A mathematical exploration of space

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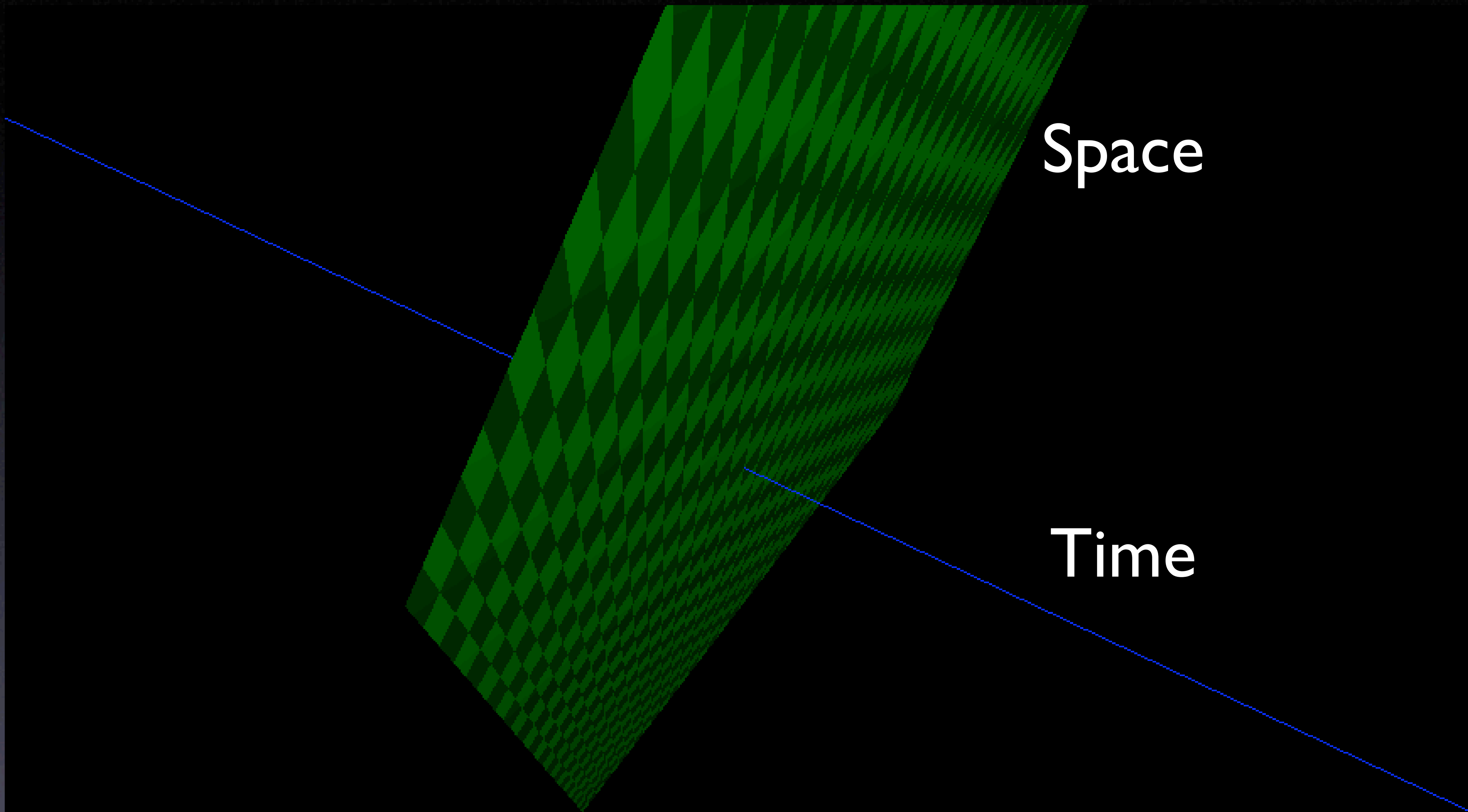
The Newtonian Universe

"I do not define time, space, place and motion, since they are well known to all."

"Absolute space, in its own nature, without relation to anything external, remains always similar and immovable."

"Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external."

Philosophiae Naturalis
Principia Mathematica



Newton's universe is $\mathbb{A} \times \mathbb{A}^3$

Time intervals and distances between simultaneous events are independent of the observer:

$$|t_1 - t_2|$$

$$|\boldsymbol{x}_1 - \boldsymbol{x}_2| \quad \text{if} \quad t_1 = t_2$$

Mathematically, the automorphisms form the Galilean group:

$$\begin{pmatrix} A & \boldsymbol{v} & \boldsymbol{a} \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} A\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a} \\ t + \tau \\ 1 \end{pmatrix}$$

$$A \in \mathrm{O}(3) \qquad \boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3 \qquad \tau \in \mathbb{R}$$

Maxwell's equations

$$\mathrm{curl}\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\mathrm{curl}\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\mathrm{div}\boldsymbol{E} = 0$$

$$\mathrm{div}\boldsymbol{B} = 0$$

$$\Rightarrow$$

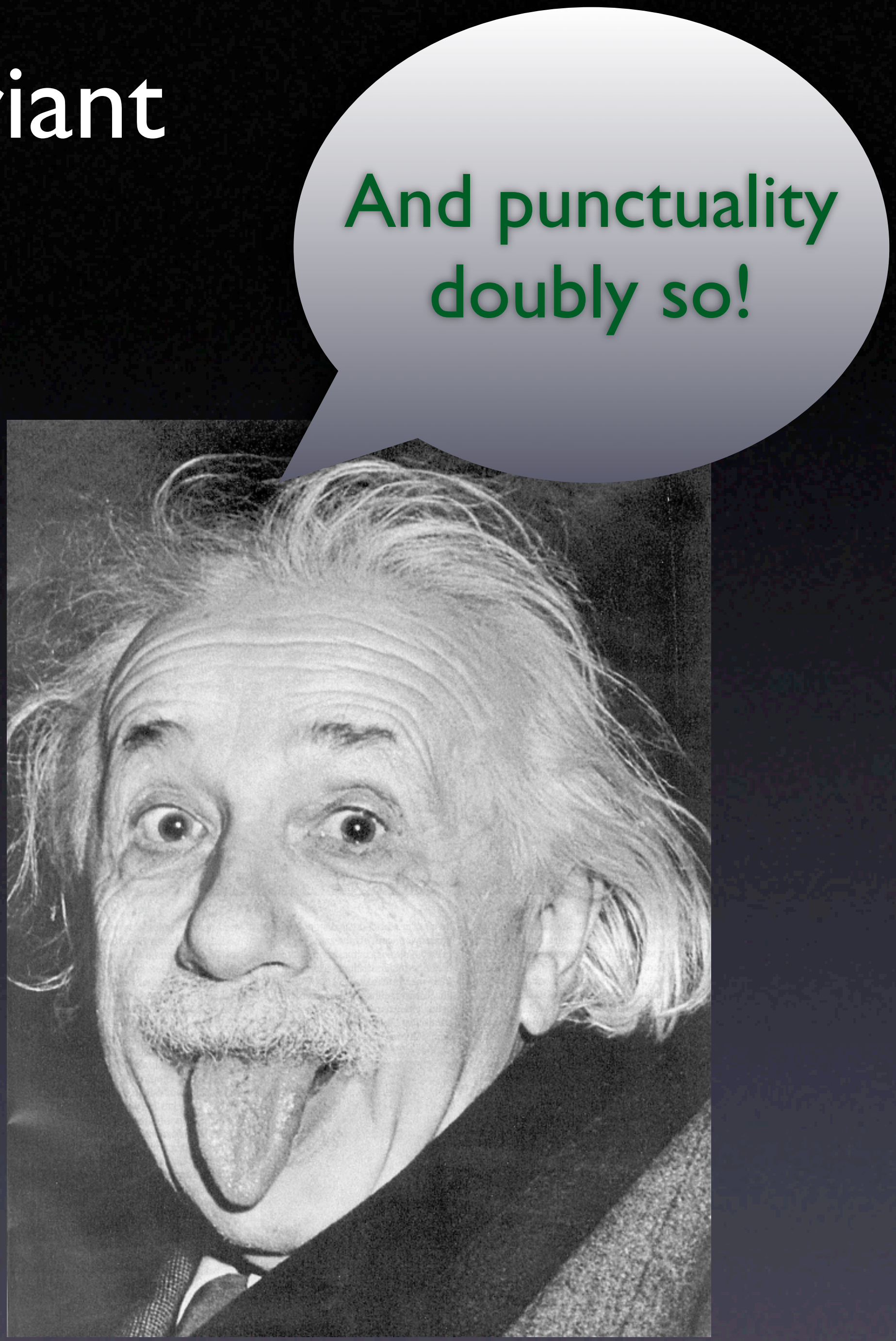
$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

$$\Rightarrow$$

Light!

But Maxwell’s equations are not invariant under the Galilean group!



Einstein’s Special Relativity:

Simultaneity is an illusion!

The invariant quantity is the `proper time`:

$$|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2 - c^2(t_1 - t_2)^2$$

between two `spacetime events` $(\boldsymbol{x}_i, t_i)$.

Minkowski spacetime

The affine space $\mathbb{A}^4$
of spacetime events

$$X = \begin{pmatrix} x \\ y \\ z \\ t \\ 1 \end{pmatrix}$$

together with a scalar product on differences

$$\langle X - X', X - X' \rangle = (x - x')^2 + (y - y')^2 + (z - z')^2 - c^2(t - t')^2$$

The automorphisms now consist of the Poincaré group:

$$\begin{pmatrix} A & \boldsymbol{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \begin{pmatrix} A\boldsymbol{x} + \boldsymbol{a} \\ 1 \end{pmatrix}$$

$$A \in \mathrm{O}(3,1) \quad \text{and} \quad \boldsymbol{a} \in \mathbb{R}^4$$

the Lorentz group, which leaves invariant the indefinite scalar product

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix}$$

"People slowly accustomed themselves to the idea that the physical states of space itself were the final physical reality."

Albert Einstein

General Relativity

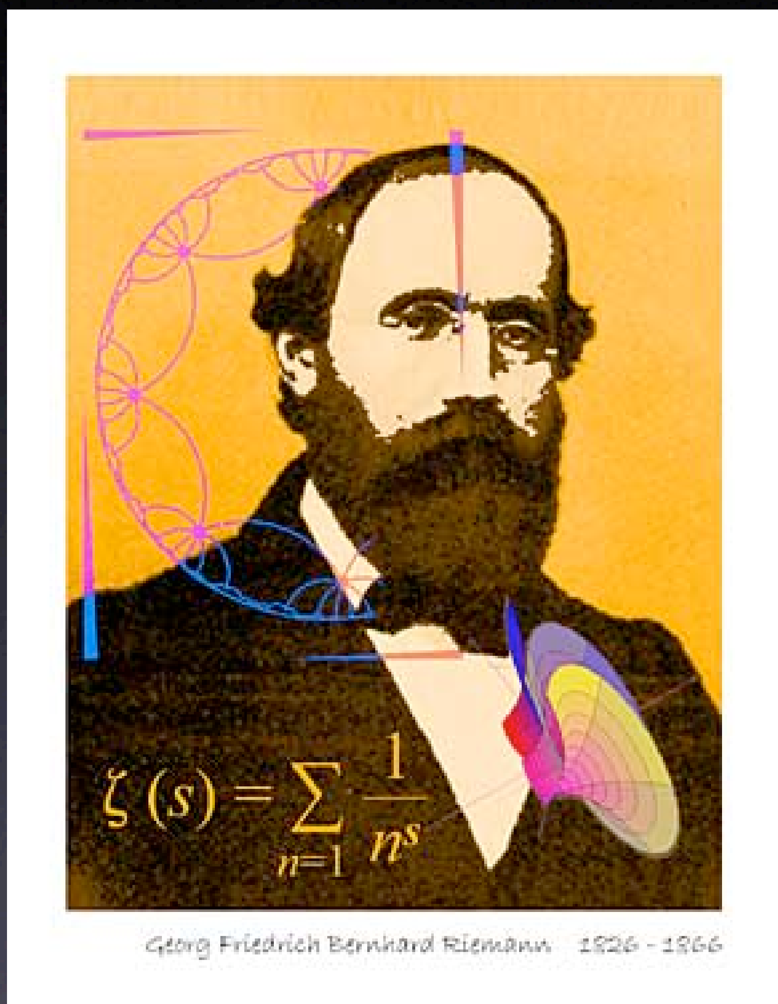
Equivalence Principle:

$$\text{inertial mass} = \text{gravitational mass}$$

$\Rightarrow$ geometrisation of gravity

"Spacetime tells matter how to move; matter tells spacetime how to curve."

Riemannian geometry



The edifice upon which our classical notion of space is built.

- a collection of points
- a notion of "closeness"
- a notion of "distance"

a topological manifold with a metric

To be concrete:

a domain $M \subset \mathbb{R}^N$

a family of inner products $g = \sum_{a,b=1}^{N} g_{ab}(x) dx^a dx^b$ parametrized by M

In GR, at every $x \in M \subset \mathbb{R}^4$, $g(x)$ has signature $(3, 1)$

(M, g) is a **lorentzian manifold**

subject to the *Einstein field equations*

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}$$

Since their inception in 1916, these equations have been extended in a number of directions:

- extra dimensions
- extra terms
- extra equations

but they are just variations on the same theme.

They do <u>not</u> alter fundamentally our notion of space.

Space is still a <u>collection of points</u>.

Quantum Mechanics

In quantum mechanics, points become fuzzy.

Quantum theory requires a notion of space which does not depend on the notion of point.

Von Neumann called this

"*pointless geometry*"

The algebra of points

X a compact Hausdorff topological space

e.g., $X \subset \mathbb{R}^N$ closed and bounded

$C(X)$, the continuous functions $f : X \to \mathbb{C}$

$C(X)$ is a commutative C* algebra

$x \in X$ defines an algebra homomorphism $C(X) \to \mathbb{C}$ by $f \mapsto f(x)$, whose kernel is a maximal ideal $\mathfrak{m}_x \subset C(X)$

and conversely...

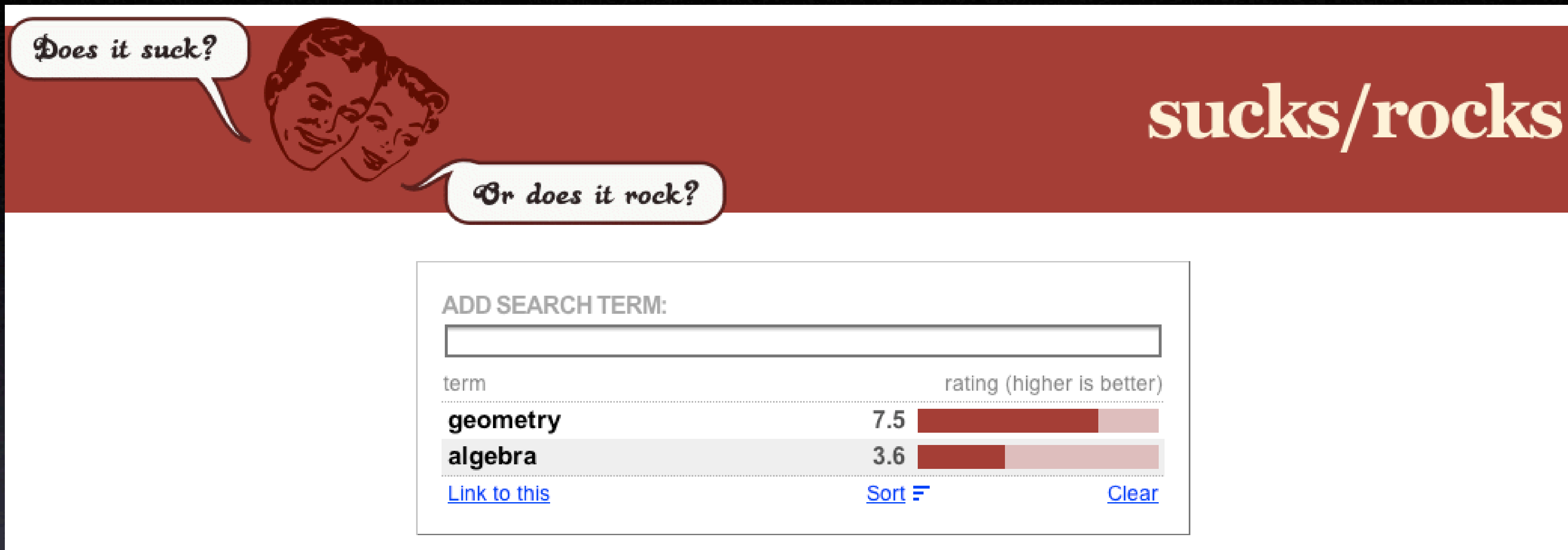
Gelfand's Theorem

Every commutative C* algebra $\mathcal{A}$ <u>is</u> $C(X)$ for some X!

points of X $\leftrightarrow$ maximal ideals $\mathfrak{m} \subset \mathcal{A}$

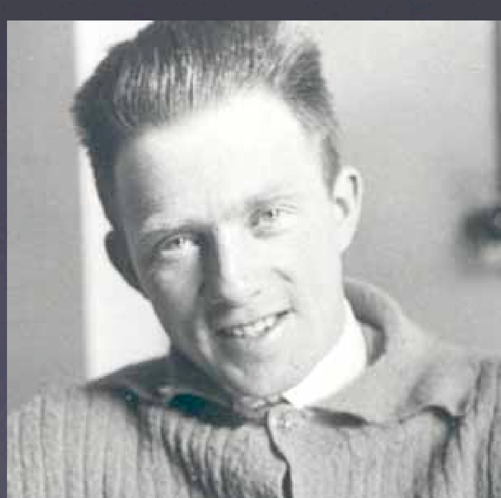
"closeness" in X $\leftrightarrow$ Zariski topology in $\mathrm{Spec}(\mathcal{A})$

Geometry becomes algebra!



http://sucks-rocks.com

	Classical Mechanics	Quantum Mechanics
States	Phase space	Hilbert space
Observables	Functions	Self-adjoint operators

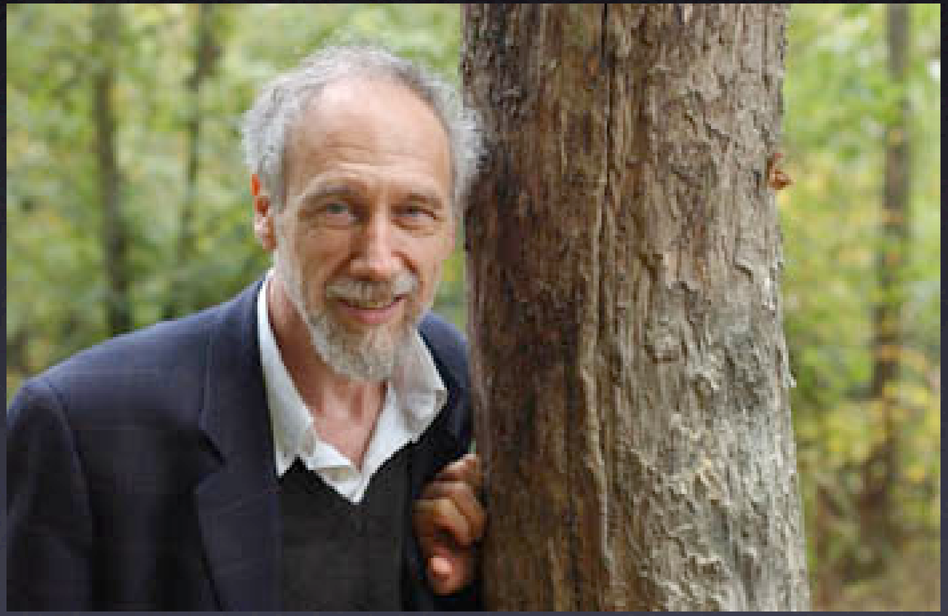


Heisenberg relation:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

Noncommutative

Noncommutative Differential Geometry



Alain Connes

Replace differential and integral calculus with the following:

a Hilbert space $\mathcal{H}$

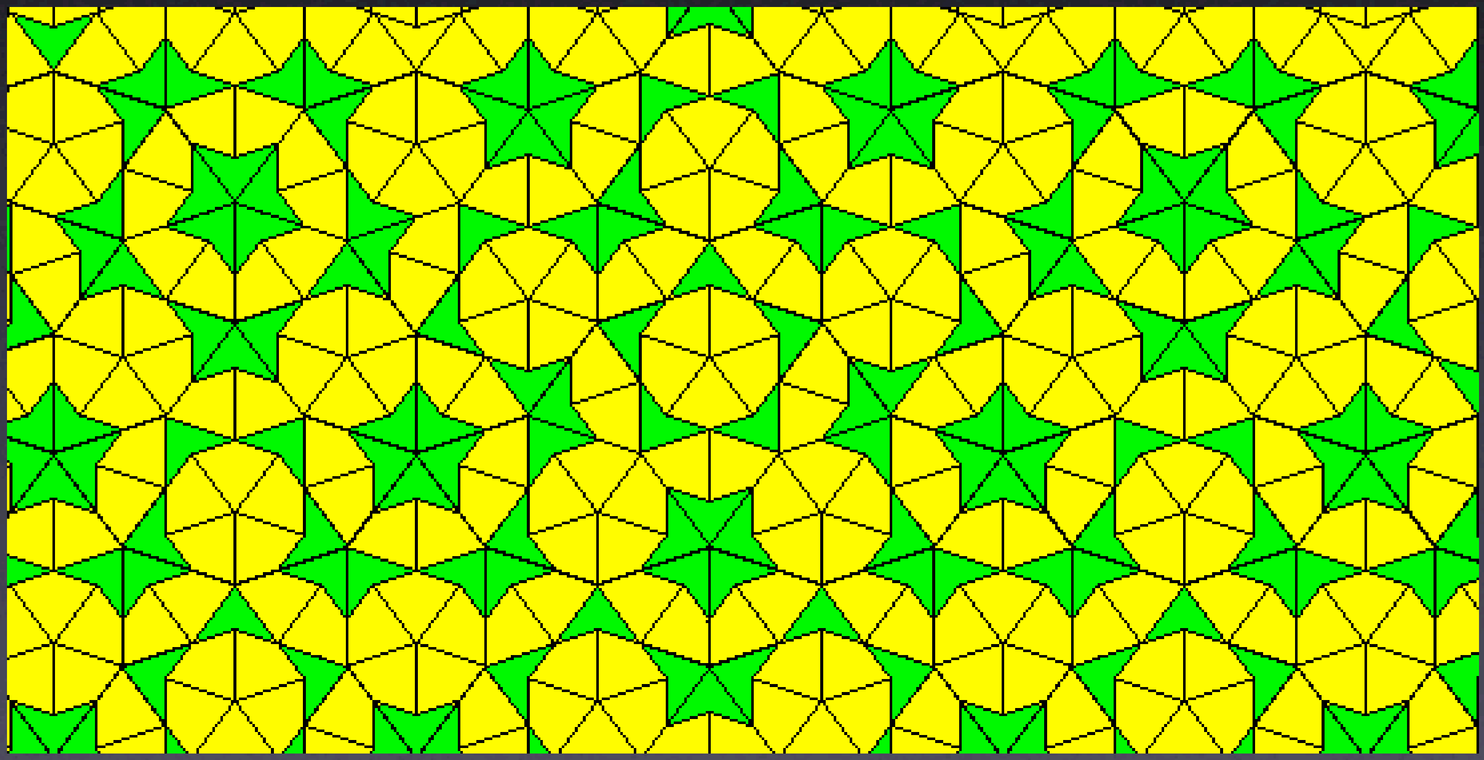
$$F : \mathcal{H} \to \mathcal{H} \quad F^{\star} = F \quad F^2 = 1$$

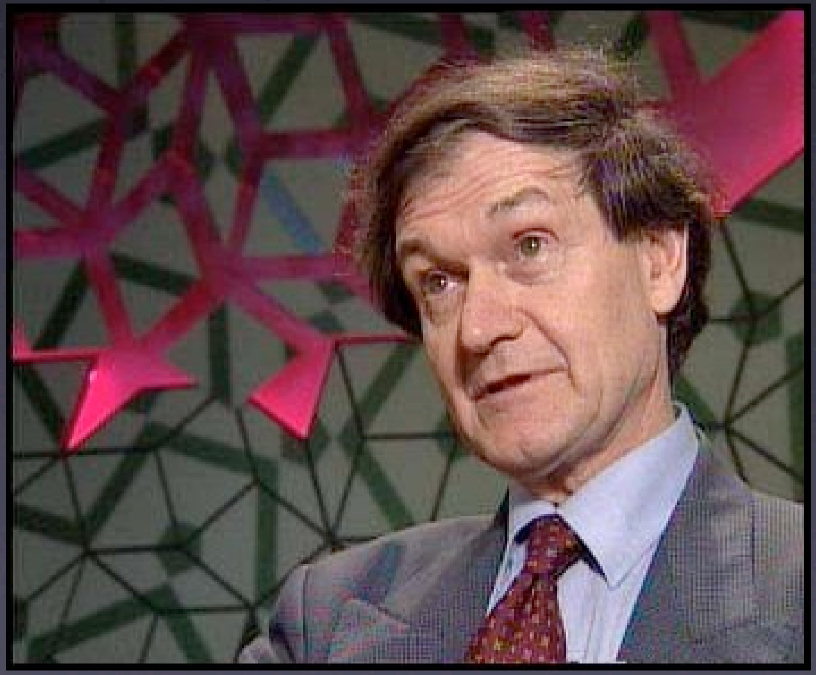
and the following dictionary

Classical	Noncommutative
Complex variable	operator in $\mathcal{H}$
Real variable	self-adjoint operator in $\mathcal{H}$
Infinitesimal	compact operator in $\mathcal{H}$
Differentiation	$df = [F, f] = Ff - fF$
Integration	Dixmier trace

All geometric constructions have noncommutative analogues, including the metric properties.

We can do calculations in noncommutative spaces such as the Penrose tilings:





To recap:

1680s	Newton's $\mathbb{A} \times \mathbb{A}^3$
1900s	Minkowski spacetime $\mathbb{A}^4$
1920s	Lorentzian manifolds (M, g)
Today	$\begin{cases} \text{suped-up lorentzian manifolds} & (M, g, \dots) \\ \text{non-commutative spaces} & (\mathcal{A}, \mathcal{H}, \dots) \end{cases}$
Tomorrow ?	

Live long and prosper.