$$

for some constant c_1. From this equation we can solve for $f'(x)$ and then integrate again to solve for $f(x)$.

✍ **Exercise 15.** *Complete the above analysis and prove that the graph $y = f(x)$ traces a circle of radius λ passing through $(0, 0)$ and $(1, 0)$ and with centre at the point $(\frac{1}{2}, -\sqrt{\lambda^2 - \frac{1}{4}})$. Finally show that λ is determined from the arclength ℓ by the transcendental equation $2\lambda \sin(\ell/2\lambda) = 1$.*

✍ **Exercise 16.** *Solve the original isoperimetric problem stated at the start of this section. Deduce that for any closed plane curve the area A of the enclosed region and the perimeter ℓ of the curve satisfy the following **isoperimetric inequality** $A \leq \ell^2/4\pi$, with equality if and only if the curve is a circle.*
(Hint: Extremise the modified action

$$\tilde{S}[x] = \int_0^1 \left(\tfrac{1}{2}(x^1\dot{x}^2 - x^2\dot{x}^1) - \lambda\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2} \right)\, dt \ ,$$

and deduce that the resulting curve is a circle of radius λ.)

☞ A similar "isoperimetric" inequality exists between the surface area S of a closed surface in $\mathbb{R}^3$ and the volume V it encloses. In this case, one can prove the bound $V^2 \leq S^3/36\pi$, which is now saturated by the sphere.

Finally, let us point out that if there are more than one constraint, one must introduce an equal number of Lagrange multipliers.

11. Lagrange multipliers

The method of Lagrange multipliers in the calculus of variations extends to other types of constrained extremisation, where the subsidiary condition is not a functional but actually a function; that is, rather than a constraint of the form $\int_0^1 K(x, \dot{x}, t)dt = 0$, we have one of the form $G(x, \dot{x}, t) = 0$. There are two types of problems where these constraints appear naturally:

- Finding geodesics on a surface defined as the zero locus of a function, say, $G(x) = 0$; and
- Reducing higher-order lagrangians to first-order lagrangians. For example, given a lagrangian $L(x, \dot{x}, \ddot{x}, t)$ depending on the second derivative of the function x, it can be replaced by a first-order lagrangian $L(x, \dot{x}, \dot{y}, t)$ with the subsidiary condition $y = \dot{x}$.

Without proof, let us simply outline the method. Suppose we want to extremise a functional

$$S[x] = \int_0^1 L(x, \dot{x}, t) \, dt$$

for C^2 functions $x : [0, 1] \to \mathbb{R}^n$ subject to a condition $G(x, \dot{x}, t) = 0$, where $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^k$ is a differentiable function. The idea is to introduce a new *function* $\lambda : [0, 1] \to \mathbb{R}^k$ and extremise the modified functional

$$\tilde{S}[x, \lambda] = \int_0^1 \left(L(x, \dot{x}, t) - \langle \lambda(t), G(x, \dot{x}, t) \rangle \right) \, dt \ ,$$

where $\langle , \rangle$ is the dot product in $\mathbb{R}^k$.

Let us apply this to derive the Euler–Lagrange equation for a lagrangian $L(x, \dot{x}, \ddot{x}, t)$ depending on the second derivative of a C^3 function $x : [0, 1] \to \mathbb{R}$. As outlined above, this problem is the same as extremising the action with lagrangian $L(x, \dot{x}, \dot{y}, t)$ subject to the constraint $y = \dot{x}$. As suggested above, we construct the modified action

$$S[x, y, \lambda] = \int_0^1 \left(L(x, \dot{x}, \dot{y}, t) - \lambda(t)(y - \dot{x}) \right) \, dt \ .$$

The Euler–Lagrange equations are

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} + \lambda \right) \ , \qquad -\lambda = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} \quad \text{and} \quad y = \dot{x} \ .$$

Solving for λ from the second equation and inserting the result in the second, we obtain

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} = 0 \ ,$$

where we have used that $y = \dot{x}$. This result should be compared with your result for the first part in Exercise 4.

☞   We can now also explain the Hint to Exercise 4 concerning the fact that the variations and their first derivatives should both vanish at the endpoints. In the above formulation there are variations for both x and y. Let's call them ε and η respectively. We have that $\varepsilon(0) = \varepsilon(1) = \eta(0) = \eta(1) = 0$. If the variations are to preserve the constraint $y = \dot{x}$, then we must have that $\eta = \dot{\varepsilon}$; whence $\dot{\varepsilon}$ should vanish at the endpoints.

✏ **Exercise 17** (Geodesics on the sphere revisited)**.** *Using the method of Lagrange multipliers prove that the geodesics on a sphere are give by great circles.*
(Hint: Extremise $S[x] = \int_0^1 \|\dot{x}\| \, dt$ for $x : [0,1] \to \mathbb{R}^3$ subject to $\|x\|^2 = 1$.)

12. SOME VARIATIONAL PDEs

Thus far we have considered functionals defined on curves; that is, on functions of one variable. The Euler–Lagrange equations obtained in this way are always ordinary differential equations. In the same way, one can obtain partial differential equations by varying functionals of functions of several variables. In fact, many of the interesting partial differential equations arise in this way.

By way of introduction let us consider the problem of extremising functionals defined on surfaces as opposed to curves. Let $x : D \subset \mathbb{R}^2 \to \mathbb{R}^n$ be a C^2 function defined on a bounded set D in the plane. Let ∂D denote the boundary of the set D, which we will assume to be smooth or at least piecewise smooth. We will let t^1, t^2 denote the coordinates on the plane, so that $x(t^1, t^2) \in \mathbb{R}^n$ for every $(t^1, t^2) \in D$. Consider a lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^{2n} \times \mathbb{R}^2 \to \mathbb{R}$, and the corresponding action

$$ S[x] = \int_D L(x, Dx, t) \, d^2t \; , $$

where Dx denotes collectively the $2n$ partial derivatives $\partial x^i / \partial t^\mu$, for $i = 1, 2, \ldots, n$ and $\mu = 1, 2$. The boundary conditions are specified by asking that $x(t) = \phi(t)$ for $t \in \partial D$, where $\phi : \partial D \to \mathbb{R}^n$ is a given function.

✏ **Exercise 18.** *Convince yourself that this agrees with what we did above in the one-dimensional case. Notice that in that case, $D = [0,1]$ and $\partial D = \{0\} \cup \{1\}$ consists of two points.*

The variations for this problem are now C^2 functions $\varepsilon : D \to \mathbb{R}^n$ such that ε vanishes on the boundary: $\varepsilon(t) = 0$ for $t \in \partial D$. The condition that a function $x : D \to \mathbb{R}^n$ be a critical point of the action $S[x]$ is then that

$$ \frac{d}{ds} S[x + s\varepsilon] \bigg|_{s=0} = 0 \qquad \text{for all variations } \varepsilon \text{ with } \varepsilon|_{\partial D} = 0. $$

To derive the Euler–Lagrange equation in this case we will find it convenient to introduce some notation. We will let x_μ, for $\mu = 1, 2$ denote the derivatives of x with respect to t^μ; that is $x_\mu = \partial x / \partial t^\mu$. Similarly we will let ∂_μ denote the derivative operator $\partial / \partial t^\mu$.

As before, let us differentiate under the integral sign to find

$$
\begin{aligned}
0 &= \frac{d}{ds} S[x + s\varepsilon]\Big|_{s=0} \\
&= \int_D \frac{d}{ds} L(x + s\varepsilon, Dx + sD\varepsilon, t)\Big|_{s=0} d^2t \\
&= \int_D \left(\sum_{i=1}^n \frac{\partial L}{\partial x^i} \varepsilon^i + \sum_{i=1}^n \sum_{\mu=1}^2 \frac{\partial L}{\partial x^i_\mu} \partial_\mu \varepsilon^i \right) d^2t \\
&= \int_D \sum_{i=1}^n \left(\frac{\partial L}{\partial x^i} - \sum_{\mu=1}^2 \partial_\mu \frac{\partial L}{\partial x^i_\mu} \right) \varepsilon^i d^2t + \int_D \sum_{i=1}^n \sum_{\mu=1}^2 \partial_\mu \left(\frac{\partial L}{\partial x^i_\mu} \varepsilon^i \right) d^2t \ .
\end{aligned}
$$

The Divergence Theorem (see below) allows us to rewrite the last integral as an integral over the boundary ∂D, which is then seen to vanish since $\varepsilon|_{\partial D} = 0$. The generalisation of the Fundamental Lemma to functions of more than one variable (see below) then allows us to deduce that the first term above vanishes for all variations ε if and only if the Euler–Lagrange equations

$$
\frac{\partial L}{\partial x^i} = \sum_{\mu=1}^2 \partial_\mu \frac{\partial L}{\partial x^i_\mu}
$$

are satisfied.

There is no reason why we are restricted to functions of only two variables. In fact, if $D \subset \mathbb{R}^m$ is a bounded region with (piecewise) smooth boundary and $L(x, Dx)$ is a lagrangian for maps $x : D \to \mathbb{R}^n$, the very same manipulations would yield the general multidimensional Euler–Lagrange equations:

$$
\boxed{\frac{\partial L}{\partial x^i} = \sum_{\mu=1}^m \partial_\mu \frac{\partial L}{\partial x^i_\mu}} \ . \tag{8}
$$

It remains to discuss the Divergence Theorem and the generalisation of the Fundamental Lemma. We start with the the Divergence Theorem. It states the following.

Theorem 3 (Divergence Theorem). *Let $D \subset \mathbb{R}^m$ be a bounded open set with (piecewise) smooth boundary ∂D. Let $X = (X^1, \dots X^m)$ be a smooth vector field defined on $D \cup \partial D$. Let N be unit outward-pointing normal of ∂D. Then*

$$
\int_D \sum_{\mu=1}^m \partial_\mu X^\mu dV = \int_{\partial D} \langle X, N \rangle \, dA \ ,
$$

where dV is the volume element in $\mathbb{R}^m$ and dA is the area element in ∂D.

🛑 The divergence theorem is usually taught in PDE. Michael Singer has written an excellent set of notes on it. They are available from the following URL:

`http://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/divthm.pdf`

We will not say more about this theorem here, but we will use it freely.

Finally we turn to the generalisation of the Fundamental Lemma to the case of multidimensional integrals. The lemma now says the following.

Theorem 4 (Multidimensional version of the Fundamental Lemma). *Let $D \subset \mathbb{R}^m$ be a bounded open set with (piecewise) smooth boundary ∂D. Let $f : D \to \mathbb{R}^n$ be a continuous function which obeys*

$$\int_D \langle f(t), h(t) \rangle \, dt = 0$$

for all C^2 functions $h : D \to \mathbb{R}^n$ vanishing on the boundary; that is, with $h|_{\partial D} = 0$. Then $f \equiv 0$.

Proof for $n = 1$. As before we will prove the case $n = 1$ and leave the trivial extension to general n as an exercise. Mutatis mutandis, the proof is the same as if $m = 1$, which was done in Section 4, so we will be brief.

Assume for a contradiction that there exists a point $t_0 \in D$ where $f(t_0) \neq 0$. Without loss of generality we will assume that $f(t_0) > 0$. Then there is an open ball B centred at t_0 contained in D with the property that, for all $t \in B$, $f(t) > c > 0$ for some constant c. We will now construct a function $h : D \to \mathbb{R}$ with the usual properties: it vanishes outside B and it has positive integral $\int_D h dV = \int_B h dV > 0$. Then, as in the proof of Theorem 1, it follows that $\int_D f h dV > 0$, violating the hypothesis.

The construction of the function h is again very similar to what was done before. Inside the open ball B there is a hypercube centred at t_0 with sides of length 2δ for some $\delta > 0$. Explicitly, the hypercube is the cartesian product

$$[t_0^1 - \delta, t_0^1 + \delta] \times [t_0^2 - \delta, t_0^2 + \delta] \times \cdots \times [t_0^m - \delta, t_0^m + \delta] \,,$$

where $t_0 = (t_0^1, t_0^2, \ldots, t_0^m)$. Now define $h(t)$ to be the product of the m functions

$$h(t^1, t^2, \ldots, t^m) = \varphi_{t_0^1 - \delta, t_0^1 + \delta}(t^1) \varphi_{t_0^2 - \delta, t_0^2 + \delta}(t^2) \cdots \varphi_{t_0^m - \delta, t_0^m + \delta}(t^m) \,,$$

where $\varphi_{a,b}(t)$ are defined by equation (4). We leave it as an exercise to the reader to convince herself that this function does the trick. $\square$

✍ **Exercise 19.** *Prove the multidimensional version of the Fundamental Lemma for $n > 1$.*

Let us now do an example. We will consider $u : \mathbb{R}^2 \to \mathbb{R}$, $(x, t) \mapsto u(x, t)$ with lagrangian

$$L(u, Du) = \tfrac{1}{2}(u_x)^2 - \tfrac{1}{2}(u_t)^2 \,,$$

where $u_x = \partial u / \partial x$ and $u_t = \partial u / \partial t$. The resulting Euler–Lagrange equation is then the wave equation:

$$u_{tt} = u_{xx} \,,$$

where $u_{tt} = \partial^2 u / \partial t^2$ and $u_{xx} = \partial^2 u / \partial x^2$.

✎

Exercise 20 (Toy electrodynamics). *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field. Let $A_\mu(t)$, for $\mu = 1, 2$ denote its component functions. Define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, for all $\mu, \nu = 1, 2$. Consider the following lagrangian*

$$L(A, DA) = \tfrac{1}{4} \sum_{\mu,\nu=1}^{2} F_{\mu\nu} F_{\mu\nu} \ .$$

Prove that the Euler–Lagrange equation implies that $F_{\mu\nu}$ is constant.
Repeat this for $A : \mathbb{R}^m \to \mathbb{R}^m$ with the same definitions for $F_{\mu\nu}$ and $L(A, DA)$. Show that the Euler–Lagrange equation implies that $F_{\mu\nu}$ is harmonic; *that is,*

$$\Box F_{\mu\nu} = 0 \qquad \text{for all } \mu, \nu,$$

where $\Box = \sum_{\mu=1}^{m} \partial_\mu \partial_\mu$.

✎ **Exercise 21** (Minimal surfaces). *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be a twice differentiable function. The graph $z = f(x, y)$ defines a surface $\Sigma \subset \mathbb{R}^3$. The area of this surface is the functional of f given by*

$$S[f] = \int_D \sqrt{1 + f_x^2 + f_y^2} \, dx dy \ ,$$

where $f_x = \partial f / \partial x$ and $f_y = \partial f / \partial y$. Prove that f is a critical point of $S[f]$ if and only if it obeys the following second-order nonlinear partial differential equation:

$$(1 + f_y^2) f_{xx} + (1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} = 0 \ .$$

If f satisfies this equation then $\Sigma \subset \mathbb{R}^3$ is said to be a minimal surface.

13. NOETHER'S THEOREM REVISITED

In this section we revisit Noether's theorem for the case of multidimensional lagrangians. This is the version of Noether's theorem which is of most relevance to modern physics, particularly to relativistic field theories and to theories of gravity.

The set up is the following. Let $D \subset \mathbb{R}^m$ be a bounded set with (piecewise) smooth boundary ∂D and let $x : D \to \mathbb{R}^n$ be a C^2 function which is also defined on the boundary. Let $L(x, Dx)$ be a lagrangian which is invariant under a one-parameter group of diffeomorphisms $\varphi_s : \mathbb{R}^n \to \mathbb{R}^n$, for $s \in \mathbb{R}$. Let $y : D \times \mathbb{R} \to \mathbb{R}^n$ be defined by $y(s, t) = \varphi_s(x(t))$. Since the lagrangian is invariant, the same argument as in the proof of Lemma 1 says that if $x(t)$ solves the Euler–Lagrange equations, then so does $y(s, t)$ for all s.

Since $L(y, Dy)$ is actually independent of s, taking the derivative with respect to s we get zero:

$$0 = \frac{\partial}{\partial s} L(y, \dot{y}) = \sum_{i=1}^{n} \frac{\partial L}{\partial y^i} \frac{\partial y^i}{\partial s} + \sum_{i=1}^{n} \sum_{\mu=1}^{m} \frac{\partial L}{\partial y_\mu^i} \frac{\partial y_\mu^i}{\partial s} \ ,$$

where $y_\mu^i = \partial y^i / \partial t^\mu$. Using the Euler–Lagrange equations (8), we can rewrite this as

$$
\begin{aligned}
0 &= \sum_{i=1}^n \sum_{\mu=1}^m \partial_\mu \frac{\partial L}{\partial y_\mu^i} \frac{\partial y^i}{\partial s} + \sum_{i=1}^n \sum_{\mu=1}^m \frac{\partial L}{\partial y_\mu^i} \frac{\partial y_\mu^i}{\partial s} \\
&= \sum_{\mu=1}^m \partial_\mu \sum_{i=1}^n \frac{\partial L}{\partial y_\mu^i} \frac{\partial y^i}{\partial s} .
\end{aligned}
$$

Finally we evaluate at $s = 0$, using that $y(0, t) = x(t)$, to arrive at

$$
\sum_{\mu=1}^m \partial_\mu \sum_{i=1}^n \frac{\partial L}{\partial x_\mu^i} \frac{\partial y^i}{\partial s} \bigg|_{s=0} = 0 .
$$

We can state this result in the form of a theorem.

Theorem 5 (Multidimensional Noether's Theorem)**.** *Let $S[x] = \int_D L(x, Dx) d^k x$ be an action for C^2 maps $x : D \subset \mathbb{R}^m \to \mathbb{R}^n$, and let L be invariant under a one-parameter group of diffeomorphisms $\{\varphi_s\}$. Then the **Noether current** $J : D \to \mathbb{R}^m$, with components J^μ given by*

$$
J^\mu(x, Dx) = \sum_{i=1}^n \frac{\partial L}{\partial x_\mu^i} \frac{\partial y^i}{\partial s} \bigg|_{s=0}
$$

is conserved; that is, its divergence vanishes:

$$
\sum_{\mu=1}^m \partial_\mu J^\mu = 0 .
$$

You may recognise this conservation law as a **continuity equation**. Let us try to understand what this says. Let us assume that $m = 2$ so that the Noether current $J(x, Dx)$ is a vector field defined in a region $D \subset \mathbb{R}^2$ in the plane. Let C be any smooth simple closed curve in D. Let N denote the outward normal to C. Then using the Divergence Theorem and the fact that J is conserved, we deduce that

$$
\int_C \langle N, J \rangle \, ds = 0 ,
$$

where ds is the infinitesimal arclength element in C. We can interpret $\langle N, J \rangle$ as the flux per unit length of the current J. The conservation law of the current simply says that the net flux is zero: as much flux comes into the region enclosed by C as comes out.

The following exercise is a continuation of Exercise 20.

✍ **Exercise 22** (Toy gauge symmetry). *In the notation of Exercise 20, let $A : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $(t, s) \mapsto A_\mu(t, s)$, be a one-parameter family of vector fields in $\mathbb{R}^2$ defined as follows*

$$A_\mu(t, s) = A_\mu(t) + s\partial_\mu\Lambda(t) ,$$

where $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Prove that the lagrangian

$$L(A, DA) = \tfrac{1}{4} \sum_{\mu,\nu=1}^{2} F_{\mu\nu}(t, s)F_{\mu\nu}(t, s)$$

is independent of s and derive the expression for the corresponding Noether current. Prove directly (without appealing to Noether's theorem) that it is conserved when the Euler–Lagrange equations are satisfied.
(Hint: The answer is

$$J^\mu = \sum_\nu F_{\mu\nu}\partial_\nu\Lambda ,$$

and its conservation follows trivially using the equations of motion.)

14. CLASSICAL FIELDS

In this section we consider briefly variational problems where we extremise functionals defined by improper integrals. Variational problems of this type lie at the heart of many of the models used in Physics to describe the fundamental forces of Nature.

As a motivating example consider C^2 functions $\phi : \mathbb{R} \to \mathbb{R}$, $x \mapsto \phi(x)$, defined on the whole real line. Let $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a lagrangian and consider the action

$$S[\phi] = \int_{\mathbb{R}} L(\phi, \phi', x)dx ,$$

where $\phi'(x)$ denotes the derivative of $\phi(x)$ with respect to x. The main difference with what we have done until now, apart from the change of notation of the independent variable (x instead of t), is that the action is now an improper integral, being an integral over all of the real line and not over some bounded subset. As usual we define the integral by the limit

$$S[\phi] = \lim_{R\to\infty} \int_{-R}^{R} L(\phi, \phi', x)dx ,$$

provided the limit exists. The existence of the limit restricts the boundary conditions that we impose on ϕ. Depending on the precise form of L we will have to demand that ϕ or ϕ' decrease sufficiently fast at infinity. For example, if $L(\phi, \phi', x) = \frac{1}{2}\phi'(x)^2$, then we must impose that as $|x| \to \infty$, $\phi'(x) \to 0$ faster than $|x|^{-1/2}$. This allows for the possibility that $\phi(x)$ should tend to a constant ϕ_0 at infinity. If the lagrangian also contains a potential term: $L(\phi, \phi', x) = \frac{1}{2}\phi'(x)^2 - V(\phi)$, then the constant ϕ_0 to which $\phi(x)$ tends at infinity must be such that $V(\phi_0) = 0$.

☞ Functions ϕ are called (classical) fields.

More generally we can consider fields $\phi : \mathbb{R}^m \to \mathbb{R}^n$ and lagrangians $L : \mathbb{R}^n \times \mathbb{R}^{nk} \times \mathbb{R}^m \to \mathbb{R}$ giving rise to actions

$$S[\phi] = \int_{\mathbb{R}^m} L(\phi, D\phi, x) \, d^k x ,$$

where the integral is now defined by taking the limit

$$S[\phi] = \lim_{R \to \infty} \int_{B_R} L(\phi, D\phi, x) \, d^k x \ ,$$

where B_R is the open ball of radius R centred at the origin. If $S_R = \partial B_R$ denotes the sphere of radius R, then the boundary conditions on the fields are now given on the "sphere at infinity": $S_\infty = \lim_{R \to \infty} S_R$. Variations will now be functions $\varepsilon : \mathbb{R}^m \to \mathbb{R}^n$ vanishing sufficiently fast at infinity. In fact, in varying the action we will get a term

$$\lim_{R \to \infty} \int_{B_R} \sum_{\mu=1}^{m} \partial_\mu \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \phi_\mu^i} \varepsilon^i \right) d^m x \ ,$$

where $\phi_\mu^i = \partial_\mu \phi^i$. By the Divergence Theorem this can can be rewritten as

$$\lim_{R \to \infty} \int_{S_R} \sum_{\mu=1}^{m} \sum_{i=1}^{n} \frac{\partial L}{\partial \phi_\mu^i} \varepsilon^i \, N^\mu \, dS \ ,$$

where N^μ is the outward pointing normal to S_R and dS is the induced infinitesimal surface element on S_R. Variations must be chosen in such a way that this integral vanishes. There are two common choices: either we demand that ε should decrease fast enough as $|x| \to \infty$, or else we demand that ε should have compact support; that is, that they vanish outside some large enough ball B_R. Either choice will result in the Euler–Lagrange equations

$$\frac{\partial L}{\partial \phi^i} = \sum_{\mu=1}^{m} \partial_\mu \frac{\partial L}{\partial \phi_\mu^i} \ , \tag{9}$$

because the Fundamental Lemma holds for such functions. In fact, it follows already from the proof of, say, Theorem 4 that the function h constructed there already has compact support.

Noether's Theorem still holds for actions defined by improper integrals, provided that the one parameter of fields is such that the action integral still converges; but, in fact, in applications one often meets a stronger form of Noether's Theorem. In the proof of, say, Theorem 5 it was assumed that the lagrangian was invariant under the one-parameter group of diffeomorphisms. In fact, this assumption is too strong: it is enough that the action be invariant. Let us see how this comes about.

Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a field and let $\Phi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, $(x, s) \to \Phi(x, s)$, be a one-parameter family of such fields, varying smoothly with s and with the property that $\Phi(x, 0) = \phi(x)$. Let us assume that action integral

$$S[\Phi] = \int_{\mathbb{R}^m} L(\Phi, D\Phi, x) \, d^m x$$

exists and moreover is independent of s. Although sufficient, it is certainly not necessary that the lagrangian be independent of s. For instance, suppose that, instead of vanishing,

$$\frac{\partial}{\partial s} L(\Phi, D\Phi, x) = \sum_{\mu=1}^{m} \partial_\mu K^\mu(\Phi, D\Phi, x) \ ,$$

where $K : \mathbb{R}^n \times \mathbb{R}^{kn} \times \mathbb{R}^m \to \mathbb{R}^m$ is some differentiable function. Differentiating inside the integral sign and using the above equation we find

$$
\begin{aligned}
\frac{\partial}{\partial s} S[\Phi] &= \int_{\mathbb{R}^m} \frac{\partial}{\partial s} L(\Phi, D\Phi, x) \, d^m x \\
&= \int_{\mathbb{R}^m} \sum_{\mu=1}^{m} \partial_\mu K^\mu(\Phi, D\Phi, x) \, d^m x \\
&= \int_{S_\infty} K^\mu(\Phi, D\Phi, x) N_\mu \, dS ,
\end{aligned}
$$

where in the last line we have used the Divergence Theorem. The action is invariant provided that this integral vanishes, say, if K is such that with suitable decay properties of Φ and $D\Phi$ at infinity, $K(\Phi, D\Phi, x)$ vanishes fast enough at infinity. The proof of Noether's Theorem goes through as before with the only difference that the Noether current picks up an extra term:

$$
J^\mu = \sum_{i=1}^{n} \frac{\partial L}{\partial \phi^i_\mu} \frac{\partial \Phi^i}{\partial s} \bigg|_{s=0} - K^\mu .
$$

✍ **Exercise 23.** *State and prove the generalisation of Noether's Theorem outlined above.*

As a typical example consider the following lagrangian for fields $\phi : \mathbb{R}^m \to \mathbb{R}^n$,

$$
L(\phi, D\phi) = \tfrac{1}{2} \|D\phi\|^2 = \tfrac{1}{2} \sum_{\mu=1}^{m} \sum_{i=1}^{n} \partial_\mu \phi^i \partial_\mu \phi^i .
$$

✍ **Exercise 24.** *Show that the Euler–Lagrange equations of this lagrangian say that ϕ is a harmonic function: $\Box \phi^i = 0$, where $\Box = \sum_{\mu=1}^{m} \partial_\mu \partial_\mu$.*

Consider the following one-parameter family of fields

$$
\Phi(x, s) = \phi(x + sa) ,
$$

where $a \in \mathbb{R}^m$ is a constant vector.

🛑 Notice that this one-parameter family of fields is *not* induced by a one-parameter family of diffeomorphisms of the "target space" $\mathbb{R}^n$, but rather by a one-parameter family of diffeomorphisms of the domain $\mathbb{R}^m$, in fact, by translations.

✍ **Exercise 25.** *Show that the lagrangian depends on s; indeed, show that*

$$
\frac{\partial L}{\partial s} = \sum_{\mu=1}^{m} \partial_\mu \left(a^\mu L \right) .
$$

The conserved Noether current is given by

$$
J^\mu = \sum_{\nu=1}^{m} \sum_{i=1}^{n} \partial_\mu \phi^i \partial_\nu \phi^i a^\nu - L a^\mu .
$$

We write it as

$$J^\mu = \sum_{\nu=1}^{m} T_{\mu\nu} a^\nu \ ,$$

where

$$T_{\mu\nu} = \sum_{i=1}^{n} \partial_\mu \phi^i \partial_\nu \phi^i - \delta_{\mu\nu} L \ ,$$

is the **energy-momentum tensor** associated to the above lagrangian. Notice that it is symmetric: $T_{\mu\nu} = T_{\nu\mu}$.

✍ **Exercise 26.** *Without appealing to Noether's Theorem, prove directly that the energy-momentum tensor is conserved when the Euler–Lagrange equations are satisfied; that is,*

$$\sum_{\mu=1}^{m} \partial_\mu T_{\mu\nu} = 0 \ .$$

In practice, the domain $\mathbb{R}^m$ where the fields are defined is interpreted physically as "space-time". This means that one of the coordinates is singled out to be "time" and the remaining $m-1$ coordinates are thought of as "space". The Euler–Lagrange equations are then interpreted as determining the time evolution of some initial data specified at a given initial time slice. In more detail, let $x^n = t$ denote the time coordinate and let x^i for $i = 1, \ldots, n-1$ denote the space coordinates. Then the Euler–Lagrange equations (9), supplemented by initial conditions $\phi|_{t=\text{constant}} = f(x^1, \ldots, x^{n-1})$ and $\partial_t \phi|_{t=\text{constant}} = g(x^1, \ldots, x^{n-1})$, tell us how this initial data propagates in time.

In this situation Noether's Theorem can again be interpreted as a certain quantity being constant in time. Indeed, consider the integral of the time component J^n of the Noether current on a hypersurface at constant time. If the integral converges, this defines the *Noether charge* associated to the current and is usually denoted Q:

$$Q = \int_{t=\text{constant}} J^n d^{m-1} x \ .$$

To see how Q evolves in time, we compute the derivative of Q with respect to time

$$\frac{dQ}{dt} = \int_{t=\text{constant}} \frac{\partial}{\partial t} J^n d^{m-1} x \ .$$

The conservation of the Noether current says that

$$\frac{\partial J^n}{\partial t} = \frac{\partial J^n}{\partial x^n} = -\sum_{\mu=1}^{n-1} \frac{\partial J^\mu}{\partial x^\mu} \ ,$$

whence

$$\frac{dQ}{dt} = -\int_{t=\text{constant}} \sum_{\mu=1}^{n-1} \frac{\partial J^\mu}{\partial x^\mu} d^{m-1} x \ ,$$

which, using the Divergence Theorem, vanishes provided that the spatial components J^μ, $\mu = 1, \ldots, n-1$, of the Noether current decrease fast enough at *spatial* infinity.

Appendix A. Extra problems

In this appendix we collect further problems and applications of the calculus of variations. They are in the form of exercises.

A.1. **Probability and maximum entropy.** Let X be a random variable taking values in the real line. The probability that X takes a value less than or equal to a given real number x is obtained by integrating the probability density ρ:

$$P(X \leq x) = \int_{-\infty}^{x} \rho(y) dy \ .$$

Since X must take some value, we have that

$$\int_{\mathbb{R}} \rho(x) dx = 1 \ .$$

In many problems one is interested in determining the probability density ρ, based on knowledge of certain expectation values. For instance, suppose that we know that the variance of ρ is given by σ^2, for some $\sigma \in \mathbb{R}$. In other words, we know that

$$\sigma^2 = \int_{\mathbb{R}} x^2 \rho(x) dx \ .$$

Which is the "least-biased" probability distribution ρ which satisfies this? The answer is provided by a variational principle called the principle of maximum entropy. This principle states that ρ is obtained by maximising the entropy

$$S[\rho] = - \int_{\mathbb{R}} \rho(x) \log \rho(x) dx \ ,$$

subject to the constraints

$$\int_{\mathbb{R}} \rho(x) dx = 1 \qquad \text{and} \qquad \sigma^2 = \int_{\mathbb{R}} x^2 \rho(x) dx \ .$$

✎ **Exercise 27** (Maximum entropy and the normal distribution)**.** *Prove that the solution to the above variational problem is the normal distribution:*

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \ .$$

Now suppose that the random variable X takes non-negative values, and let $\rho : [0, \infty) \to \mathbb{R}$ be the corresponding probability density.

✎ **Exercise 28.** *Prove that the probability density which maximises entropy subject to the condition that the expectation value of X is μ, is the exponential distribution:*

$$\rho(x) = \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right) \ .$$

A.2. **Maximum entropy in statistical mechanics.** The maximum entropy principle originates in the branch of Physics known as Statistical Mechanics. The aim of statistical mechanics is to provide a microscopic explanation for thermodynamic phenomena. Given a physical system consisting of a large number of randomly distributed particles (e.g., a gas) one would like to explain thermodynamic properties of the system (e.g., pressure, temperature, heat content) starting from the dynamics of the individual particles. When there are many particles, solving for the dynamics of each individual particle is not practical, and this is where statistical mechanics steps in. Taking its cue from the theory of probability, it interprets thermodynamic quantities as expectation values of certain functions relative to a probability density which has to be determined. The principle of maximum entropy can be used to determine a probability density subject to constraints involving the thermodynamic quantities.

As an example, let us consider the Maxwell velocity distribution of a gas. The probability density is a function $\rho : \mathbb{R}^3 \to \mathbb{R}$, $v \mapsto \rho(v)$, where $v \in \mathbb{R}^3$ is to be interpreted as the velocity vector of a gas molecule. The probability density ρ can be used to determine the probability that a gas molecule has velocity in a given subset of $\mathbb{R}^3$. The Maxwell probability density is obtained by applying the principle of maximum entropy to ρ subject to the following constraints:

$$\int_{\mathbb{R}^3} \rho(v)d^3v = 1$$

$$\int_{\mathbb{R}^3} v^i \rho(v)d^3v = 0 \qquad \text{for } i = 1,2,3$$

$$\int_{\mathbb{R}^3} \tfrac{1}{2}\|v\|^2 \rho(v)d^3v = E \ ,$$

where E is the *internal energy*.

✍ **Exercise 29** (Maxwell distribution)**.** *Prove that the Maxwell probability density is given by*

$$\rho(v) = \left(\frac{3}{4\pi E}\right)^{3/2} \exp\left(-\frac{3\|v\|^2}{4E}\right) \ .$$

What is the formula for the Maxwell distribution of a d-dimensional gas?
Answer:

$$\rho(v) = \left(\frac{d}{4\pi E}\right)^{d/2} \exp\left(-\frac{d\|v\|^2}{4E}\right) \ .$$

A.3. **Geodesics, harmonic maps and Killing vectors.** Let us consider C^2 curves $x : [0,1] \to \mathbb{R}^n$ and a function g defined on $\mathbb{R}^n$ which assigns to every point in $\mathbb{R}^n$ an invertible symmetric matrix with entries g_{ij}. Consider the following lagrangian

$$L(x,\dot{x}) = \tfrac{1}{2} \sum_{i,j=1}^{n} g_{ij}(x)\dot{x}^i\dot{x}^j dt \ . \tag{10}$$

☞ This lagrangian defines the one-dimensional sigma model.

✎ **Exercise 30.** *Prove that the Euler–Lagrange equations associated to this lagrangian are given by the* geodesic equation

$$\ddot{x}^i + \sum_{j,k=1}^n \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 ,$$

where the Christoffel symbols Γ^i_{jk} *are defined by*

$$\Gamma^i_{jk} = \tfrac{1}{2} \sum_{\ell=1}^n g^{i\ell} \left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right) ,$$

where g^{ij} are the entries of the matrix inverse to g.

More generally, let $D \subset \mathbb{R}^m$ be a bounded set of $\mathbb{R}^m$ and let $x : D \to \mathbb{R}^n$ be a C^2 function. Consider now the lagrangian

$$L(x, Dx) = \tfrac{1}{2} \sum_{\mu=1}^m \sum_{i,j=1}^n g_{ij}(x) \partial_\mu x^i \partial_\mu x^j dt . \tag{11}$$

☞ Not surprisingly, this lagrangian defines the m-dimensional sigma model.

✎ **Exercise 31.** *Prove that the Euler–Lagrange equations associated to this lagrangian are given by the* harmonic map equation

$$\Box x^i + \sum_{\mu=1}^m \sum_{j,k=1}^n \Gamma^i_{jk} \partial_\mu x^j \partial_\mu x^k = 0 .$$

Now let us study symmetries of the lagrangian given by (10). Let $\varphi_s : \mathbb{R}^n \to \mathbb{R}^n$ be a one-parameter family of diffeomorphisms of $\mathbb{R}^n$, and let $y(t, s) = \varphi_s(x(t))$ be a one parameter of C^2 curves $y : [0, 1] \times \mathbb{R} \to \mathbb{R}^n$, $(t, s) \mapsto y(t, s)$.

✎ **Exercise 32.** *Prove that φ_s is a symmetry of the lagrangian (10) if the following equation holds*

$$\sum_{i=1}^n \left(g_{ij} \frac{\partial K^i}{\partial x^k} + g_{ik} \frac{\partial K^i}{\partial x^j} + K^i \frac{\partial g_{jk}}{\partial x^i} \right) = 0 , \tag{12}$$

where K^i are the components of the vector field $K : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$K^i(x) = \left. \frac{\partial y^i(t, s)}{\partial s} \right|_{s=0} .$$

☞ Equation (12) is called Killing's equation and K is said to be a Killing vector field.

According to Noether's Theorem, there is a conserved charge associated to every Killing vector.

✎ **Exercise 33.** *Find the expression for the conserved charge associated to the Killing vector K and prove directly (i.e., without recourse to Noether's Theorem) that it is conserved provided that the Euler–Lagrange equations are satisfied.*
Answer: The Noether charge Q is given by

$$Q = \sum_{i,j=1}^n g_{ij} \dot{x}^i K^j .$$

The study of symmetries of the lagrangian given by (11) follows in the same way.

✍ **Exercise 34.** *Fill in the details of the previous statement.*

A.4. Geodesics on surfaces of revolution.

In this section we will consider geodesic curves on surfaces of revolution. Given a differentiable function $f : \mathbb{R} \to \mathbb{R}$ defined on (some interval of) the real line, one defines a surface $S_f \subset \mathbb{R}^3$ parametrised as follows

$$(u, v) \mapsto (x(u,v), y(u,v), z(u,v)) = (f(v) \cos u, f(v) \sin u, v) \ ,$$

where $0 \leq u < 2\pi$ and v takes values in the domain of definition of f. The surface S_f is called the **surface of revolution with profile** f. Examples of surfaces of revolution include the cylinder, the cone and the sphere.

☞ It is often convenient to let the "angle" u range over all of the real line, resulting in an infinitely redundant parametrisation of S_f.

Let $\gamma : [0, 1] \to S_f$, $t \mapsto \gamma(t)$ be a curve on S_f. Equivalently we can think of it as a curve in the parameter space $t \mapsto (u(t), v(t))$, where $\gamma(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$.

✍ **Exercise 35.** *Show that this curve has arclength given by*

$$I[u, v] = \int_0^1 \sqrt{(1 + f'^2)\dot{v}^2 + f^2 \dot{u}^2} \, dt \ ,$$

where $'$ denotes the derivative with respect to v. Write down the associated Euler–Lagrange equations.

By analogy with the sphere, we will call the image in S_f of a curve of constant u (resp. v) a **meridian** (resp. **parallel**).

✍ **Exercise 36.** *Show that all meridians are geodesic, and that a parallel $v = v_0$ is geodesic if and only if $f'(v_0) = 0$. Re-examine in this light the result of Exercise 9.*

Let γ be a geodesic on the surface of revolution S_f. Suppose that the curve traced by γ on S_f can be expressed in terms of a functional relation $u = u(v)$ between the parameters of the surface. The arclength functional then becomes

$$I[u] = \int_{v_0}^{v_1} \sqrt{1 + f'^2 + f^2 u'^2} \, dv \ ,$$

where $'$ again denotes the derivative with respect to v.

✍ **Exercise 37.** *Derive the Euler–Lagrange equations associated to the above functional and give an integral expression for $u = u(v)$ in terms of f.*
(Hint: Notice that the lagrangian does not depend on u, but only on u', whence $\partial L / \partial u'$ is constant.)