ORDINARY DIFFERENTIAL EQUATIONS
SPRING AND SUMMER TERMS 2002

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Contents

1. Overview and basic concepts

1.1. Initial value problems. The simplest ODE is

$$\frac{dx}{dt} = ax \ , \tag{1}$$

where $a \in \mathbb{R}$ is a constant and $x : \mathbb{R} \to \mathbb{R}$ is an unknown function taking $t \mapsto x(t)$. This equation can be written equivalently as

$$x' = ax \qquad \text{or} \qquad x'(t) = ax(t) \ .$$

All solutions of (1) are of the form

$$x(t) = Ce^{at} \ ,$$

where $C \in \mathbb{R}$ is some constant. This is easily proven: if $x(t)$ is any solution of (1) then consider $C(t) = x(t)e^{-at}$. Taking the derivative, one shows that $C' = 0$, whence it is a constant.

We fix this constant by specifying the value of x at some given point t_0: if $x(t_0) = x_0$, then $C = x_0 e^{-at_0}$, so that

$$x(t) = x_0 e^{a(t-t_0)} \ .$$

Without loss of generality we can take $t_0 = 0$, for if $u(t)$ solves (1) with $u(0) = u_0$, then $v(t) = u(t - t_0)$ obeys (1) with $v(t_0) = u_0$.

Therefore we often recast ODEs as **initial value problems**:

$$\boxed{x' = ax \qquad \text{and} \qquad x(0) = C} \ .$$

We have just proven that such initial value problems have a unique solution.

1.2. Systems and phase portraits. We will spend most of this course studying systems of ODEs, e.g.,

$$\begin{aligned} x_1' &= a_1 x_1 \\ x_2' &= a_2 x_2 \ , \end{aligned} \tag{2}$$

where $a_i, a_2 \in \mathbb{R}$ are constants and $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are the unknown functions of a real variable t. This particular system is decoupled, and the most general solution is of the form

$$\begin{aligned} x_1(t) &= C_1 e^{a_1 t} \\ x_2(t) &= C_2 e^{a_2 t} \ , \end{aligned}$$

for some constants $C_1, C_2 \in \mathbb{R}$ which can be determined from initial conditions: $C_1 = x_1(0)$ and $C_2 = x_2(0)$.

Let us describe this system geometrically. The functions $x_1(t)$ and $x_2(t)$ specify a **curve** $x(t) = (x_1(t), x_2(t))$ in the (x_1, x_2)-plane $\mathbb{R}^2$. The derivative $x'(t)$ is called the **tangent vector** to the curve at t (or at $x(t)$). We can rewrite (2) as

$$x' = Ax = (a_1 x_1, a_2 x_2) \ ,$$

where Ax is a vector based at x. The map $A : \mathbb{R}^2 \to \mathbb{R}^2$ sending $x \mapsto Ax$ is a **vector field** in the plane $\mathbb{R}^2$. In other words, it is an assignment to every point x in the plane, of a vector Ax based at x. We picture this by drawing at the point $x \in \mathbb{R}^2$, a directed line segment from x to $x + Ax$.

Solving the system (2) consists in finding curves $x : \mathbb{R} \to \mathbb{R}^2$ in the plane, whose tangent vector $x'(t)$ agree with the vector field $Ax(t)$ for all t. Initial conditions are of the form

$$x(0) = u \ ,$$

where $u \in \mathbb{R}^2$ is a point in the plane. This just means that at $t = 0$, the curve passes through $u \in \mathbb{R}^2$.

Existence and uniqueness of solutions to (2) means that through every point $u \in \mathbb{R}^2$ there passes *one and only one* solution curve.

Notice that for this to be the case, the trivial solution $x(t) = (0, 0)$ for all t is also considered a curve.

The family of all solution curves as subsets of $\mathbb{R}^2$ is called the **phase portrait** of the ODE.

1.3. **Linear systems with constant coefficients.** Generalising the above, consider the linear system

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\ \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \ .
\end{aligned} \tag{3}$$

Here $a_{ij} \in \mathbb{R}$ are n^2 constants, whereas $x_i : \mathbb{R} \to \mathbb{R}$ is an unknown function of a real variable t.

At the most primitive level, solving this system consists of finding n differentiable functions x_i which obey (3).

More conceptually, a solution of (3) is a curve in $\mathbb{R}^n$.

The Cartesian space $\mathbb{R}^n$ is the space of ordered n-tuples of real numbers. An element of $\mathbb{R}^n$ is a point $x = (x_1, x_2, \ldots, x_n)$, and the number x_i is called the i-th **coordinate** of the point. Points can be added coordinatewise

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \ ,$$

and can also be multiplied coordinatewise by **scalars** $\lambda \in \mathbb{R}$

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n) \ .$$

The **distance** between points x and y is

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \ ,$$

and the **length** of a point x is the distance to the origin:

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \ .$$

The **origin**, of course, is the point

$$0 = (0, \ldots, 0) \in \mathbb{R}^n \ .$$

A vector **based at** $x \in \mathbb{R}^n$ is an ordered pair of points x, y in $\mathbb{R}^n$ and denoted $\vec{xy}$. We visualise it as an arrow from x to y. A vector $\vec{0x}$ based at the origin can be identified with the point $x \in \mathbb{R}^n$. To a vector $\vec{xy}$ based at x is associated the vector $y - x$ based at the origin. We say that $\vec{xy}$ and $y - x$ are translates of each other.

We will reserve the name vector for vectors based at the origin. Their translates are useful in visualisation, but for computations we always use vectors based at the origin, as only those can be added and multiplied by scalars.

A candidate solution of (3) is a **curve** in $\mathbb{R}^n$,

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \ ,$$

by which we mean a map $x : \mathbb{R} \to \mathbb{R}^n$ sending $t \mapsto x(t)$. This map is called **differentiable** if and only if each of the coordinate functions x_i is differentiable. The derivative is defined by

$$\frac{dx}{dt} = x'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)) \ ,$$

and it is also a map from $\mathbb{R}$ to $\mathbb{R}^n$.

The translate $v(t)$ of $x'(t)$ which is based at $x(t)$ can be interpreted geometrically as the tangent vector to the curve at t (or at $x(t)$). Its length $\|x'(t)\|$ is interpreted physically as the speed of the particle whose motion is described by the curve $x(t)$.

We can use matrices to write (3) more succinctly. Let

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

For each $x \in \mathbb{R}^n$ we define a vector $Ax \in \mathbb{R}^n$ whose i-th coordinate is

$$a_{i1}x_1 + \cdots + a_{in}x_n \ ,$$

so that a matrix A is interpreted as a map $A : \mathbb{R}^n \to \mathbb{R}^n$ sending $x \mapsto Ax$. In this notation we write (3) simply as

$$\boxed{x' = Ax} \ . \tag{4}$$

The map $A : \mathbb{R}^n \to \mathbb{R}^n$ is a **vector field** on $\mathbb{R}^n$: to each point $x \in \mathbb{R}^n$ it assigns the vector based at x which is a translate of Ax. A solution of (4) is a curve $x : \mathbb{R} \to \mathbb{R}^n$ whose tangent vector at any given t is the vector $Ax(t)$ (translated to $x(t)$).

The vector equation (4) can be supplemented by an initial condition $x(0) = x_0 \in \mathbb{R}^n$, and we will see that such an initial value problem has a unique solution. We have already proved this in the case $n = 1$, and the general case is not any harder.

1.4. **Autonomous equations.** We need not restrict ourselves to linear equations, i.e., equations where the vector field $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. A large part of this course will concern itself with the more general equation

$$x' = f(x)$$

where $f : U \to \mathbb{R}^n$ is a continuous map from a subset $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Such equations are called **autonomous** because f does not depend explicitly on t. On the other hand an equation like

$$x' = f(x, t)$$

is called **non-autonomous** and are harder to interpret geometrically. We will focus on autonomous systems for the most part. We will not prove in this course any results on existence and uniqueness of nonlinear initial value problems. There are such results but they are harder to state. Some of them are discussed in Foundations of Analysis.

Problems

(These problems are taken from Hirsch & Smale, Chapter 1.)

Problem 1.1. For each of the matrices A which follow, sketch the phase portrait of the corresponding differential equation $x' = Ax$, using Maple if so desired.

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} \frac{1}{2} & -2 \\ 2 & 0 \end{pmatrix}$ **(e)** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **(f)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ **(h)** $\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$ **(i)** $\begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}$

Problem 1.2. Consider the one-parameter family of differential equations

$$\begin{aligned} x_1' &= 2\,x_1\ , \\ x_2' &= a\,x_2\ ; \qquad a \in \mathbb{R}\ . \end{aligned}$$

(a) Find all solutions $(x_1(t), x_2(t))$.
(b) Sketch the phase portrait for the following values of a: $-1, 0, 1, 2, 3$.

Problem 1.3. For each of the following matrices A sketch the vector field $x \mapsto Ax$ in $\mathbb{R}^3$. (Any missing entries are treated as 0.) Maple is of indirect use here. Although it will not plot the vector field itself, it will plot the solutions.

(a) $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & & \\ & -2 & \\ & & 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & & \\ & -2 & \\ & & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ **(e)** $\begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & -\frac{1}{2} \end{pmatrix}$ **(f)** $\begin{pmatrix} -1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$

Problem 1.4. For A as in (a), (b), (c) of Problem 1.3, solve the initial value problem

$$x' = Ax \ , \qquad x(0) = (k_1, k_2, k_3) \ .$$

Problem 1.5. Let A be as in (e), Problem 1.3. Find constants a, b, and c such that the curve $(a\cos t, b\sin t, ce^{-t/2})$ is a solution of $x' = Ax$ with $x(0) = (1, 0, 3)$.

Problem 1.6. Find two different matrices A and B such that the curve

$$x(t) = (e^t, 2e^{2t}, 4e^{2t})$$

satisfies both the differential equations

$$x' = Ax \qquad \text{and} \qquad x' = Bx \ .$$

Problem 1.7. Let A be an $n \times n$ diagonal matrix. Show that the differential equation

$$x' = Ax$$

has a unique solution for every initial condition.

Problem 1.8. Let A be an $n \times n$ diagonal matrix. Find conditions on A guaranteeing that

$$\lim_{t\to\infty} x(t) = 0$$

for all solutions of $x' = Ax$.

Problem 1.9. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Denote by $-A$ the matrix $[-a_{ij}]$.

(a) What is the relation between the vector fields $x \mapsto Ax$ and $x \mapsto -Ax$?

(b) What is the geometric relation between the solution curves of $x' = Ax$ and $x' = -Ax$?

Problem 1.10.

(a) Let $u(t)$ and $v(t)$ be solutions to $x' = Ax$. Show that the curve $w(t) = \alpha u(t) + \beta v(t)$ is a solution for all real numbers α, β.

(b) Let $A = \begin{pmatrix} 1 & \\ & -2 \end{pmatrix}$. Find solutions $u(t)$, $v(t)$ of $x' = Ax$ such that every solution can be expressed in the form $\alpha u(t) + \beta v(t)$ for suitable constants α, β.

2. Ordinary differential equations of higher order

2.1. **Second order equations.** The study of differential equations derives from the study of Newton's second law (with unit mass):

$$F(x) = x'' \ , \tag{5}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ (typically $n \leq 3$) is the position of some object as a function of time and F is a **force field** $F : \mathbb{R}^n \to \mathbb{R}^n$.

Newton's equation is a *second order* differential equation, the **order** of an ODE being defined as the order of the highest derivative of x which appears in the equation.

Second order equations (in fact, equations of any order > 1) are equivalent to first-order equations by the following trick: define $y : \mathbb{R} \to \mathbb{R}^n$ by $y = x'$. Then $x'' = y'$, and Newton's equation becomes the system

$$y' = F(x) \qquad \text{and} \qquad x' = y \ . \tag{6}$$

Clearly if x is any solution of (5), then (x, y) with $y = x'$ is a solution of (6); and conversely if (x, y) is any solution of (6), then x solves (5). We say that (6) and (5) are **equivalent**.

2.2. **Some several variable calculus.** We summarise some results from Several Variable Calculus.

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be vectors in $\mathbb{R}^n$. Their **inner product** (or *dot product*) is denoted $\langle x, y\rangle$ and is defined as

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = x_1y_1 + x_2y_2 + \cdots + x_ny_n \ .$$

The **norm** of x is $\|x\|$ where $\|x\|^2 = \langle x, x\rangle$. If $x, y : \mathbb{R} \to \mathbb{R}^n$ are differentiable functions, then we have the following version of the Leibniz rule:

$$\boxed{\langle x, y\rangle' = \langle x', y\rangle + \langle x, y'\rangle} \ . \tag{7}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. The **gradient** of f, denoted $\operatorname{grad} f$, is the vector field $\operatorname{grad} f : \mathbb{R}^n \to \mathbb{R}^n$ sending x to $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. The **level sets** of f are the sets of the form $f^{-1}(c)$ for $c \in \mathbb{R}$:

$$f^{-1}(c) = \{x \in \mathbb{R}^n \mid f(x) = c\} \ .$$

A point $x \in \mathbb{R}^n$ is called a **regular point** if $\operatorname{grad} f(x) \neq 0$. The gradient is perpendicular to the level set at all regular points.

Now consider the composition of two differentiable maps:

$$\mathbb{R} \xrightarrow{x} \mathbb{R}^n \xrightarrow{f} \mathbb{R} \ ,$$

which sends t to $f(x(t))$. The **chain rule** becomes

$$\boxed{\frac{d}{dt}f(x(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x(t))\frac{dx_i}{dt}(t) = \langle \operatorname{grad} f\,(x(t)), x'(t)\rangle} \,. \tag{8}$$

2.3. **Energy.** A force field $F : \mathbb{R}^n \to \mathbb{R}^n$ is called **conservative** if there exists some function $V : \mathbb{R}^n \to \mathbb{R}$, called a **potential**, such that $F = -\operatorname{grad} V$.

Newton's equation (5) for a conservative force field is

$$x'' = -\operatorname{grad} V(x) \ . \tag{9}$$

Define the **energy** of a curve $x : \mathbb{R} \to \mathbb{R}^n$ by

$$E(x(t)) = \tfrac{1}{2}\|x'(t)\|^2 + V(x(t)) \ .$$

The energy is constant along solution curves $x : \mathbb{R} \to \mathbb{R}^n$ of Newton's equation. Indeed, using the Leibniz rule (7) and the chain rule (8), we have

$$\frac{d}{dt}E(x(t)) = \langle x'', x'\rangle + \langle \operatorname{grad} V(x), x'\rangle = 0,$$

where we have used Newton's equation (9). This is called *conservation of energy*.

2.4. **Hamiltonian vector fields.** Newton's equation (9) is equivalent to the following system:

$$(x, y)' = (x', y') = (y, -\operatorname{grad} V(x)) \ ,$$

whose solutions can be interpreted as curves in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ associated to the vector field defined by

$$(x, y) \mapsto (y, -\operatorname{grad} V(x)) \ . \tag{10}$$

Such a vector field is called hamiltonian. More precisely, a **hamiltonian** vector field in $\mathbb{R}^{2n}$ is one of the form

$$(x, y) \mapsto (\operatorname{grad}_y H, -\operatorname{grad}_x H) \ ,$$

where

$$\operatorname{grad}_x H = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right) \quad \text{and} \quad \operatorname{grad}_y H = \left(\frac{\partial H}{\partial y_1}, \ldots, \frac{\partial H}{\partial y_n}\right) \ ,$$

for some function $H : \mathbb{R}^{2n} \to \mathbb{R}$ called the **hamiltonian (function)**.

The vector field (10) defined by Newton's equation is hamiltonian with hamiltonian function

$$H(x, y) = \tfrac{1}{2}\|y\|^2 + V(x) \ .$$

Notice that the hamiltonian is constant along any solution curve and equal to the energy. This means that solution curves lie in the level sets of the hamiltonian.

2.5. **Gradient vector fields.** A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathbb{R}^n$ is called a **gradient** vector field if $F = \operatorname{grad} U$ for some function $U : \mathbb{R}^n \to \mathbb{R}$.

Solution curves of the equation $x' = \operatorname{grad} U(x)$ are orthogonal to the level sets of the function U. This contrasts with hamiltonian systems, in which the solution curves lie in the level sets.

Problems

(Some of the problems are taken from Hirsch & Smale, Chapter 2.)

Problem 2.1. Which of the following force fields $F : \mathbb{R}^2 \to \mathbb{R}^2$ are conservative? If the field is conservative, find $V : \mathbb{R}^2 \to \mathbb{R}$ such that $F = -\operatorname{grad} V$.

(a) $F(x, y) = (-x^2, -2y^2)$
(b) $F(x, y) = (x^2 - y^2, 2xy)$
(c) $F(x, y) = (x, 0)$

Problem 2.2. An ODE of order n is said to be in **standard form** if it can be written as

$$x^{(n)} = f(x, x', x'', \ldots, x^{(n-1)}) \ ,$$

where $x : \mathbb{R} \to \mathbb{R}$. Prove that such an equation is equivalent to a first-order system

$$X' = F(X) \ ,$$

for some function $X : \mathbb{R} \to \mathbb{R}^n$ and some vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. How are x and X related? How are f and F related?

Problem 2.3. Prove the Leibniz rule (7).

Problem 2.4. Which of the following vector fields $F : \mathbb{R}^2 \to \mathbb{R}^2$ are hamiltonian? Which ones are gradient?

(a) $F(x, y) = (y, x - x^3)$
(b) $F(x, y) = (y^2 - x^2, xy,)$
(c) $F(x, y) = (-x, y)$

For those which are, find the function which makes them so.

Problem 2.5. Suppose that a vector field in $\mathbb{R}^2$ is both gradient and hamiltonian. Show that the hamiltonian function H is harmonic; that is,

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0 \ .$$

Problem 2.6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$F : (x, y) \mapsto (ax + by, cx + dy) \ .$$

For which values of the real parameters a, b, c, d is it hamiltonian, gradient or both?

Problem 2.7. A force field $F : \mathbb{R}^n \to \mathbb{R}^n$ is called **central** if $F(x) = h(x)x$, for some function $h : \mathbb{R}^n \to \mathbb{R}$. Let F be central and conservative, so that $F = -\operatorname{grad} V$, for some function $V : \mathbb{R}^n \to \mathbb{R}$. Show that $V(x) = f(\|x\|)$, for some $f : \mathbb{R} \to \mathbb{R}$; in other words, the potential is constant on spheres $S_R = \{x \in \mathbb{R}^n \mid \|x\| = R\}$.
Hint: Show that V is constant along any curve on the the sphere S_R, and use that any two points on the sphere can be joined by curve. (Can you prove this last assertion?)

Problem 2.8. Sketch the phase portraits of the following hamiltonian vector fields in $\mathbb{R}^2$, by first finding a hamiltonian function and sketching its level sets.

(a) $(x, y) \mapsto (y, 2(1 - x))$
(b) $(x, y) \mapsto (2y - x, y - 2x)$
(c) $(x, y) \mapsto (2y - 3x, 3y - 2x)$
(d) $(x, y) \mapsto (y - x, y - x)$

Problem 2.9. Sketch the phase portraits of each of the following gradient vector fields $F : \mathbb{R}^2 \to \mathbb{R}^2$, by first finding a function U such that $F = -\operatorname{grad} U$ and then sketching its level sets.

(a) $(x, y) \mapsto (2(x - 1), y)$
(b) $(x, y) \mapsto (2x - y, 2y - x)$
(c) $(x, y) \mapsto (2x - 3y, 2y - 3x)$
(d) $(x, y) \mapsto (x - y, y - x)$

3. Linear vector fields

3.1. **Some linear algebra.** We collect some concepts which were covered in Linear Algebra and Differential Equations.

$\mathbb{R}^n$ is an example of an n-dimensional real vector space, and moreover every n-dimensional real vector space is isomorphic to $\mathbb{R}^n$, the isomorphism being given by a choice of basis.

A map $A : \mathbb{R}^n \to \mathbb{R}^n$ is **linear** if

(i) $A(x + y) = Ax + Ay$, for all $x, y \in \mathbb{R}^n$; and
(i) $A(\lambda\, x) = \lambda\, Ax$, for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Any linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \dots, x_n) \mapsto ((Ax)_1, \dots, (Ax)_n)$ is characterised by n^2 real numbers $[a_{ij}]$ defined by

$$Ax_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ .$$

Writing points in $\mathbb{R}^n$ as column vectors (i.e., as $n \times 1$ matrices), we can write the n^2 numbers $[a_{ij}]$ as the entries in an $n \times n$ matrix, and in this way the action of A is given simply by matrix multiplication:

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{pmatrix} \ .$$

Henceforth we will not distinguish between a linear transformation and its associated matrix. Notice that $n \times n$ matrices can be added and multiplied by real numbers, so they too form a vector space isomorphic to $\mathbb{R}^{n^2}$. In addition, matrices can be multiplied, and matrix multiplication and composition of linear transformations correspond.

I assume familiarity with the notions of **trace** and **determinant** of a matrix.

A subset $E \subset \mathbb{R}^n$ is a **(vector) subspace** if

(i) $x + y \in E$ for every $x, y \in E$, and
(ii) $\lambda\, x \in E$ for every $x \in E$ and $\lambda \in \mathbb{R}$.

The **kernel** of a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is the subspace defined by

$$\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n \ .$$

Similarly, the **image** is the subspace defined by

$$\operatorname{im} A = \{y \in \mathbb{R}^n \mid y = Ax,\ \exists x \in \mathbb{R}^n\} \subset \mathbb{R}^n \ .$$

A linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a linear transformation $A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ obeying $AA^{-1} = A^{-1}A = I$, where $I : \mathbb{R}^n \to \mathbb{R}^n$, $Ix = x$, is the identity. The following statements are equivalent:

(a) A is invertible
(b) $\ker A = 0$
(c) $\det A \neq 0$

(d) $\operatorname{im} A = \mathbb{R}^n$

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A nonzero vector $x \in \mathbb{R}^n$ is called a (real) **eigenvector** if $Ax = \alpha x$ for some real number α, which is called a **(real) eigenvalue**. The condition that α be a real eigenvalue of A means that the linear transformation $\alpha I - A : \mathbb{R}^n \to \mathbb{R}^n$ is not invertible. Its kernel is called the α-**eigenspace** of A: it consists of all eigenvectors of A with eigenvalue α together with the 0 vector. The real eigenvalues of A are precisely the real roots of the **characteristic polynomial** of A:

$$p_A(\lambda) = \det(\lambda I - A) \ .$$

A complex root of $p_A(\lambda)$ is called a **complex eigenvalue** of A.

An $n \times n$ matrix $A = [a_{ij}]$ is **diagonal** if $a_{ij} = 0$ for $i \neq j$, and it is called **diagonalisable** (over $\mathbb{R}$) if there exists an invertible $n \times n$ (real) matrix S, such that $SAS^{-1} = D$, with D diagonal. A sufficient (but *not* necessary) condition for A to be diagonalisable, is that its characteristic polynomial should factorise as

$$p_A(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\cdots(\lambda - \alpha_n) \ ,$$

where the α_i are real and distinct. In other words, this means that $p_A(\lambda)$ should have n distinct real roots. We will summarise this condition as "A has real, distinct eigenvalues". Another sufficient condition for diagonalisability is that A be **symmetric**: $a_{ij} = a_{ji}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Its trace and determinant are, respectively, $\operatorname{tr} A = a + d$ and $\det A = ad - bc$. The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A \ .$$

Therefore the condition for A to have real, distinct roots is the positivity of the discriminant

$$(\operatorname{tr} A)^2 - 4 \det A > 0 \ .$$

3.2. **Real eigenvalues.** Let $A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$, be a linear vector field and consider the associated differential equation: $x' = Ax$, where $x : \mathbb{R} \to \mathbb{R}^n$.

If A is diagonal, $A = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we know from Problem 1.7 that this equation has a unique solution for each choice of initial value $x(0)$. In fact, the solutions are

$$x_i(t) = x_i(0) e^{\alpha_i t} \ .$$

The solution depends continuously on the initial conditions (see Problem 3.5).

Suppose that A is diagonalisable, then there exists some constant invertible matrix S such that $D := SAS^{-1}$ is diagonal. Consider the equation $x' = Ax = S^{-1}DSx$. Define $y : \mathbb{R} \to \mathbb{R}^n$ by $y = Sx$. Then

$y' = Dy$ and the previous paragraph applies. In particular, this system has a unique solution for each choice of initial value $y(0)$. Finally, $x(t) = S^{-1}y(t)$ is the desired solution.

In particular, if A has real, distinct eigenvalues then the equation $x' = Ax$ has a unique solution for each choice of initial conditions $x(0)$. Moreover the solution is easy to construct explicitly: all we need to do is to construct the matrix S, and this is done as follows:

1. find the eigenvectors $\{v_1, v_2, \ldots, v_n\}$;
2. construct a matrix V whose columns are the eigenvectors:

$$V = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} ;$$

3. and invert to obtain the matrix $S = V^{-1}$.

For many problems we will not need the explicit form of S, and only a knowledge of the eigenvalues of A will suffice.

3.3. **Complex eigenvalues.** It may happen that a real matrix A has complex eigenvalues; although they always come in complex conjugate pairs (see Problem 3.8). In this case it is convenient to think of A as a linear transformation in a complex vector space.

The canonical example of a complex vector space is $\mathbb{C}^n$, the space of ordered n-tuples of complex numbers: $(z_1, z_2, \ldots, z_n)$.

Let $A = [a_{ij}]$ be a $n \times n$ matrix. It defines a complex linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ as follows:

$$A \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} a_{11}z_1 + \cdots + a_{1n}z_n \\ \vdots \\ a_{n1}z_1 + \cdots + a_{nn}z_n \end{pmatrix} .$$

If A is real, then it preserves the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$ consisting of real n-tuples: $z_i = \bar{z}_i$.

A sufficient (but *not* necessary) condition for a complex $n \times n$ matrix A to be diagonalisable is that its characteristic polynomial should factorise into distinct linear factors:

$$p_A(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2) \cdots (\lambda - \mu_n) ,$$

where the $\mu_i \in \mathbb{C}$ are distinct.

Suppose that A is diagonalisable, but with eigenvalues which might be complex. This means that there is an invertible $n \times n$ complex matrix S such that $SAS^{-1} = D = \operatorname{diag}\{\mu_1, \ldots, \mu_n\}$. The μ_i are in general complex, but if A is real, they are either real or come in complex conjugate pairs (see Problem 3.8).

The equation $x' = Ax$, where $x : \mathbb{R} \to \mathbb{R}^n \subset \mathbb{C}^n$, can be easily solved by introducing $y : \mathbb{R} \to \mathbb{C}^n$ by $x(t) = S^{-1}y(t)$, where

$$y_i(t) = y_i(0)e^{\mu_i t} .$$

3.4. **The exponential of a matrix.** Let A be an $n \times n$ matrix. The **exponential of** A is the matrix defined by the following series:

$$\boxed{e^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j = I + A + \tfrac{1}{2}A^2 + \cdots} \ .$$

This series converges absolutely for all A.

Proposition 3.1. *Let A, B and C be $n \times n$ matrices. Then*

(a) *if $B = CAC^{-1}$, then $e^B = Ce^AC^{-1}$;*
(b) *if $AB = BA$, then $e^{A+B} = e^A e^B$;*
(c) *$e^{-A} = \left(e^A\right)^{-1}$;*
(d) *if $n = 2$ and $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ then*

$$e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} \ .$$

If $x \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue α, then x is also an eigenvector of e^A with eigenvalue e^α.

Let $L(\mathbb{R}^n)$ denote the vector space of $n \times n$ real matrices. Consider the map $\mathbb{R} \to L(\mathbb{R}^n)$ which to $t \in \mathbb{R}$ assigns the matrix e^{tA}. This can be thought of as a map $\mathbb{R} \to \mathbb{R}^{n^2}$, so it makes sense to speak of the derivative of this map. We have the following

$$\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA}A \ .$$

Theorem 3.2. *Let $x \mapsto Ax$ be a linear vector field on $\mathbb{R}^n$. The solution of the initial value problem*

$$x' = Ax \qquad x(0) = K \in \mathbb{R}^n$$

is

$$x(t) = e^{tA}K \ ,$$

and there are no other solutions.

Although the solution of a linear ordinary differential equation is given very explicitly in terms of the matrix exponential, exponentiating a matrix—especially if it is of suffficiently large rank—is not practical in many situations. A more convenient way to solve a linear equation is to change basis to bring the matrix to a normal form which can be easily exponentiated, and then change basis back.

3.5. **The case** $n = 2$**.** Recall the following result from linear algebra.

Theorem 3.3. *Given any 2×2 real matrix A, there is an invertible real matrix S such that $B := SAS^{-1}$ takes one (and only one) of the following forms*

$$\text{(I)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{(II)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad \text{(III)} \begin{pmatrix} \nu & 0 \\ 1 & \nu \end{pmatrix} ,$$

where $\lambda \leq \mu$ in (I) and $b \neq 0$ in (II).

Case (I) corresponds to matrices which are diagonalisable over $\mathbb{R}$. Case (II) corresponds (since $b \neq 0$) to matrices which are not diagonalisable over $\mathbb{R}$ but are diagonalisable over $\mathbb{C}$. Finally, case (III) corresponds to matrices which are not diagonalisable.

The exponential of A is then given by

$$e^A = S^{-1} e^B S ,$$

where e^B is given by the following matrices:

$$\text{(I)} \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix} \qquad \text{(II)}\ e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} \qquad \text{(III)}\ e^\nu \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .$$

From these it is easy to reconstruct the phase portrait of any linear vector field $x \mapsto Ax$ on $\mathbb{R}^2$.

There are seventeen different types, which can be identified by the trace τ and determinant Δ of A:

1. Case (I)
   (a) *Foci* [$\tau^2 = 4\Delta$, A diagonal]
      (i) $\lambda = \mu < 0$: sink [$\tau < 0$, $\Delta > 0$]
      (ii) $0 < \lambda = \mu$: source [$\tau > 0$, $\Delta > 0$]
      (iii) $0 = \lambda = \mu$: degenerate [$\tau = 0$, $\Delta = 0$]
   (b) *Nodes* [$\tau^2 > 4\Delta$]
      (i) $\lambda < \mu = 0$: degenerate [$\tau < 0$, $\Delta = 0$]
      (ii) $\lambda < \mu < 0$: sink [$\tau < 0$, $\Delta > 0$]
      (iii) $0 < \lambda < \mu$: source [$\tau > 0$, $\Delta > 0$]
      (iv) $0 = \lambda < \mu$: degenerate [$\tau > 0$, $\Delta = 0$]
   (c) *Saddles* [$\Delta < 0$]
      (i) $\lambda < 0 < \mu$
2. Case (II) [$\tau^2 < 4\Delta$]
   (a) *Centres* [$\tau = 0$]
      (i) $a = 0$, $b < 0$: periodic, counterclockwise
      (ii) $a = 0$, $0 < b$: periodic, clockwise
   (b) *Spirals*
      (i) $a < 0$, $b < 0$: sink, counterclockwise [$\tau < 0$]
      (ii) $a < 0$, $0 < b$: sink, clockwise [$\tau < 0$]
      (iii) $0 < a$, $b < 0$: source, counterclockwise [$\tau > 0$]
      (iv) $0 < a$, $0 < b$: source, clockwise [$\tau > 0$]
3. Case (III) [$\tau^2 = 4\Delta$, A not diagonal]

(a) *Improper Nodes*
  (i) $\nu = 0$: degenerate $[\tau = 0]$
  (ii) $\nu < 0$: sink $[\tau < 0]$
  (iii) $0 < \nu$: source $[\tau > 0]$

In a sink, the flows approach the origin, whereas in a source they move away from it.

3.6. **Inhomogeneous systems.** Consider the following **inhomogeneous** nonautonomous linear differential equation for $x : \mathbb{R} \to \mathbb{R}^n$,

$$x' = Ax + B \;, \tag{11}$$

where $B : \mathbb{R} \to \mathbb{R}^n$ is a continuous map. The method of *variation of parameters* consists of seeking a solution of the form

$$x(t) = e^{tA} f(t) \;, \tag{12}$$

where $f : \mathbb{R} \to \mathbb{R}^n$ is some differentiable curve. (This represents no loss of generality since e^{tA} is invertible for all t.) Differentiating (12) using the Leibniz rule yields

$$x'(t) = Ae^{tA} f(t) + e^{tA} f'(t) = Ax(t) + e^{tA} f'(t) \;.$$

Since x is assumed to be a solution of (11), we see that

$$f'(t) = e^{-tA} B(t) \;,$$

which can be integrated to yield

$$f(t) = \int_0^t e^{-sA} B(s)\, ds + K \;,$$

for some $K \in \mathbb{R}^n$. Solving for $x(t)$ we finally have

$$\boxed{x(t) = e^{tA} \left[\int_0^t e^{-sA} B(s)\, ds + K \right]} \;,$$

for some $K \in \mathbb{R}^n$. Moreover every solution is of this form. Indeed, if $y : \mathbb{R} \to \mathbb{R}^n$ is another solution of (11), then $x - y$ solves the homogeneous equation $(x-y)' = A(x-y)$. By Theorem 3.2, it has the form $x - y = e^{tA} K_0$ for some constant $K_0 \in \mathbb{R}^n$, whence y has the same form as x with possibly a different constant $K \in \mathbb{R}^n$.

Theorem 3.4. *Let $u(t)$ be a particular solution of the inhomogeneous linear differential equation*

$$x' = Ax + B \;. \tag{11}$$

Then every solution of (11) *has the form $u(t) + v(t)$ where $v(t)$ is a solution of the homogeneous equation*

$$x' = Ax \;. \tag{4}$$

Conversely, the sum of a solution of (11) *and a solution of* (4) *is a solution of* (11).

3.7. **Higher order equations.** Consider the nth order linear differential equation

$$s^{(n)} + a_1 s^{(n-1)} + \cdots + a_{n-1}s' + a_n s = 0 \ , \tag{13}$$

where $s : \mathbb{R} \to \mathbb{R}$. This equation is equivalent to the first order equation $x' = Ax$, where $x : \mathbb{R} \to \mathbb{R}^n$ sends $t \mapsto (s, s', \ldots, s^{(n-1)})$, and where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & \cdots & 1 \\ -a_n & -a_{n-1} & & \cdots & -a_1 \end{pmatrix} . \tag{14}$$

Proposition 3.5. *The characteristic polynomial of the matrix A in* (14) *is*

$$p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n \ .$$

This results says that we can read off the eigenvalues of the matrix A directly from the differential equation.

Notice that if $s(t)$ and $q(t)$ solve (13) then so do $s(t)+q(t)$ and $ks(t)$ where $k \in \mathbb{R}$. In other words, the solutions of (13) form a vector space. This is an n-dimensional vector space, since the n initial conditions $s(0)$, $s'(0)$,..., $s^{(n-1)}(0)$ uniquely determine the solution.

At a conceptual level this can be understood as follows. Let $\mathcal{F}$ denote the (infinite-dimensional) vector space of smooth (i.e., infinitely differentiable) functions $s : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{L} : \mathcal{F} \to \mathcal{F}$ denote the linear map

$$s \mapsto s^{(n)} + a_1 s^{(n-1)} + \cdots + a_{n-1}s' + a_n s \ .$$

A function $s(t)$ solves equation (13) if and only if it belongs to the kernel of $\mathcal{L}$, which is a subspace of $\mathcal{F}$.

Higher order inhomogeneous equations

$$s^{(n)} + a_1 s^{(n-1)} + \cdots + a_{n-1}s' + a_n s = b \ , \tag{15}$$

for $b : \mathbb{R} \to \mathbb{R}$ can also be solved by solving the associated first order system

$$x' = Ax + B \ , \tag{11}$$

where A is given by (14) and $B : \mathbb{R} \to \mathbb{R}^n$ is given by

$$B(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix} .$$

Problems

(Some of the problems are taken from Hirsch & Smale, Chapters 3,4 and 5.)

Problem 3.1. Solve the following initial value problems:

(a)
$$x' = -x$$
$$y' = x + 2y$$
$$x(0) = 0 \quad y(0) = 3$$

(b)
$$x_1' = 2x_1 + x_2$$
$$x_2' = x_1 + x_2$$
$$x_1(1) = 1 \quad x_2(1) = 1$$

(c)
$$x' = Ax$$
$$x(0) = (0, 3)$$
$$A = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$$

(d)
$$x' = Ax$$
$$x(0) = (0, -b, b)$$
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -3 \end{pmatrix}$$

Problem 3.2. Find a 2×2 matrix A such that one solution to $x' = Ax$ is

$$x(t) = (e^{2t} - e^{-t}, e^{2t} + 2e^{-t}) \ .$$

Problem 3.3. Show that the only solution to

$$x_1' = x_1$$
$$x_2' = x_1 + x_2$$
$$x_1(0) = a \quad x_2(0) = b$$

is

$$x_1(t) = a\, e^t$$
$$x_2(t) = (a\, t + b)e^t \ .$$

(*Hint*: If $(y_1(t), y_2(t))$ is any other solution, consider the functions $e^{-t}y_1(t)$ and $e^{-t}y_2(t)$.)

Problem 3.4. Let a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ have real, distinct eigenvalues. What condition on the eigenvalues is equivalent to $\lim_{t\to\infty} \|x(t)\| = \infty$ for every solution $x(t)$ to $x' = Ax$?

Problem 3.5. Suppose that the $n \times n$ matrix A has real, distinct eigenvalues. Let $t \mapsto \phi(t, x_0)$ be the solution to $x' = Ax$ with initial value $\phi(0, x_0) = x_0$. Show that for each fixed t,

$$\lim_{y_0 \to x_0} \phi(t, y_0) = \phi(t, x_0) \ .$$

This means the solutions depend continuously on the initial conditions. (*Hint*: Suppose that A is diagonal.)

Problem 3.6. Consider the second order differential equation

$$x'' + bx' + cx = 0 \qquad b \text{ and } c \text{ constant.} \tag{16}$$

(a) By examining the equivalent first order system

$$x' = y \qquad y' = -cx - by \ ,$$

show that if $b^2 - 4c > 0$, then (16) has a unique solution $x(t)$ for every initial condition of the form $x(0) = u$ and $x'(0) = v$.

(b) If $b^2 - 4c > 0$, what assumption about b and c ensures that $\lim_{t\to\infty} x(t) = 0$ for every solution $x(t)$?

(c) Sketch the graphs of the three solutions of

$$x'' - 3x' + 2x = 0$$

for the initial conditions

$$x(0) = 1 \qquad \text{and} \qquad x'(0) = -1, 0, 1 \ .$$

Problem 3.7. Let a 2×2 matrix A have real, distinct eigenvalues λ and μ. Suppose that an eigenvector of eigenvalue λ is $(1, 0)$ and an eigenvector of eigenvalue μ is $(1, 1)$. Sketch the phase portraits of $x' = Ax$ for the following cases:

(a) $0 < \lambda < \mu$ **(b)** $0 < \mu < \lambda$ **(c)** $\lambda < \mu < 0$

(d) $\lambda < 0 < \mu$ **(e)** $\lambda = 0 \quad \mu > 0$

Problem 3.8. Let A be a $n \times n$ real matrix. Prove that its eigenvalues are either real or come in complex conjugate pairs.

Problem 3.9. Solve the following initial value problems and sketch their phase portraits:

(a) $x' = -y$, $y' = x$, $x(0) = 1 \quad y(0) = 1$

(b) $x_1' = -2x_2$, $x_2' = 2x_1$, $x_1(c) = 0 \quad x_2(0) = 2$

(c) $x' = y$, $y' = -x$, $x(0) = 1 \quad y(0) = 1$

(d) $x' = Ax$, $x(0) = (3, -9)$, $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

Problem 3.10. Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and let $x(t)$ be a solution of $x' = Ax$, not identically zero. Show that the curve $x(t)$ is of the following form:

(a) a circle if $a = 0$;

(b) a spiral inward toward $(0, 0)$ if $a < 0$ and $b \neq 0$;

(c) a spiral outward away from $(0, 0)$ if $a > 0$ and $b \neq 0$.

What effect has the sign of b on the spirals in (b) and (c)? What is the phase portrait if $b = 0$?

Problem 3.11. Sketch the portraits of:

$$\textbf{(a)}\ \begin{aligned} x' &= -2x \\ y' &= 2z \\ z' &= -2y \end{aligned} \qquad \textbf{(b)}\ \begin{aligned} x' &= -x + z \\ y' &= 3y \\ z' &= -x - z \end{aligned}$$

Which solutions tend to 0 as $t \to \infty$?

Problem 3.12. Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Prove that the solutions of $x' = Ax$ depend continuously on initial values. (See Problem 3.5.)

Problem 3.13. Solve the initial value problem

$$\begin{aligned} x' &= -4y \\ y' &= x \\ x(0) = 0 \qquad & y(0) = -7\ . \end{aligned}$$

Problem 3.14. Solve the equation $x' = Ax$, for each of the following matrices A:

$$\textbf{(a)}\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \qquad \textbf{(b)}\ \begin{pmatrix} 0 & 0 & 15 \\ 1 & 0 & -17 \\ 0 & 1 & 7 \end{pmatrix}$$

Problem 3.15. Let $A(t)$ and $B(t)$ be two $n \times n$ matrices depending differentiably on t. Prove the following version of the Leibniz rule:

$$(A\,B)' = A'\,B + A\,B'\ .$$

Deduce from this that if $A(t)$ is invertible for all t, then

$$\left(A^{-1}\right)' = -A^{-1}\,A'\,A^{-1}\ .$$

Problem 3.16. Compute the exponentials of the following matrices ($i = \sqrt{-1}$):

$$\textbf{(a)}\ \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \qquad \textbf{(b)}\ \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \qquad \textbf{(c)}\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\textbf{(d)}\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \textbf{(e)}\ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \textbf{(f)}\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\textbf{(g)}\ \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \qquad \textbf{(h)}\ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \textbf{(i)}\ \begin{pmatrix} i+1 & 0 \\ 2 & 1+i \end{pmatrix}$$

$$\textbf{(j)}\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Problem 3.17. For each matrix T in Problem 3.16 find the eigenvalues of e^T.

Problem 3.18. Find an example of two linear transformations A, B on $\mathbb{R}^2$ such that

$$e^{A+B} \neq e^A e^B \ .$$

Problem 3.19. If $AB = BA$, prove that $e^A e^B = e^B e^A$ and $e^A B = B e^A$.

Problem 3.20. Let a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ leave invariant a subspace $E \subset \mathbb{R}^n$ (that is, $Ax \in E$ for all $x \in E$). Show that e^A also leaves E invariant.

Problem 3.21. Show that there is no real 2×2 matrix S such that $e^S = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$.

Problem 3.22. Find the general solution to each of the following systems:

(a) $$\begin{cases} x' &= 2x - y \\ y' &= 2y \end{cases}$$

(b) $$\begin{cases} x' &= 2x - y \\ y' &= x + 2y \end{cases}$$

(c) $$\begin{cases} x' &= y \\ y' &= x \end{cases}$$

(d) $$\begin{cases} x' &= -2x \\ y' &= x - 2y \\ z' &= y - 2z \end{cases}$$

(e) $$\begin{cases} x' &= y + z \\ y' &= z \\ z' &= 0 \end{cases}$$

Problem 3.23. In (a), (b) and (c) of Problem 3.22, find the solutions satisfying each of the following initial conditions:
(a) $x(0) = 1$, $y(0) = -2$;
(b) $x(0) = 0$, $y(0) = -2$;
(c) $x(0) = 0$, $y(0) = 0$.

Problem 3.24. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that leaves a subspace $E \subset \mathbb{R}^n$ invariant. Let $x : \mathbb{R} \to \mathbb{R}^n$ be a solution of $x' = Ax$. If $x(t_0) \in E$ for some $t_0 \in \mathbb{R}$, show that $x(t) \in E$ for all $t \in \mathbb{R}$.

Problem 3.25. Prove that if the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ has a real eigenvalue $\lambda < 0$, then the equation $x' = Ax$ has at least one nontrivial solution $x(t)$ such that $\lim_{t\to\infty} x(t) = 0$.

Problem 3.26. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation and suppose that $x' = Ax$ has a nontrivial *periodic solution*, $u(t)$: this means that $u(t + p) = u(t)$ for some $p > 0$. Prove that every solution is periodic with the same period.

Problem 3.27. If $u : \mathbb{R} \to \mathbb{R}^n$ is a nontrivial solution of $x' = Ax$, then show that

$$\frac{d}{dt}\|u\| = \frac{1}{\|u\|}\langle u, Au\rangle \ .$$

Problem 3.28. Classify and sketch the phase portraits of planar differential equations $x' = Ax$, with $A : \mathbb{R}^2 \to \mathbb{R}^2$ linear, where A has zero as an eigenvalue.

Problem 3.29. For each of the following matrices A consider the corresponding differential equation $x' = Ax$. Decide whether the origin is a sink, source, saddle or none of these. Identify in each case those vectors u such that $\lim_{t\to\infty} x(t) = 0$, where $x(t)$ is the solution of $x' = Ax$ with $x(0) = u$:

(a) $\begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$ **(e)** $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

Problem 3.30. Which values (if any) of the parameter k in the following matrices makes the origin a sink for the corresponding differential equation $x' = Ax$?

(a) $\begin{pmatrix} a & -k \\ k & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 0 \\ k & -4 \end{pmatrix}$

(c) $\begin{pmatrix} k^2 & 1 \\ 0 & k \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & k \end{pmatrix}$

Problem 3.31. Let $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow corresponding to the equation $x' = Ax$. (That is, $t \mapsto \phi_t(x)$ is the solution passing through x at $t = 0$.) Fix $\tau > 0$, and show that ϕ_τ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. Then show that ϕ_τ preserves area if and only if $\operatorname{tr} A = 0$, and that in this case the origin is neither a sink nor a source.
(*Hint*: A linear transformation is area-preserving if and only if its determinant is ± 1.)

Problem 3.32. Describe in words the phase portraits of $x' = Ax$ for the following matrices A:

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$.

Problem 3.33. Let T be an invertible linear transformation on $\mathbb{R}^n$, n odd. Show that $x' = Tx$ has a nonperiodic solution.

Problem 3.34. Let Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ have nonreal eigenvalues. Show that $b \neq 0$ and that the nontrivial solution curves to $x' = Ax$ are spirals or ellipses that are oriented clockwise if $b > 0$ and counterclockwise if $b < 0$.

Problem 3.35. Find all solutions of the following equations:

(a) $x' - 4x - \cos t = 0$ **(b)** $x' - 4x - t = 0$ **(c)** $\begin{aligned} x' &= y \\ y' &= 2 - x \end{aligned}$

(d) $\begin{aligned} x' &= y \\ y' &= -4x + \sin 2t \end{aligned}$ **(e)** $\begin{aligned} x' &= x + y + z \\ y' &= -2y + t \\ z' &= 2z + \sin t \end{aligned}$

Problem 3.36. Which of the following functions satisfy an equation of the form $s'' + as' + bs = 0$ (a, b constant)?

(a) te^t **(b)** $t^2 - t$ **(c)** $\cos 2t + 3 \sin 2t$

(d) $\cos 2t + 2 \sin 3t$ **(e)** $e^{-t} \cos 2t$ **(f)** $e^t + 4$

(g) $3t - 9$

Problem 3.37. Find solutions of the following equations having the specified initial values:

(a) $s'' + 4s = 0$; $s(0) = 1$, $s'(0) = 0$.

(b) $s'' - 3s' - 6s = 0$; $s(1) = 0$, $s'(1) = -1$.

Problem 3.38. For each of the following equations find a *basis* for the solutions; that is, find two solutions $s_1(t)$ and $s_2(t)$ such that every solution has the form $\alpha s_1(t) + \beta s_2(t)$ for suitable constants α, β:

(a) $s'' + 3s = 0$ **(b)** $s'' - 3s = 0$

(c) $s'' - s' - 6s = 0$ **(d)** $s'' + s' + s = 0$

Problem 3.39. Suppose that the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$ have negative real parts. Prove that every solution of the differential equation

$$s'' + as' + bs = 0$$

satisfies

$$\lim_{t \to \infty} s(t) = 0 .$$

State and prove a generalisation of this result for nth order differential equations

$$s^{(n)} + a_1 s^{(n-1)} + \cdots + a_n s = 0 ,$$

where the polynomial

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$

has n distinct roots with negative real parts.

Problem 3.40. Under what conditions on the constants a, b is there a nontrivial solution to $s''+as'+bs=0$ such that the equation $s(t)=0$ has

(a) no solution;
(b) a positive finite number of solutions;
(c) infinitely many solutions?

Problem 3.41. For each of the following equations sketch the phase portrait of the corresponding first order system. Then sketch the graph of several solutions $s(t)$ for different initial conditions:

(a) $s''+s=0$ **(b)** $s''-s=0$ **(c)** $s''+s'+s=0$
(d) $s''+2s'=0$ **(e)** $s''-s'+s=0$

Problem 3.42. Which equations $s''+as'+bs=0$ have a nontrivial periodic solution? What is the period?

Problem 3.43. Find all solutions to

$$s'''-s''+4s'-4s=0\ .$$

Problem 3.44. Find all pairs of functions $x(t)$, $y(t)$ that satisfy the system of differential equations

$$\begin{aligned} x' &= -y \\ y'' &= -x-y+y'\ . \end{aligned}$$

Problem 3.45. Find a real-valued function $s(t)$ such that

$$s''+4s=\cos 2t$$
$$s(0)=0, \quad s'(0)=1.$$

Problem 3.46. Let $q(t)$ be a polynomial of degree m. Show that any equation

$$s^{(n)}+a_1s^{(n-1)}+\cdots+a_ns=q$$

has a solution which is a polynomial of degree $\leq m$.

4. Stability of nonlinear systems

Having studied linear vector fields in the previous section, we now start the study of the general equation $x' = f(x)$. Explicit solutions for such equations usually cannot be found, so it behoves us to ask different questions which can be answered. Two approaches suggest themselves: approximating the solutions (either analytically or numerically) or studying those qualitative properties of the equation which do not require knowing the solution explicitly. This is what we do in this section.

4.1. Topology of $\mathbb{R}^n$.

We first recall some basic definitions concerning subsets of $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ be any point and $\varepsilon > 0$ a real number. The **(open) ε-ball about** x, denoted $B_\varepsilon(x)$, is the set

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon\}$$

consisting of points in $\mathbb{R}^n$ which are a distance less than ε from x. Similarly, one defines the **closed ε-ball about** x, denoted $\overline{B_\varepsilon(x)}$, to be the set

$$\overline{B_\varepsilon(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq \varepsilon\} \ .$$

A subset U of $\mathbb{R}^n$ is said to be **open** if U contains some ε-ball about every point. In other words, if for every $x \in U$, there is some ε (which may depend on x) such that $B_\varepsilon(x) \subset U$. A subset of $\mathbb{R}^n$ is **closed** if its complement is open. By convention, the empty set and $\mathbb{R}^n$ itself are both open and closed. They are the only two subsets of $\mathbb{R}^n$ with this property. Notice that a subset need not be either open or closed.

By a **neighbourhood** of a point $x \in \mathbb{R}^n$ we will mean any open subset of $\mathbb{R}^n$ containing the point x.

The **closure** of a subset $A \subset \mathbb{R}^n$ is defined as the intersection of all closed sets containing A. It is denoted $\bar{A}$. Equivalently $x \in \bar{A}$ if every neighbourhood of x intersects A.

Let A be a closed subset of $\mathbb{R}^n$ and let $U \subset A$. We say that U is **dense** in A if $\bar{U} = A$.

A subset of $\mathbb{R}^n$ is **bounded** if it is contained in some closed ball about the origin. A subset which is both closed and bounded is called **compact**. A continuous function defined on a compact set always attains its maximum and minimum inside the set.

A subset U of $\mathbb{R}^n$ is **connected** if there are continuous curves in U joining any two points in U. More precisely, U is connected if given $x, y \in U$ there is a continuous curve $c : [0, 1] \to U$ such that $c(0) = x$ and $c(1) = y$.

A subset U of $\mathbb{R}^n$ is **simply connected** if any closed continuous curve in U is continuously deformable to a constant curve.

We will need the following plausible result from the topology of the plane $\mathbb{R}^2$.

Theorem 4.1 (Jordan Curve Theorem)**.** *A closed curve in* $\mathbb{R}^2$ *which does not intersect itself separates* $\mathbb{R}^2$ *into two connected regions, a bounded one (the* interior *of the curve) and an unbounded one (the* exterior *of the curve).*

4.2. **Existence and uniqueness.** Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}^n$ be a continuous vector field. A solution of the differential equation

$$x' = f(x) \tag{17}$$

is a curve $u : I \to U$, where $I \subset \mathbb{R}$ is some interval in the real line, satisfying

$$u'(t) = f(u(t)) \qquad \text{for all } t \in I.$$

The interval I need not be finite and need not be either open nor closed: $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $(-\infty, b]$, $(-\infty, b)$, (a, ∞) and $[a, \infty)$ are all possible.

Let $U \subset \mathbb{R}^n$ be an open set. A vector field $f : U \to \mathbb{R}^n$ defined on U is said to be C^1, if it is continuously differentiable; that is, all the n^2 partial derivatives are continuous functions $U \to \mathbb{R}$.

Theorem 4.2. *Let* $U \subset \mathbb{R}^n$ *be open, let* $f : U \to \mathbb{R}^n$ *be a* C^1 *vector field and let* $x_0 \in U$*. Then there exists* $a > 0$ *and a unique solution*

$$x : (-a, a) \to U$$

of (17) *with* $x(0) = x_0$*.*

There are two significant differences from the linear case: we may not be able to take U to be all of $\mathbb{R}^n$ *and* we may not be able to extend the solution from $(-a, a)$ to the whole real line.

To illustrate this second point, consider the vector field $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1 + x^2$. The solution of $x' = f(x)$ is

$$x(t) = \tan(t - c) \ ,$$

where c is some constant. Clearly this solution cannot be extended beyond $|t - c| < \pi/2$. Such vector fields are said to be **incomplete**.

The differentiability condition on the vector field is necessary. For example, consider the vector field $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^{2/3}$. Then both $x(t) = 0$ and $x(t) = t^3$ solve $x' = f(x)$ with $x(0) = 0$. Thus there is no unique solution. This does not violate the theorem because $f'(x) = 2x^{-1/3}$ is not continuously differentiable at $x = 0$.

The proof of this theorem is given in Foundations of Analysis, but we can sketch the idea. Briefly, suppose $x(t)$ solves the initial value problem

$$x' = f(x) \qquad x(0) = x_0 \ . \tag{18}$$

Then let us integrate the equation $x'(s) = f(x(s))$ from $s = 0$ to t:

$$x(t) = x_0 + \int_0^t f(x(s))\,ds\ . \tag{19}$$

This integral equation is equivalent to the initial value problem (18), as can be seen by evaluating at $t = 0$ and by differentiating with respect to t both sides of the equation. Therefore it is enough to show that equation (19) has a solution. Let P denote the operator which takes a function $u : I \to U$ to the function $Pu : I \to U$, defined by

$$(Pu)(t) = x_0 + \int_0^t f(u(s))\,ds\ ,$$

where I is some interval in the real line. Then u solves (19) if and only if it is a **fixed point** of the operator P. This suggests the following iterative scheme (called **Picard's iteration method**). One defines a sequence $x_1, x_2, \ldots$ of functions where

$$\begin{aligned}
x_1(t) &= x_0 + \int_0^t f(x_0)\,ds = x_0 + f(x_0)t \\
x_2(t) &= x_0 + \int_0^t f(x_1(s))\,ds \\
&\vdots \\
x_{k+1}(t) &= x_0 + \int_0^t f(x_k(s))\,ds\ .
\end{aligned}$$

It can be shown that for f a C^1 vector field, the sequence of functions (x_i) converges uniformly in $-a < t < a$, for some a, to a unique fixed point x of P.

In fact, for $f(x) = Ax$ a linear vector field, it is easy to see that

$$x_k(t) = \left(I + tA + \cdots + \frac{t^k}{k!}A^k\right) x_0\ ,$$

so that (x_k) converges to $e^{tA}x_0$ as expected.

Notice however that despite the "constructive" nature of the proof, it is impractical for most f to use Picard's method to find the solution. Moreover even when the iterates x_k approximate the solution, they will still miss important qualitative behaviour. For example, if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $x' = Ax$ and $x(0) = x_0$ has unique solution

$$x(t) = e^{tA}x_0 = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} x_0\ ,$$

which is periodic with period 2π. However no term in the sequence of functions (x_k) obtained from Picard's method is ever periodic.

4.3. **Linearisation about equilibrium points.** Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be a C^1 vector field defined on U. Suppose that $f(\bar{x}) = 0$ for some $\bar{x} \in U$. Then the constant curve $x(t) = \bar{x}$ solves the differential equation (17) and by Theorem 4.2 it is the unique solution passing through $\bar{x}$. Therefore if $x(t) = \bar{x}$ at some time t then it will always be (and has always been) at $\bar{x}$. Such a point $\bar{x}$ is called an **equilibrium point** of the differential equation. It is also called a **zero** or a **critical point** of the vector field f.

We will be mostly interested in **isolated** equilibrium points, which are those equilibrium points $\bar{x}$ such that some ε-ball about $\bar{x}$ contains no other equilibrium points.

For example, a linear vector field $f(x) = Ax$ has a zero at the origin. This is an isolated critical point provided that A is nondegenerate; that is, $\det A \neq 0$. In fact, in this case, it is the only critical point. (See Problem 4.2.)

A general vector field may have more than one critical point. For example, the *simple pendulum* is described by the vector field $f(x,y) = (y, -\sin x)$ which has an infinite number of zeros $(n\pi, 0)$ for $n \in \mathbb{Z}$.

When faced with a general vector field, our strategy will be to try and piece together the phase portrait of the differential equation from the phase portraits near each of the critical points. The basic idea here is to *linearise* the equation at each of the critical points.

Let $\bar{x} \in U$ be a critical point of a C^1 vector field $f : U \to \mathbb{R}^n$ defined on some open subset U of $\mathbb{R}^n$. By changing coordinates in $\mathbb{R}^n$ to $y = x - \bar{x}$ if necessary, we can assume that $\bar{x} = 0$. Since f is continuously differentiable, the derivative matrix Df exists and in particular

$$\lim_{x \to 0} \frac{\|f(x) - Df(0)x\|}{\|x\|} = 0 ,$$

where we have used that $f(0) = 0$. We see that $f(x)$ is approximated by the linear vector field $Ax = Df(0)x$ at 0. We call $A = Df(0)$ the **linear part** (or the **linearisation**) of f around 0. More generally, if $\bar{x}$ is any critical point, then

$$\lim_{x \to \bar{x}} \frac{\|f(x) - Df(\bar{x})(x - \bar{x})\|}{\|x - \bar{x}\|} = 0 ,$$

hence $Df(\bar{x})$ linearises f near $\bar{x}$.

Notice that a vector field f may have an isolated zero $\bar{x}$, but its linearisation $Df(\bar{x})$ does not have an isolated zero (cf. Problem 4.3).

4.4. **Stability.** Roughly speaking an equilibrium point is stable if trajectories which start near the equilibrium point remain nearby. A more precise definition is the following.

Definition 4.3. Let $f : U \to \mathbb{R}^n$ be a C^1 vector field defined on an open subset U of $\mathbb{R}^n$. Let $\bar{x} \in U$ be a zero of the vector field. Then

$\bar{x}$ is **stable** if for every neighbourhood $W \subset U$ of $\bar{x}$, there is a neighbourhood $W_1 \subset W$ such that every solution to (17) with $x(0) \in W_1$ is defined and in W for all $t > 0$. If W_1 can be chosen so that *in addition* $\lim_{t\to\infty} x(t) = \bar{x}$, then $\bar{x}$ is **asymptotically stable**.

If $\bar{x}$ is not stable, it is **unstable**. This means that there exists one neighbourhood W of $\bar{x}$ such that *for every* neighbourhood $W_1 \subset W$ of $\bar{x}$, there is at least one solution $x(t)$ starting at $x(0) \in W_1$ which does not lie entirely in W.

An equivalent ε-δ definition of (asymptotic) stability is given in Problem 4.4.

Stable equilibria which are *not* asymptotically stable are sometimes called **neutrally stable**.

One should note that $\lim_{t\to\infty} x(t) = \bar{x}$ on its own does *not* imply stability. (There are counterexamples, but they are quite involved.)

Let $x \mapsto Ax$ be a linear vector field on $\mathbb{R}^n$. Then the origin is called a **(linear) sink** if all the eigenvalues of A have negative real parts. More generally, a zero $\bar{x}$ of a C^1 vector field $f : U \to \mathbb{R}^n$ is called a **(nonlinear) sink** if all the eigenvalues of the linearisation $Df(\bar{x})$ have negative real parts.

A linear sink is asymptotically stable, whereas a centre is stable but not asymptotically stable. Saddles and sources, for example, are unstable.

The following theorems tell us to what extent we can trust the stability properties of the linearisation of a nonlinear vector field.

Theorem 4.4. *Let $f : U \to \mathbb{R}^n$ be a C^1 vector field defined on an open subset of $\mathbb{R}^n$ and let $\bar{x} \in U$ be a sink. Then there is a neighbourhood $W \subset U$ of $\bar{x}$ such that if $x(0) \in W$ then $x(t)$ is defined and in W for all $t > 0$ and such that $\lim_{t\to\infty} x(t) = \bar{x}$.*

Theorem 4.5. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be a C^1 vector field. Suppose that $\bar{x}$ is a stable equilibrium point of the equation* (17)*. Then no eigenvalue of $Df(\bar{x})$ has positive real part.*

Morally speaking, these two theorems say that if the linearised systems is unstable or asymptotically stable, then so will be the nonlinear system in a small enough neighbourhood of the equilibrium point. If the linearised system is stable but not asymptotically stable, then we cannot say anything about the nonlinear system. (See Problem 4.12.)

4.5. **Liapunov stability.** In those cases where linearisation about an equilibrium point sheds no light on its stability properties (because the linearisation is neutrally stable, say) a method due to Liapunov can help. Throughout this section we will let $f : U \to \mathbb{R}^n$ be a C^1 vector field defined on an open subset U of $\mathbb{R}^n$, and we will let $\bar{x} \in U$ be such that $f(\bar{x}) = 0$.

Let $E : W \to \mathbb{R}$ be a differentiable function defined in a neighbourhood $W \subset U$ of $\bar{x}$. We denote $\dot{E} : W \to \mathbb{R}$ the function defined by

$$\dot{E}(x) = DE(x)f(x) \ .$$

Here the right-hand side is simply the operator $DE(x)$ applied to the vector $f(x)$. If we let $\phi_t(x)$ denote the solution to (17) passing through x when $t = 0$, then

$$\dot{E}(x) = \frac{d}{dt}E(\phi_t(x))\bigg|_{t=0}$$

by the chain rule.

Definition 4.6. A **Liapunov function** for $\bar{x}$ is a continuous function $E : W \to \mathbb{R}$ defined on a neighbourhood $W \subset U$ of $\bar{x}$, differentiable on $W - \bar{x}$, such that

(a) $E(\bar{x}) = 0$ and $E(x) > 0$ if $x \neq \bar{x}$;
(b) $\dot{E} \leq 0$ in $W - \bar{x}$.

If in addition, E satisfies

(c) $\dot{E} < 0$ in $W - \bar{x}$,

then it is said to be a **strict** Liapunov function for $\bar{x}$.

We can now state the stability theorem of Liapunov.

Theorem 4.7 (Liapunov Stability Theorem)**.** *Let $\bar{x}$ be an equilibrium point for* (17)*. If there exists a (strict) Liapunov function for $\bar{x}$ then $\bar{x}$ is (asymptotically) stable.*

Proof. Let $\delta > 0$ be so small that the closed δ-ball about $\bar{x}$ lies entirely in W. Let α be the minimum value of E on the boundary of this δ-ball, the sphere $S_\delta(\bar{x})$ of radius δ centred at $\bar{x}$. From (a) we know that $\alpha > 0$. Let

$$W_1 = \left\{ x \in \overline{B_\delta(\bar{x})} \mid E(x) < \alpha \right\} \ .$$

Then no solution starting inside W_1 can meet the $S_\delta(\bar{x})$ since, by (b), E is non-increasing on solution curves. Therefore $\bar{x}$ is stable.

Now assume that (c) also holds so that E is strictly decreasing on solution curves in $W - \bar{x}$. Let $x(t)$ be a solution starting in $W_1 - \bar{x}$ and and consider $E(x(t))$. Showing that $\lim_{t\to\infty} E(x(t)) = 0$ is equivalent to showing that $\lim_{t\to\infty} x(t) = \bar{x}$. Since $E(x(t))$ is strictly decreasing and bounded below by 0, $L := \lim_{t\to\infty} E(x(t))$ exists. We claim that $L = 0$.

Assume for a contradiction that $L > 0$ instead. Then by the same argument as in the first part, we deduce that there is some smaller sphere S_ρ ($\rho < r$) such that $E(x) < L$ for all points x inside S_ρ. Since $\dot{E}$ is continuous, it attains a maximum $-M$ in the spherical shell $A_{\rho,R}$ bounded by S_ρ and S_R. Because $\dot{E}$ is negative definite, $-M$ is negative. Now consider any solution curve starting inside S_r at time 0, but outside S_ρ. By stability it remains inside S_R, and since E is strictly decreasing and $\lim_{t\to\infty} E(x(t)) = L$, it remains outside the sphere S_ρ. Therefore at any later time t,

$$\begin{aligned} E(x(t)) &= E(x(0)) + \int_0^t \dot{E}(x(s))ds \\ &\leq E(x(0)) - M\,t\ ; \end{aligned}$$

but no matter how small M is, for t large enough the right hand side will eventually be negative, contradicting the positive-definiteness of E. □

One can picture this theorem in the following way. Near $\bar{x}$, a Liapunov function has level sets which look roughly like ellipsoids containing $\bar{x}$. One can interpret the condition that E is decreasing geometrically, as saying that at any point on the level set of E, the vector field $f(x)$ points to the inside of the ellipsoid. If E is merely non-increasing, then the vector field may also point tangential to the ellipsoid; but in either case, once inside such an ellipsoid, a solution curve can never leave again.

There is also a similar result (which we state without proof) concerning the instability of a critical point.

Theorem 4.8 (Liapunov Instability Theorem)**.** *Let $E : W \to \mathbb{R}$ be a continuous function defined on a neighbourhood $W \subset U$ of $\bar{x}$ and differentiable in $W - \bar{x}$. Then if*

(a) $\dot{E} > 0$ *in $W - \bar{x}$, and*
(b) *every closed ball centred at $\bar{x}$ and contained in W contains a point x where $E(x) > 0$,*

then $\bar{x}$ is an unstable equilibrium point.

4.6. **Stability and gradient fields.** Let $U \subset \mathbb{R}^n$ be open and let $V : U \to \mathbb{R}$ be a C^2 function (twice continuously differentiable). Let $f : U \to \mathbb{R}^n$ be the associated gradient vector field $f(x) = -\operatorname{grad} V(x)$, as discussed in Section 2.5.

It follows from the chain rule that

$$\dot{V}(x) = -\|f(x)\|^2 \leq 0\ .$$

Moreover $\dot{V}(x) = 0$ if and only if $\operatorname{grad} V(x) = 0$, so that x is an equilibrium point of the gradient system $x' = -\operatorname{grad} V(x)$. This, together with the observations in Section 2.5, allows us to characterise the gradient flows geometrically.

Theorem 4.9. *Consider the gradient dynamical system*

$$x' = -\operatorname{grad} V(x)\ .$$

At regular points, where $\operatorname{grad} V(x) \neq 0$, *the solution curves cross level surfaces orthogonally. Nonregular points are equilibria of the system. Isolated minima are asymptotically stable.*

4.7. **Limit cycles and the Poincaré–Bendixson theorem.** This section deals exclusively with *planar* vector fields.

We saw in Problem 3.26 that if a linear vector field $x \mapsto Ax$ in the plane admits a nontrivial periodic solution, then all solutions are periodic with the same period. The situation is very different for nonlinear planar vector fields. These may admit nontrivial periodic solutions which are limits of other nonperiodic trajectories. These solutions are known as limit cycles.

Let $t \mapsto \phi_t(u)$ denote the solution curve to $x' = f(x)$ which passes by u at $t = 0$. We define the **future limit point set** of u to be all those $x \in U$ such that there is a increasing infinite sequence $(t_i) \to \infty$ such that in this limit $\phi_{t_i} u \to x$. Similarly we define the **past limit point set** of u by taking instead decreasing sequences $(t_i) \to -\infty$.

Definition 4.10. By a **closed orbit** γ of a dynamical system we mean the image of a nontrivial periodic solution. That is, one which is not an equilibrium point and for which $\phi_p(x) = x$ for some $x \in \gamma$ and $p \neq 0$. It follows that $\phi_{np}(y) = y$ for all $y \in \gamma$ and $n \in \mathbb{Z}$.

A **limit cycle** for the planar system $x' = f(x)$ is a closed orbit with the property that it is contained either in the past or future limit point set of some point *not* contained in γ.

We mention several results without proofs concerning the existence of limit cycles in planar nonlinear differential equations.

Theorem 4.11. *A limit cycle encloses an equilibrium point.*

Let $f : U \to \mathbb{R}^2$ be a C^1 vector field defined on an open subset U of the plane.

Theorem 4.12 (Bendixson Negative Criterion)**.** *If $W \subset U$ is simply-connected* and *the divergence of f does not change sign in W, the there is no limit cycle in W.*

Theorem 4.13 (Poincaré–Bendixson Theorem)**.** *Let $K \subset U$ be a compact subset which contains no equilibrium points. If $t \mapsto x(t)$ is a solution curve which starts in K and remains in K for all $t > 0$, then it is itself a limit cycle or spirals towards one. In either case the subset K contains a limit cycle.*

Problems

(Some of the problems are taken from Hirsch & Smale, Chapters 8 and 9.)

Problem 4.1. Write the first few terms of the Picard iteration scheme for each of the following initial value problems. Where possible, use any method to find explicit solutions and discuss the domain of the solution.

(a) $x' = x + 2$, $x(0) = 2$

(b) $x' = x^{4/3}$, $x(0) = 0$
(c) $x' = x^{4/3}$, $x(0) = 1$
(d) $x' = \sin x$, $x(0) = 0$
(e) $x' = 1/(2x)$, $x(1) = 1$

Problem 4.2. Consider the linear equation $x' = Ax$. Show that if $\det A = 0$ then there are an infinite number of critical points, none of which is isolated.

Problem 4.3. Write down an example of a C^1 vector field f having an isolated zero, say $\bar{x}$, but whose linearisation $Df(\bar{x})$ about $\bar{x}$ does not.

Problem 4.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be a C^1 vector field and $\bar{x} \in U$ a zero of f. Prove that $\bar{x}$ is *stable* if and only if given any $\varepsilon > 0$ such that the open ε-ball about $\bar{x}$ is contained in U, there exists a $0 < \delta \leq \varepsilon$ such that for any solution of $x' = f(x)$,

$$\|x(0) - \bar{x}\| < \delta \implies \|x(t) - \bar{x}\| < \varepsilon ,$$

for all $t \geq 0$.

Problem 4.5. Show by example that if f is a nonlinear C^1 vector field and $f(0) = 0$, it is possible that $\lim_{t\to\infty} x(t) = 0$ for all solutions to $x' = f(x)$, without the eigenvalues of $Df(0)$ having negative real parts.

Problem 4.6. **(a)** Let $\bar{x}$ be a stable equilibrium of a differential equation corresponding to a C^1 vector field on an open subset $U \subset \mathbb{R}^n$. Show that for every neighbourhood W of $\bar{x}$ in U, there is a neighbourhood W' of $\bar{x}$ in W such that every solution curve $x(t)$ with $x(0) \in W'$ is defined an in W' for all $t > 0$.

(b) If $\bar{x}$ is asymptotically stable, the neighbourhood W' in (a) can be chosen with the additional property that $\lim_{t\to\infty} x(t) = \bar{x}$ if $x(0) \in W'$.

(*Hint:* Consider the set of all points of W whose trajectories for $t \geq 0$ enter the set W_1 in Definition 4.3.)

Problem 4.7. For which of the following linear operators A on $\mathbb{R}^n$ is $0 \in \mathbb{R}^n$ a stable equilibrium of $x' = Ax$?

(a) $A = O$

(b) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

Problem 4.8. Each of the following linear autonomous systems has the origin as an isolated critical point. Determine the nature and stability properties of that critical point.

(a) $\begin{cases} x' = 2x \\ y' = 3y \end{cases}$ **(b)** $\begin{cases} x' = -x - 2y \\ y' = 4x - 5y \end{cases}$

(c) $\begin{cases} x' = -3x + 4y \\ y' = -2x + 3y \end{cases}$ **(d)** $\begin{cases} x' = 5x + 2y \\ y' = -17x - 5y \end{cases}$

(e) $\begin{cases} x' = -4x - y \\ y' = x - 2y \end{cases}$ **(f)** $\begin{cases} x' = 4x - 2y \\ y' = 5x + 2y \end{cases}$

Problem 4.9. Consider the autonomous system

$$\begin{cases} x' = ax + by + e \\ y' = cx + dy + f \ , \end{cases}$$

where $ad - bc \neq 0$ and e, f are constants.

(a) Show that it has an isolated critical point, say (x_0, y_0).

(b) Introduce new variables $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$, and show that in terms of these variables the system is of the form

$$\begin{cases} \bar{x}' = \bar{a}\bar{x} + \bar{b}\bar{y} \\ \bar{y}' = \bar{c}\bar{x} + \bar{d}\bar{y} \ . \end{cases}$$

Find $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ in terms of a, b, c, d. Deduce therefore that this system has a critical point at the origin.

(c) Find the critical point of the system

$$\begin{cases} x' = 2x - 2y + 10 \\ y' = 11x - 8y + 49, \end{cases}$$

and determine its nature and stability properties.

Problem 4.10. Consider the linear dynamical system $x' = Ax$ in the plane $\mathbb{R}^2$. Discuss the stability of the origin as an equilibrium point if:

(a) A is symmetric ($A^T = A$) and positive definite;

(b) A is symmetric and negative definite;

(c) A is skew-symmetric ($A^T = -A$);

(d) $A = B + C$, where B is symmetric negative definite and C is skew-symmetric.

Problem 4.11. Show that the dynamical system in $\mathbb{R}^2$ whose equations in polar coordinates are

$$\theta' = 1 \qquad r' = \begin{cases} r^2 \sin(1/r) & r > 0 \\ 0 & r = 0 \end{cases}$$

has a stable equilibrium at the origin.
(*Hint:* Every neighbourhood of the origin contains a solution curve encircling the origin.)

Problem 4.12. This problem illustrates that if the linearised system has a centre, then the nonlinear stability is undecided. Consider the two autonomous systems

$$\textbf{(a)} \begin{cases} x' = y + x(x^2+y^2) \\ y' = -x + y(x^2+y^2) \end{cases} \qquad \textbf{(b)} \begin{cases} x' = y - x(x^2+y^2) \\ y' = -x - y(x^2+y^2) \end{cases} .$$

Both systems have a critical point at the origin. Linearise the system and show that that both systems have a centre at the critical point. Solve the nonlinear systems by changing coordinates to polar coordinates (r, θ), and show that the two systems have opposite stability properties. Which system is stable and which is unstable?

Problem 4.13. Sketch the phase portrait of the equation $x'' = 2x^3$ and show that it has an unstable critical point at the origin.

Problem 4.14. Consider the following second order equation:

$$x'' + 2\mu x' + \omega^2 x = 0 \ ,$$

where $\mu \geq 0$ and $\omega > 0$ are constants.

(a) Write down the equivalent autonomous system and show that it has a unique critical point at the origin of the phase plane.
(b) For each of the following cases, describe the nature and stability properties of the critical point:
(i) $\mu = 0$; ***(ii)*** $0 < \mu < \omega$; ***(iii)*** $\mu = \omega$; and ***(iv)*** $\mu > \omega$.

Problem 4.15. Consider the following second order equation:

$$x'' + x + x^3 = 0 \ .$$

Find the trajectories, critical points and their nature and stability properties.

Problem 4.16. For each of the following functions $V(u)$, sketch the phase portrait of the gradient flow $u' = -\operatorname{grad} V(u)$. Identify the equilibria and classify them as to stability or instability. Sketch the level surfaces of V on the same diagram.

(a) $x^2 + 2y^2$ **(b)** $x^2 - y^2 - 2x + 4y + 5$
(c) $y \sin x$ **(d)** $2x^2 - 2xy + 5y^2 + 4x + 4y + 4$
(e) $x^2 + y^2 - z$ **(f)** $x^2(x-1) + y^2(y-2) + z^2$

Problem 4.17. Find the type of critical point at the origin of the following system:

$$\begin{cases} x' = x + y - x(x^2+y^2) \\ y' = -x + y - y(x^2+y^2) \end{cases} .$$

Derive an equation for $E = x^2 + y^2$ and show that $\dot{E} = 0$ for $E = 1$. Solve the system and sketch the trajectories spiralling out and in to this limit cycle.

Problem 4.18. Show that the following system has a periodic solution:

$$\begin{cases} x' = 3x - y - xe^{x^2+y^2} \\ y' = x + 3y - ye^{x^2+y^2} \end{cases} .$$

Problem 4.19. Consider the equation

$$x'' + x - \lambda x^2 - 1 = 0 ,$$

where $0 < \lambda < 1$. Sketch the phase portrait of the system for $0 < \lambda < \frac{1}{4}$ and for $\frac{1}{4} < \lambda < 1$ and compare the nature of the trajectories.

Problem 4.20. Consider a linear system

$$x' = Ax , \quad \text{with } A \text{ an } n \times n \text{ constant invertible matrix.}$$

This system has a unique critical point at the origin.

(a) If A is skew-symmetric, show that the energy function $E = \|x\|^2$ is conserved along trajectories.

(b) If there exists M such that $MA + A^T M$ is negative definite, show that the energy function $E = \langle x, Mx \rangle$ is decreasing ($\dot{E} < 0$) along trajectories. If in addition M is positive-definite, use Liapunov stability to deduce that the origin is a stable critical point, at least when $n = 2$.

(c) Let $A = S^{-1}NS$, so that relative to the new variables $y = Sx$, the system becomes

$$y' = Ny .$$

Suppose that there exists a positive-definite matrix M satisfying $MA + A^T M$ is negative definite (resp. negative semi-definite). Show that there exists a positive-definite matrix P such that $PN + N^T P$ is negative definite (resp. negative semi-definite).

(d) For each of the stable critical points of 2×2 autonomous systems find a suitable Liapunov function which detects the precise type of stability: neutral or asymptotic.
Hint: Use part (c) to argue that it is enough to look at the normal forms discussed in the lecture:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} \nu & 0 \\ 1 & \nu \end{pmatrix} \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} ,$$

where in this case $\lambda \leq \mu < 0$, $\nu < 0$ and $\alpha \leq 0$. Then use (b) to find Liapunov functions for the normal forms.

Problem 4.21. Consider the second order equation

$$x'' + \kappa x' + x = 0 , \quad \text{where } \kappa \geq 0.$$

(a) Write the corresponding linear system and show that it has an isolated critical point at the origin.
(b) Show that the function $E(x,y) = x^2 + y^2$ is a Liapunov function. Deduce that the origin is stable.
(c) Identify the type of critical point and its stability property for $\kappa = 0$, $0 < \kappa < 2$, $\kappa = 2$ and $\kappa > 2$. In particular, show that the origin is asymptotically stable for $\kappa > 0$.

This problem shows that a given Liapunov function may fail to detect asymptotic stability. (It can be shown that there exists one which does.) *Moral*: Liapunov functions are not unique and knowing that one exists is not the same thing as finding one!

Problem 4.22. For each of the following systems, show that the origin is an isolated critical point, find a suitable Liapunov function, and prove that the origin is asymptotically stable:

$$\textbf{(a)} \begin{cases} x' = -3x^3 - y \\ y' = x^5 - 2y^3 \end{cases} \qquad \textbf{(b)} \begin{cases} x' = -2x + xy^3 \\ y' = -x^2y^2 - y^3 \end{cases}$$

$$\textbf{(c)} \begin{cases} x' = y^2 + xy^2 - x^3 \\ y' = -xy + x^2y - y^3 \end{cases} \qquad \textbf{(d)} \begin{cases} x' = x^3y + x^2y^3 - x^5 \\ y' = -2x^4 - 6x^3y^2 - 2y^5 \end{cases} .$$

(*Hint:* Try Liapunov functions of the form $a\,x^{2p} + b\,y^{2q}$, for suitable a, b, p, q.)

Problem 4.23. Consider the van der Pol equation:

$$x'' + \mu(x^2 - 1)x' + x = 0 \ .$$

Write the equivalent linear system and show that it has an isolated critical point at the origin. Investigate the stability properties of this critical point for $\mu > 0$, $\mu = 0$ and $\mu < 0$.

Problem 4.24. Consider the following autonomous system:

$$\begin{cases} x' = -y + x\,f(x,y) \\ y' = x + y\,f(x,y) \end{cases} ,$$

where f is a function on the phase plane which is continuous and continuously differentiable on some disk D about the origin.

(a) Show that the origin is an isolated critical point.
(b) By constructing a Liapunov function or otherwise, show that the origin is asymptotically stable if f is negative definite on D.

Problem 4.25. Discuss the stability of the limit cycles and critical points of the following systems. (Here $r^2 = x^2 + y^2$.)

(a) $\begin{cases} x' = x + y + x(r^2 - 3r + 1) \\ y' = -x + y + y(r^2 - 3r + 1) \end{cases}$ **(b)** $\begin{cases} x' = y + x\sin(1/r) \\ y' = -x + y\sin(1/r) \end{cases}$

(c) $\begin{cases} x' = x - y + x(r^3 - r - 1) \\ y' = x + y + y(r^3 - r - 1) \end{cases}$ **(d)** $\begin{cases} x' = y + x(\sin r)/r \\ y' = -x + y(\sin r)/r \end{cases}$

Problem 4.26. Does any of the following differential equations have limit cycles? Justify your answer.

(a) $x'' + x' + (x')^5 - 3x^3 = 0$
(b) $x'' - (x^2 + 1)x' + x^5 = 0$
(c) $x'' - (x')^2 - 1 - x^2 = 0$

Problem 4.27. Prove that the following systems have a limit cycle, by studying the behaviour of the suggested Liapunov function and applying the Poincaré–Bendixson theorem.

$$\textbf{(a)} \quad \begin{cases} x' = 2x - y - 2x^3 - 3xy^2 \\ y' = 2x + 4y - 4y^3 - 2x^2y \end{cases} .$$

(*Hint:* Try $E(x, y) = 2x^2 + y^2$.)

$$\textbf{(b)} \quad \begin{cases} x' = 8x - 2y - 4x^3 - 2xy^2 \\ y' = x + 4y - 2y^3 - 3x^2y \end{cases} .$$

(*Hint:* Try $E(x, y) = x^2 + 2y^2$.)

5. Rudiments of the theory of distributions

It is not unusual to encounter differential equations with inhomogeneous terms which are not continuous functions: infinitesimal sources, instantaneous forces,... There is a nifty formalism which allows us to treat these peculiar differential equations on the same footing as the regular ones. This formalism is called the theory of distributions. The introduction to this topic is a little more technical than the rest of the course, so please be patient.

5.1. **Test functions.**

Definition 5.1. A **test function** $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. We shall let $\mathcal{D}$ denote the space of test functions.

Smooth means that φ is continuous and infinitely differentiable, so that all its derivatives $\varphi', \varphi'', \varphi^{(3)}, \ldots$ are continuous. One often says that φ is of class C^∞.

A function φ has compact support if there is a real number R such that $\varphi(t) = 0$ for all $|t| > R$. More precisely, if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function, we can define its **support** $\operatorname{supp}\varphi$ to be the closure of the set of points on which φ does *not* vanish:

$$\operatorname{supp}\varphi = \overline{\{t \mid \varphi(t) \neq 0\}} \ .$$

Alternatively, $\operatorname{supp}\varphi$ is the complement of the largest open set on which φ vanishes. In other words, a function will always vanish outside its support, but may also vanish at some points at the boundary of its support.

A useful class of test functions are the so-called bump functions. To construct them, consider first the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) := \begin{cases} e^{-1/t} \ , & t > 0 \\ 0 \ , & t \leq 0 \end{cases}$$

This function is clearly smooth everywhere except possibly at $t = 0$; but L'Hôpital's rule removes that doubt: h is everywhere smooth. Clearly, $\operatorname{supp} h = [0, \infty)$. Define the function

$$\varphi(t) := h(t)h(1-t) \ .$$

It is clearly smooth, being the product of two smooth functions. Its support is now $[0, 1]$, so that it is a test function. Notice that $\varphi(t) \geq 0$. If you plot it, you find that it is a small bump between 0 and 1. We can rescale φ to construct a small bump between a and b:

$$\varphi_{[a,b]}(t) := \varphi\left(\frac{t-a}{b-a}\right) \ .$$

Clearly $\varphi_{[a,b]}$ is a test function with support $[a, b]$. Somewhat loosely we will call a nowhere negative test function which is not identically zero, a **bump function**. The integral of a bump function is always positive. For instance, one has

$$\int_{\mathbb{R}} \varphi_{[a,b]}(t)dt \approx 0.00703\,(b-a) \ .$$

Proposition 5.2. *The space $\mathcal{D}$ of test functions has the following easily proven properties:*

1. *$\mathcal{D}$ is a real vector space; so that if $\varphi_1, \varphi_2 \in \mathcal{D}$ and $c_1, c_2 \in \mathbb{R}$, then $c_1\varphi_1 + c_2\varphi_2 \in \mathcal{D}$.*
2. *If f is smooth and $\varphi \in \mathcal{D}$, then $f\varphi \in \mathcal{D}$.*
3. *If $\varphi \in \mathcal{D}$, then $\varphi' \in \mathcal{D}$. Hence all the derivatives of a test function are test functions.*

We are only considering real-valued functions of a real variable, but *mutatis mutandis* everything we say also holds for complex-valued functions of a real variable.

Definition 5.3. A function $f : \mathbb{R} \to \mathbb{R}$ is called **(absolutely) integrable** if

$$\int_{\mathbb{R}} |f(t)|dt < \infty \ .$$

We say that $f : \mathbb{R} \to \mathbb{R}$ is **locally integrable** if

$$\int_a^b |f(t)|dt < \infty$$

for any finite interval $[a, b]$. A special class of locally integrable functions are the (piecewise) continuous functions.

Test functions can be used to *probe* other functions.

Proposition 5.4. *If f is locally integrable and φ is a test function, then the following integral is finite:*

$$\int f\,\varphi := \int_{\mathbb{R}} f(t)\varphi(t)dt \ .$$

Proof. Since $\varphi \in \mathcal{D}$, it vanishes outside some closed interval $[-R, R]$. Moreover since it is continuous, it is bounded: $|\varphi(t)| \leq M$ for some M. Therefore

$$\int_{\mathbb{R}} f(t)\varphi(t)dt = \int_{-R}^{R} f(t)\varphi(t)dt \ ;$$

whence we can estimate

$$\left|\int_{-R}^{R} f(t)\varphi(t)dt\right| \leq \int_{-R}^{R} |f(t)||\varphi(t)|dt \leq M \int_{-R}^{R} |f(t)|dt < \infty \ ,$$

where in the last inequality we have used that f is locally integrable. $\square$

As the next result shows, test functions are pretty good probes. In fact, they can distinguish continuous functions.

Theorem 5.5. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that*

$$\int f\,\varphi = \int g\,\varphi \quad \forall \varphi \in \mathcal{D}\ .$$

Then $f = g$.

Proof. We prove the logically equivalent statement: if $f \neq g$, then there exists some test function φ for which $\int f\,\varphi \neq \int g\,\varphi$.

If $f \neq g$ there is some point t_0 for which $f(t_0) \neq g(t_0)$. Without loss of generality, let us assume that $f(t_0) > g(t_0)$. By continuity this is also true in a neighbourhood of that point. That is, there exist $\varepsilon > 0$ and $\delta > 0$, such that

$$f(t) - g(t) \geq \varepsilon \qquad \text{for } |t - t_0| \leq \delta.$$

Let φ be a bump function supported in $[t_0 - \delta, t_0 + \delta]$. Then, since bumps have nonzero area,

$$\int f\,\varphi - \int g\,\varphi = \int_{t_0-\delta}^{t_0+\delta} (f(t) - g(t))\,\varphi(t) dt \geq \varepsilon \int_{t_0-\delta}^{t_0+\delta} \varphi(t) dt > 0\ ,$$

whence $\int f\,\varphi \neq \int g\,\varphi$. □

Remark 5.6. Continuity of f and g is essential; otherwise we could change f, say, in just one point without affecting the integral.

The next topic is somewhat technical. It is essential for the logical development of the subject, but we will not stress it. In particular, it will not be examinable.

Definition 5.7. A sequence of functions $f_m : \mathbb{R} \to \mathbb{R}$ is said to **converge uniformly** to zero, if given any $\varepsilon > 0$ there exists N (depending on ε) so that $|f_m(t)| < \varepsilon$ for all $m > N$ *and for all* t. In other words, the crucial property is that N depends on ε but not on t.

We now define a "topology" on the space of test functions; i.e., a notion of when two test functions are close to each other. Because $\mathcal{D}$ is a vector space, two test functions are close to each other if their difference is close to zero. It is therefore enough to say when a sequence of test functions approaches zero.

Definition 5.8. A sequence of test functions φ_m for $m = 1, 2, \ldots,$ is said to **converge to zero** (written $\varphi_m \to 0$) if the following two conditions are satisfied:

1. all the test functions φ_m vanish outside some common finite interval $[-R, R]$ (in other words, R does not depend on m); and
2. for a fixed k, the k-th derivatives $\varphi_m^{(k)}$ converge uniformly to zero.

We can also talk about test functions converging to a function $\varphi_m \to \varphi$ in the same way. One can show that φ is itself a test function. In other words, the space of test functions is complete.

5.2. **Distributions.** Finally we can define a distribution.

Definition 5.9. A **distribution** is a continuous linear functional on the space $\mathcal{D}$ of test functions. The space of distributions is denoted $\mathcal{D}'$.

In other words, a distribution T is a linear map which, acting on a test function φ, gives a real number denoted by $\langle T, \varphi\rangle$.

Linearity means that

$$\langle T, c_1\varphi_1 + c_2\varphi_2\rangle = c_1 \langle T, \varphi_1\rangle + c_2 \langle T, \varphi_2\rangle \ ,$$

for all $\varphi_1, \varphi_2 \in \mathcal{D}$ and $c_1, c_2 \in \mathbb{R}$.

Continuity simply means that if a sequence φ_m of test functions converges to zero, then so does the sequence of real numbers $\langle T, \varphi_m\rangle$:

$$\varphi_m \to 0 \implies \langle T, \varphi_m\rangle \to 0 \ .$$

Proposition 5.10. *Every locally integrable function f defines a distribution T_f defined by*

$$\langle T_f, \varphi\rangle = \int f\,\varphi \qquad \textit{for all } \varphi \in \mathcal{D}.$$

Proof. The map T_f is clearly linear. To show that it is also continuous, suppose that $\varphi_m \to 0$. By the first convergence property in Definition 5.8, there exists a finite R such that all φ_m vanish outside of $[-R, R]$. Let C be the result of integrating $|f(t)|$ on $[-R, R]$, and let M_m be such that $|\varphi_m(t)| \le M_m$. Then

$$|\langle T_f, \varphi_m\rangle| = \left|\int_{-R}^{R} f(t)\varphi_m(t)dt\right| \le \int_{-R}^{R} |f(t)||\varphi_m(t)|dt \le M_m C \ .$$

But now notice that $\varphi_m \to 0 \implies M_m \to 0 \implies \langle T_f, \varphi_m\rangle \to 0$. ☐

A distribution of the form T_f for some locally integrable function f is called **regular**.

A famous example of a regular distribution is the one associated to the **Heaviside step function**. Let H be the following function:

$$H(t) = \begin{cases} 1 \ , & t \ge 0 \\ 0 \ , & t < 0 \ . \end{cases}$$

It is clearly locally integrable, although it is not continuous. It gives rise to a regular distribution T_H defined by

$$\langle T_H, \varphi\rangle = \int_{\mathbb{R}} H(t)\varphi(t)dt = \int_0^\infty \varphi(t)dt \ .$$

Remark 5.11. The value at $t = 0$ of the Heaviside step function is a matter of convention. It is an intrinsic ambiguity in its definition. Notice however that the regular distribution T_H does not depend on this choice. This simply reiterates the fact that we should think of the Heaviside step function not as a function, but as a distribution.

If all distributions were regular, there would be little point to this theory. In fact, the *raison d'être* of the theory of distributions is to provide a rigorous framework for the following distribution.

Definition 5.12. The **Dirac δ-distribution** is the distribution δ defined by

$$\boxed{\langle \delta, \varphi \rangle := \varphi(0) \qquad \forall \varphi \in \mathcal{D}\ .} \tag{20}$$

This distribution cannot be regular: indeed, if there were a function $\delta(t)$ such that $\int \delta\, \varphi = \varphi(0)$ it would have to satisfy that $\delta(t) = 0$ for all $t \neq 0$; but then such a function could not possibly have a nonzero integral with any test function. Nevertheless it is not uncommon to refer to this distribution as the *Dirac δ-function*.

Distributions which are not regular are called **singular**.

Distributions obey properties which are analogous to those obeyed by the test functions. In fact, dually to Proposition 5.2 we have the following result.

Proposition 5.13. *The space $\mathcal{D}'$ of distributions enjoys the following properties:*

1. *$\mathcal{D}'$ is a real vector space. Indeed, if $T_1, T_2 \in \mathcal{D}'$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 T_1 + c_2 T_2$ defined by*
$$\langle c_1 T_1 + c_2 T_2, \varphi \rangle = c_1 \langle T_1, \varphi \rangle + c_2 \langle T_2, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}$$
is a distribution.
2. *If f is smooth and $T \in \mathcal{D}'$, then fT, defined by*
$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle \qquad \forall \varphi \in \mathcal{D}$$
is a distribution.
3. *If $T \in \mathcal{D}'$, then T' defined by*
$$\boxed{\langle T', \varphi \rangle = -\langle T, \varphi' \rangle \qquad \forall \varphi \in \mathcal{D}} \tag{21}$$
is a distribution.

Notice that any test function, being locally integrable, gives rise to a (regular) distribution. This means that we have a linear map $\mathcal{D} \to \mathcal{D}'$ which is one-to-one by Theorem 5.5. On the other hand, the existence of singular distributions means that this map is not injective. Nevertheless one can approximate (in a sense to be made precise below) singular distributions by regular ones.

The space of distributions inherits a notion of convergence from the space of test functions.

Definition 5.14. We say that a sequence of distributions T_n **converges weakly** to a distribution T, written $T_n \to T$, if

$$\boxed{\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \quad \forall\, \varphi \in \mathcal{D}\ .}$$

We will often say that T_n converges to T "in the (weak) distributional sense" to mean this type of convergence. It is in this sense that singular distributions can be approximated by regular distributions. In fact, Problem 5.1 explicitly shows how to construct sequences of regular distributions which converge weakly to δ.

5.3. **Distributional derivatives and ODEs.** Let f be a locally integrable function. It need not be continuous, and certainly not differentiable. So there is little point in thinking of its derivative f' as a function: it may not even be well defined at points. Nevertheless, we can make sense of f' as a distribution: T'_f.

If f is differentiable, then $T'_f = T_{f'}$ (see Problem 5.2). On the other hand, if f is not differentiable, then T'_f need not be a regular distribution at all. Let us illustrate this with an important example.

The derivative H' of the step function is not a function: it would not be defined for $t = 0$. Nevertheless, the distributional derivative T'_H is well-defined:

$$\langle T'_H, \varphi \rangle = -\langle T_H, \varphi' \rangle = -\int_0^\infty \varphi'(t)dt = -\varphi(t)\Big|_0^\infty = \varphi(0)\ ,$$

where we have used the fact that φ has compact support. Comparing with equation (20), we see that δ is the (distributional) derivative of the step function:

$$\boxed{T'_H = \delta\ .} \tag{22}$$

Let us give some examples of distributional derivatives. We write them as if they were functions, but it is important to keep in mind that they are not. Hence, when we write f' what we really mean is the distribution T'_f and so on.

1. Let $f(t) = t\,H(t)$. Then $f' = H$ and $f'' = \delta$.
2. Let $f(t) = |t|$. Then $f'(t) = H(t) - H(-t)$ and $f'' = 2\delta$.
3. Let f be n-times continuously differentiable (i.e., of class C^n). Then

$$f\delta^{(n)} = \sum_{i=0}^{n} (-1)^i f^{(i)}(0)\delta^{(n-i)}\ . \tag{23}$$

4. In particular we have that

$$t^m \delta^{(n)} = \begin{cases} 0 & m > n \\ (-1)^m m! \delta & m = n \\ (-1)^m \frac{n!}{(n-m)!} \delta^{(n-m)} & m < n \ . \end{cases}$$

Our present interest in distributions is to be able to solve a larger class of differential equations. We will focus solely on linear ODEs. Therefore we start by showing how differential operators act on distributions.

Since the derivative of a distribution is a distribution, we can take any (finite) number of derivatives of a distribution and still get a distribution. Iterating the identity (21) we obtain for any distribution T and any test function φ:

$$\langle T'', \varphi \rangle = -\langle T', \varphi' \rangle = \langle T, \varphi'' \rangle \ ,$$

or more generally

$$\boxed{\left\langle T^{(k)}, \varphi \right\rangle = (-1)^k \left\langle T, \varphi^{(k)} \right\rangle \qquad \forall T \in \mathcal{D}' \quad \forall \varphi \in \mathcal{D} \ .} \tag{24}$$

Let L be the following linear differential operator of order n:

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 = \sum_{i=0}^{n} a_i D^i \ ,$$

where the a_i are smooth functions with $a_n \not\equiv 0$ and $D = d/dt$.

It follows from Proposition 5.13 that L acts on a distribution and yields another distribution. Indeed, if $T \in \mathcal{D}'$, then

$$\boxed{\langle L\, T, \varphi \rangle = \langle T, L^* \varphi \rangle \qquad \forall \varphi \in \mathcal{D} \ ,}$$

where L^*, the **formal adjoint** of L, is defined by

$$L^* \varphi := \sum_{i=0}^{n} (-1)^i \left(a_i \, \varphi\right)^{(i)} \ .$$

A typical linear inhomogeneous ODE can be written in the form

$$L\, x(t) = f(t) \ ,$$

where L is a linear differential operator and f is the inhomogeneous term. By analogy, we define a (linear) **distributional ODE** to be an equation of the form:

$$L\, T = \tau \ , \tag{25}$$

where τ is a fixed distribution and the equation is understood as an equation in $\mathcal{D}'$ whose solution is a distribution T. In other words, a distribution T solves the above distributional ODE if

$$\langle L\, T, \varphi \rangle = \langle T, L^* \varphi \rangle = \langle \tau, \varphi \rangle \qquad \forall \varphi \in \mathcal{D} \ .$$

The distributional ODE (25) can have two different types of solutions:

- **Classical solutions**. These are regular distributions $T = T_x$, where in addition x is sufficiently differentiable so that $L\,x$ makes sense as a function. In this case, $\tau = T_f$ has to be a regular distribution corresponding to a continuous function f.
- **Weak solutions**. These are either regular distributions $T = T_x$, where x is not sufficiently differentiable for $L\,x$ to make sense as a function; or simply singular distributions.

Suppose that the differential operator L is in standard form, so that $a_n \equiv 1$. Then it is possible to show that if the inhomogeneous term in equation (25) is the regular distribution $\tau = T_f$ corresponding to a continuous function f, then all solutions are regular, with $T = T_x$ for $x(t)$ a sufficiently differentiable function obeying $L\,x = f$ as functions.

However, a simple first order equation like

$$t^2 T' = 0 \ ,$$

which as functions would only have as solution a constant function, has a three-parameter family of distributional solutions:

$$T = c_1 + c_2 T_H + c_3 \delta \ ,$$

where c_i are constants and H is the Heaviside step function.

5.4. **Green's functions.** Solving a linear differential equation is not unlike inverting a matrix, albeit an infinite-dimensional one. Indeed, a linear differential operator L is simply a linear transformation in some infinite-dimensional vector space: the vector space of distributions in the case of an equation of the form (25). If this equation were a linear equation in a *finite-dimensional* vector space, the solution would be obtained by inverting the operator L, now realised as a matrix. In this section we take the first steps towards making this analogy precise. We will introduce the analogue of the inverse for L (the "Green's function of L") and the analogue of matrix multiplication ("convolution").

Let L be an n-th order linear differential operator in standard form:

$$L = D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \ ,$$

where a_i are smooth functions.

Definition 5.15. By a **fundamental solution for** L we mean a distribution T satisfying

$$L\,T = \delta \ .$$

Fundamental solutions are not unique, since one can add to T anything in the kernel of L. For example, we saw in (22) that (the regular distribution defined by) the Heaviside step function is a fundamental solution of $L = D$. In this case it is a weak solution. This is not the only fundamental solution, since we can always add a constant distribution:

$$\langle T_c, \varphi \rangle = c \int_{\mathbb{R}} \varphi(t)\, dt \qquad \forall\, \varphi \in \mathcal{D} \ ,$$

for some $c \in \mathbb{R}$.

One way to resolve this ambiguity is to impose boundary conditions. An important class of boundary conditions is the following.

Definition 5.16. A **(causal) Green's function** for the operator L is a fundamental solution G which in addition obeys

$$\langle G, \varphi \rangle = 0 \quad \forall\, \varphi \in \mathcal{D} \text{ such that } \operatorname{supp} \varphi \subset (-\infty, 0) \ .$$

In other words, the Green's function G is zero on any test function $\varphi(t)$ vanishing for non-negative values of t. With a slight abuse of notation, and thinking of G as a function, we can say that $G(t) = 0$ for $t < 0$.

As an example, consider the Green's function for the differential operator $L = D^k$, which is given by

$$G(t) = \frac{t^{k-1}}{(k-1)!} H(t) \ .$$

Let us consider the Green's function for the linear operator L above, where the a_i are real constants. By definition the Green's function $G(t)$ obeys $L\,G = \delta$ and $G(t) = 0$ for $t < 0$. This last condition suggests that we try $G(t) = x(t)H(t)$, where $x(t)$ is a function to be determined and H is the Heaviside step function. Computing $L\,G$ we find, after quite a little bit of algebra,

$$L\,G = (L\,x)\,H + \sum_{\ell=0}^{n-1} \left[\sum_{k=0}^{n-\ell-1} \binom{k+\ell}{\ell} a_{k+\ell+1} x^{(k)}(0) \right] \delta^{(\ell)} \ ,$$

where we have used equation (23). The distributional equation $L\,G = \delta$ is therefore equivalent to the following initial value problem for the function x:

$$L\,x = 0 \qquad x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0 \qquad x^{(n-1)}(0) = 1 \ .$$

We know from our treatment of linear vector fields that this initial value problem has a unique solution. Therefore the Green's function for L exists and is unique.

The Green's function is the analogue of an inverse of the differential operator. In fact, it is only a "right-inverse": it is a common feature of infinite-dimensional vector spaces that linear transformations may have left- or right-inverses but not both. This statement is lent further credibility by the fact there is a product relative to which the Dirac δ is the identity. This is the analogue of matrix multiplication: the convolution product. Convolutions are treated in more detail in Problems

5.10 and 5.13. Here we simply show how to solve an inhomogeneous equation, given the Green's function.

Theorem 5.17. *Let G be the Green's function for the operator L. Let T be the distribution defined by*

$$T(t) := \int_{\mathbb{R}} G(t-s)f(s)ds \ .$$

Then T obeys the inhomogeneous equation

$$LT(t) = f(t) \ .$$

Proof. This follows from the fact that G is a fundamental solution:

$$LG(t) = \delta(t) \ ,$$

whence taking L inside the integral

$$LT(t) = \int_{\mathbb{R}} LG(t-s)f(s)ds = \int_{\mathbb{R}} \delta(t-s)f(s)ds = f(t) \ .$$

□

The expression of T is an example of a *convolution* product. More details are given in Problem 5.10, which describes the convolution product of test functions. It is possible to extend the convolution product to more general functions and indeed to certain types of distributions. In the notation of Problem 5.10, $T = G \star f$.

Notice that because $G(t) = 0$ for $t < 0$, the integral in the expression for T is only until t:

$$(G \star f)(t) = \int_{-\infty}^{t} G(t-s)\, f(s)\, ds \ .$$

This definition embodies the principle of *causality*. If the above equation describes the response of a physical system to an external input $f(t)$, then one expects that the response of the system at any given time should *not* depend on the future behaviour of the input.

Problems

Problem 5.1. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function of unit area; that is,

$$\int_{\mathbb{R}} |f(t)|dt < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} f(t)dt = 1 \ .$$

Consider the sequence of functions f_n defined by $f_n(t) = nf(nt)$, where n is a positive integer. Show that $T_{f_n} \to \delta$ in the distributional sense; that is,

$$\langle T_{f_n}, \varphi\rangle \to \varphi(0) \quad \forall \varphi \in \mathcal{D} \ .$$

(*Hint*: First show that

$$\langle T_{f_n}, \varphi \rangle - \varphi(0) = \int_{\mathbb{R}} f(t)\,[\varphi(t/n) - \varphi(0)]\,dt\ ,$$

estimate the integral and show that it goes to zero for large n.)

Problem 5.2. Let f be a continuous function whose derivative f' is also continuous (i.e., f is of class C^1). Let T_f denote the corresponding regular distribution. Prove that $T_f' = T_{f'}$.

Problem 5.3. Let $a \in \mathbb{R}$ and define the shifted step function $H_a(t)$ by

$$H_a(t) = \begin{cases} 1\ , & t \geq a \\ 0\ , & t < a\ , \end{cases}$$

and let T_{H_a} be the corresponding regular distribution. Prove that $T_{H_a}' = \delta_a$, where δ_a is defined by

$$\langle \delta_a, \varphi \rangle = \varphi(a) \quad \forall\, \varphi \in \mathcal{D}\ .$$

Problem 5.4. Let f be a smooth function and T be a distribution. Then $f\,T$ is a distribution, as was shown in lecture. Prove that

$$(f\,T)' = f'\,T + f\,T'\ .$$

Problem 5.5. Let f be a smooth function. Prove that

$$f\delta^{(n)} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f^{(j)}(0)\delta^{(n-j)}\ .$$

As a corollary, prove that

$$t^m\delta^{(n)} = \begin{cases} 0\ , & n < m \\ (-1)^m m!\delta\ , & n = m \\ (-1)^m \frac{n!}{(n-m)!}\delta^{(n-m)}\ , & n > m\ ; \end{cases}$$

and that

$$e^{-\lambda t}\delta^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}\delta^{(j)}\ .$$

Problem 5.6. Prove that (the regular distribution corresponding to) the function

$$\frac{t^{k-1}}{(k-1)!}H(t)$$

is a fundamental solution for the operator D^k.

Problem 5.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the following piecewise continuous function:

$$f(t) = \begin{cases} t\ , & |t| < 1 \\ 1\ , & t \geq 1 \\ -1\ , & t \leq -1\ . \end{cases}$$

Compute its second distributional derivative.

Problem 5.8. Prove that

$$|t|' = H(t) - H(-t) \ ,$$

and that

$$\left(D^2 - k^2\right) e^{-k|t|} = -2k\delta \ .$$

Use this result to find the causal Green's function for the linear operator $L = D^2 - k^2$ and find a solution $x(t)$ of the inhomogeneous equation $Lx(t) = f(t)$, where

$$f(t) = \begin{cases} t & \text{for } 0 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 5.9. Find the Green's function for the linear second order differential operator

$$L = D^2 + aD + b \ ,$$

where $a, b \in \mathbb{R}$. Distinguish between the cases $a^2 < 4b$, $a^2 = 4b$ and $a^2 > 4b$ and write the Green's function explicitly in this case. Use this to solve the inhomogeneous initial value problem

$$x''(t) + x'(t) + x(t) = f(t) \qquad x(t), x'(t) \to 0 \text{ as } t \to -\infty \ ,$$

where $f(t)$ is the piecewise continuous function $f(t) = H(t) - H(t-1)$. Sketch (or ask Maple to sketch) $x(t)$ as a function of t. From the sketch or otherwise, is $x(t)$ smooth?

Problem 5.10. Let φ, ψ and χ be test functions. Define their *convolution* $\varphi \star \psi$ by

$$(\varphi \star \psi)(t) := \int_{\mathbb{R}} \varphi(t-s)\, \psi(s)\, ds \ .$$

(a) Show that if φ has support $[a, b]$ and ψ has support $[c, d]$, then $\varphi \star \psi$ has support $[a + c, b + d]$.
(b) Show that $(\varphi \star \psi)' = \varphi' \star \psi = \varphi \star \psi'$.
(c) Conclude that $\varphi \star \psi$ is a test function.
(d) Show that the convolution product is commutative: $\varphi \star \psi = \psi \star \varphi$, and associative

$$(\varphi \star \psi) \star \chi = \varphi \star (\psi \star \chi) \ .$$

The following problems get a little deeper into the notion of a distribution. They are not examinable, but some of you might find them interesting.

Problem 5.11. Let $\Phi : \mathcal{D} \to \mathcal{D}$ be a continuous linear map; that is,

1. $\Phi(c_1\varphi_1 + c_2\varphi_2) = c_1\Phi(\varphi_1) + c_2\Phi(\varphi_2)$ for $c_i \in \mathbb{R}$ and $\varphi_i \in \mathcal{D}$; and
2. if $\varphi_m \to 0$ then $\Phi(\varphi_m) \to 0$ in $\mathcal{D}$.

(a) Prove that the "adjoint" map Φ^* defined on linear functionals by

$$\langle \Phi^* T, \varphi \rangle = \langle T, \Phi(\varphi) \rangle \quad \forall\, \varphi \in \mathcal{D} \ , \tag{26}$$

maps $\mathcal{D}'$ to $\mathcal{D}'$.

Let $a, b \in \mathbb{R}$ with $a \neq 0$ and define the following operations on functions:

$$\left(\Delta_a \varphi\right)(t) = \varphi(ax) \quad \text{and} \quad \left(\Theta_b \varphi\right)(t) = \varphi(t - b) \ .$$

(b) Prove that Δ_a and Θ_b map test functions to test functions, and that they are linear and continuous.

Let $\Delta_a^* : \mathcal{D}' \to \mathcal{D}'$ and $\Theta_b^* : \mathcal{D}' \to \mathcal{D}'$ be their adjoints, defined by (26).

(c) If f is a locally integrable function and T_f the corresponding regular distribution, show that

$$\Delta_a^* T_f = T_{\Delta_a^* f} \quad \text{and} \quad \Theta_b^* T_f = T_{\Theta_b^* f} \ ,$$

where

$$\left(\Delta_a^* f\right)(t) = \frac{1}{|a|} f(t/a) \quad \text{and} \quad \left(\Theta_b^* f\right)(t) = f(t + b) \ .$$

Problem 5.12. Let $T \in \mathcal{D}'$. Prove that its derivative obeys

$$T' = \lim_{h \to 0} \frac{1}{h} \left[\Theta_h^* T - T\right] \ ,$$

where Θ_h^* is defined in Problem 3 and where the limit is taken in the distributional sense:

$$\left\langle \lim_{h \to 0} \frac{1}{h} \left[\Theta_h^* T - T\right], \varphi \right\rangle = \lim_{h \to 0} \frac{1}{h} \left\langle \Theta_h^* T - T, \varphi \right\rangle \quad \forall\, \varphi \in \mathcal{D} \ .$$

Problem 5.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a bump function of unit area (see Problem 5.1). Prove (without using Problem 5.1, since we have not defined the convolution of a distribution and a test function) that the sequence of test functions $\varphi_n := f_n \star \varphi$ converges to φ as test functions:

$$\varphi_n \to \varphi \ .$$

(*Hint*: You can use the results of Problem 5.10 about convolutions.) (*Remark*: Together with Problem 5.1, which illustrates that one can approximate distributions by functions, this problem illustrates that one can define the convolution of a distribution and a test function, and that δ is the identity under convolution.

6. The Laplace transform

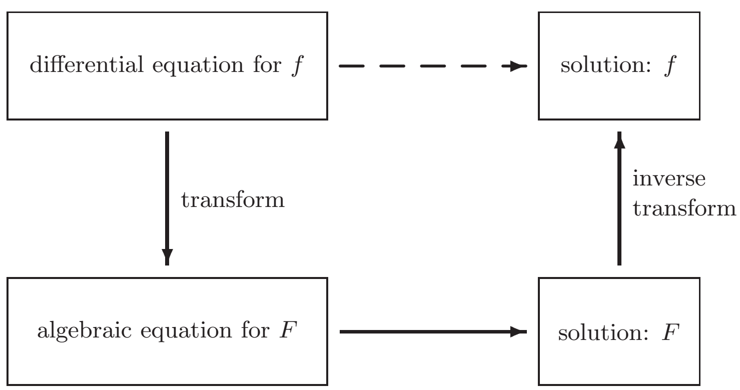
This part of the course introduces a powerful method of solving differential equations which is conceptually different from the methods discussed until now: the Laplace transform.

The Laplace transform is an example of an integral transform. Integral transforms are linear maps sending a function $f(t)$ to a function $F(s)$ defined by a formula of the form

$$F(s) = \int K(s,t)\, f(t)\, dt\ ,$$

for some function $K(s,t)$ called the **kernel** and where the integral is over some specified subset of the real line.

Integral transforms are useful in the solution of linear ordinary differential equations because these get transformed into algebraic equations which are easier to solve. One then inverts the transform to recover the solution to the differential equation. This circle (or square!) of ideas can be represented diagrammatically as follows:



The two most important integral transforms are the Fourier transform (cf. PDE) and the Laplace transform. Whereas the Fourier transform is useful in boundary value problems, the Laplace transform is useful in solving initial value problems. As this is the main topic of this course, we will concentrate solely on the Laplace transform.

6.1. Definition and basic properties.

Definition 6.1. Let $f(t)$ be a function. Its **Laplace transform** is defined by

$$\boxed{\mathcal{L}\left\{f\right\}(s) := \int_0^\infty f(t)\, e^{-st}\, dt\ ,} \tag{27}$$

provided that the integral exists. One often uses the shorthand $F(s)$ for the Laplace transform $\mathcal{L}\left\{f\right\}(s)$ of $f(t)$.

Remark 6.2. The following should be kept in mind:

1. Not every function has a Laplace transform: for example, $f(t) = e^{t^2}$ does not, since the integral in equation (27) does not exist for any value of s.
2. For some $f(t)$ the Laplace transform does not exist for all values of s. For example, the Laplace transform $F(s)$ of $f(t) = e^{ct}$ only exists for $s > c$.
3. Although t is a real variable, the logical consistency of the theory requires s to be a complex variable. Therefore the Laplace transform $\mathcal{L}\{e^{ct}\}(s)$ actually exists provided that $\mathrm{Re}(s) > c$.

A large class of functions which have a Laplace transform (at least for some values of s) are those of exponential order.

Definition 6.3. A function $f(t)$ is said to be of *exponential order* if there exist real constants M and α such that

$$|f(t)| \leq M\, e^{\alpha t} \quad \forall\, t\ .$$

Proposition 6.4. *If $f(t)$ is of exponential order, then the Laplace transform $F(s)$ of $f(t)$ exists provided that* $\mathrm{Re}(s) > \alpha$.

Proof. We estimate the integral

$$\left| \int_0^\infty f(t)\, e^{-st}\, dt \right| \leq \int_0^\infty |f(t)|\, |e^{-st}|\, dt \leq \int_0^\infty M\, e^{\alpha t}\, e^{-\mathrm{Re}(s) t}\, dt\ .$$

Provided that $\mathrm{Re}(s) > \alpha$, this integral exists and

$$|F(s)| \leq \frac{M}{\mathrm{Re}(s) - \alpha}\ .$$

□

Notice that in the above proof, in the limit $\mathrm{Re}(s) \to \infty$, $F(s) \to 0$. In fact, this can be proven in more generality: so that if a function $F(s)$ does not approach 0 in the limit $\mathrm{Re}(s) \to \infty$, it cannot be the Laplace transform of any function $f(t)$; although as we will see it can be the Laplace transform of a singular distribution!

A fundamental property of integral transforms is linearity:

$$\mathcal{L}\{f + g\}(s) = \mathcal{L}\{f\}(s) + \mathcal{L}\{g\}(s) = F(s) + G(s)\ ,$$

which makes sense for all s for which *both* $F(s)$ and $G(s)$ exist.

Let $f(t)$ be differentiable and let us compute the Laplace transform of its derivative $f'(t)$. By definition,

$$\mathcal{L}\{f'\}(s) = \int_0^\infty f'(t)\, e^{-st}\, dt\ ,$$

which can be integrated by parts to obtain

$$\begin{aligned} \mathcal{L}\{f'\}(s) = \int_0^\infty s\, f(t)\, e^{-st}\, dt + f(t)\, e^{-st}\Big|_0^\infty \\ = s\,\mathcal{L}\{f\}(s) - f(0) + \lim_{t\to\infty} f(t)\, e^{-st}\ . \end{aligned}$$

Provided the last term is zero,

$$\boxed{\mathcal{L}\left\{f'\right\}(s) = s\,\mathcal{L}\left\{f\right\}(s) - f(0)\ .}$$

We can iterate this expression in order to find the Laplace transform of higher derivatives of $f(t)$:

$$\begin{aligned}\mathcal{L}\left\{f''\right\}(s) &= s\,\mathcal{L}\left\{f'\right\}(s) - f'(0) \\ &= s^2\,\mathcal{L}\left\{f\right\}(s) - s\,f(0) - f'(0)\ ,\end{aligned}$$

provided that $f(t)\exp(-st)$ and $f'(t)\exp(-st)$ both go to zero in the limit $t \to \infty$.

The entries in Table 1 below can be obtained by straightforward integration. The last two lines will deserve more attention below.

Function	Transform	Conditions
$f(t)$	$F(s)$	convergence
e^{at}	$\frac{1}{s-a}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\omega \in \mathbb{R}$ and $\mathrm{Re}(s) > 0$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\omega \in \mathbb{R}$ and $\mathrm{Re}(s) > 0$
$\cosh\beta t$	$\frac{s}{s^2-\beta^2}$	$\beta \in \mathbb{R}$ and $\mathrm{Re}(s) > \lvert\beta\rvert$
$\sinh\beta t$	$\frac{\beta}{s^2-\beta^2}$	$\beta \in \mathbb{R}$ and $\mathrm{Re}(s) > \lvert\beta\rvert$
t^n	$\frac{n!}{s^{n+1}}$	$n = 0, 1, \ldots$ and $\mathrm{Re}(s) > 0$
$e^{at}\,f(t)$	$F(s-a)$	convergence
$t^n\,f(t)$	$(-1)^n\,F^{(n)}(s)$	same as for $F(s)$
$f_\tau(t)$	$e^{-s\tau}\,F(s)$	$\tau > 0$ and same as for $F(s)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k}\,f^{(k)}(0)$	$\lim_{t\to\infty} f^{(k)}(t)e^{-st} = 0$
$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$	same as for $F(s)$ *and* $G(s)$

Table 1. Some Laplace transforms

In this table, f_τ is the function defined by

$$f_\tau(t) := \begin{cases} f(t-\tau) & \text{for } t \geq \tau, \\ 0 & \text{otherwise.}\end{cases}$$

Definition 6.5. The **convolution** of two functions $f(t)$ and $g(t)$ is defined as

$$(f \star g)(t) = \int_0^t f(t-u)\, g(u)\, du \; .$$

Proposition 6.6. *The convolution obeys the following properties:*

- Commutativity*:* $f \star g = g \star f$*.*
- Bilinearity*:* $f \star (a\, g + b\, h) = a\, f \star g + b\, f \star h$*, where* a *and* b *are constants. Commutativity implies linearity on the first factor too.*
- Associativity*:* $f \star (g \star h) = (f \star g) \star h$*.*

Remark 6.7. The above definition of convolution agrees with that introduced in Problem 5.10 provided that the functions involved are such that they vanish for negative values of the independent variable. Indeed, if $f(t)$ and $g(t)$ are functions for which $f(t < 0) = 0$ and $g(t < 0) = 0$, then

$$\int_{-\infty}^{\infty} f(t-u)\, g(u)\, du = \int_0^t f(t-u)\, g(u)\, du \; .$$

The most important property is probably the following.

Theorem 6.8. *Let* $F(s)$ *and* $G(s)$ *be the Laplace transforms of* $f(t)$ *and* $g(t)$*, respectively. Then*

$$\boxed{\mathcal{L}\left\{f \star g\right\}(s) = F(s)\, G(s) \; .}$$

Proof. We simply compute:

$$\begin{aligned} \mathcal{L}\left\{f \star g\right\}(s) &= \int_0^\infty e^{-st}\, (f \star g)(t)\, dt \\ &= \int_0^\infty e^{-st} \left(\int_0^t f(t-u)\, g(u)\, du \right) dt \; . \end{aligned}$$

We can exchange the order of integration by noticing that

$$\int_{t=0}^{t=\infty} \int_{u=0}^{u=t} k(t,u)\, du\, dt = \int_{u=0}^{u=\infty} \int_{t=u}^{t=\infty} k(t,u)\, du\, dt \; ,$$

for any function $k(t,u)$ for which the integrals exist. Therefore,

$$\begin{aligned} \mathcal{L}\left\{f \star g\right\}(s) &= \int_0^\infty \left(\int_u^\infty e^{-st} f(t-u)\, dt \right) g(u)\, du \\ &= \int_0^\infty \left(\int_0^\infty e^{-s(u+v)} f(v)\, dv \right) g(u)\, du \qquad (v = t-u) \\ &= \mathcal{L}\left\{f\right\}(s) \int_0^\infty e^{-su} g(u)\, du \\ &= \mathcal{L}\left\{f\right\}(s)\, \mathcal{L}\left\{g\right\}(s) \; . \end{aligned}$$

∎

6.2. **Application: solving linear ODEs.** As explained in the introduction to this section, the usefulness of the Laplace transform is based on its ability to turn initial value problems into algebraic equations. We now discuss this method in more detail.

Let L be a linear differential operator in standard form:

$$L = D^n + a_{n-1}D^{n-1} + \dots a_1 D + a_0 = \sum_{i=0}^{n} a_i D^i \ ,$$

where a_i are real constants ($a_n = 1$) and where D is the derivative operator: $(Df)(t) = f'(t)$. We will consider the initial value problem

$$Lx = f \quad \text{and} \quad x(0) = c_0,\ x'(0) = c_1,\ \dots,\ x^{(n-1)}(0) = c_{n-1} \ .$$

We can solve this using the Laplace transform in three easy steps:

1. We take the Laplace transform of the equation:
$$\mathfrak{L}\{Lx\}(s) = \mathfrak{L}\{f\}(s) = F(s) \ .$$
Using linearity and the expression for the Laplace transform for $x^{(i)}$ in Table 1 one finds
$$\mathfrak{L}\{Lx\}(s) = \left(\sum_{i=0}^{n} a_i s^i\right) X(s) - P(s) \ ,$$
where $X(s) = \mathfrak{L}\{x\}(s)$ and $P(s)$ is the polynomial
$$P(s) = \sum_{i=0}^{n-1} p_i s^i \quad \text{where} \quad p_i = \sum_{j=0}^{n-i-1} a_{i+j+1} c_j \ , \tag{28}$$
which encodes all the information about the initial conditions.
2. We solve the transformed equation for $X(s)$:
$$\left(\sum_{i=0}^{n} a_i s^i\right) X(s) = F(s) + P(s) \ ,$$
to obtain
$$X(s) = T(s)\left(F(s) + P(s)\right) \ ,$$
where
$$T(s) = \left(\sum_{i=0}^{n} a_i s^i\right)^{-1} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \ .$$
3. We invert the transform to recover $x(t)$.

If the Laplace transform $F(s)$ of $f(t)$ is a rational function (i.e., a ratio of polynomials), then so is $X(s)$. The inverse transform is easy to find in principle by a partial fraction decomposition and then looking up in the tables. The resulting function $x(t)$ will then be a sum of terms each of which is made out of polynomials, exponential and trigonometric functions.

The transformed solution found above is a sum of two terms: the term $T(s)P(s)$ depends on the initial conditions, whereas the term $T(s)F(s)$ depends on the inhomogeneous term in the ODE: $f(t)$. In both terms, the function $T(s)$ only depends on L and not on the initial conditions or on the inhomogeneous term in the equation. This function is known as the **transfer function** of L. As we will show it is the Laplace transform of a function $g(t)$. This allows us to invert the term $T(s)F(s)$ using convolution.[1]

We desire a function $g(t)$ whose Laplace transform is the transfer function of the differential operator L. Let $g(t)$ solve the homogeneous equation $Lg = 0$. Solving for its Laplace transform, we find, as above,

$$\mathcal{L}\{g\}(s) = T(s)P(s) ,$$

where $P(s)$ is the polynomial defined in (28), but with $c_j = g^{(j)}(0)$. Clearly $\mathcal{L}\{g\}(s) = T(s)$ if and only if $P(s) = 1$, which is equivalent to the initial conditions $g(0) = g'(0) = \cdots = g^{(n-2)}(0) = 0$ and $g^{(n-1)}(0) = 1$. In other words, $g(t)$ is essentially the Green's function for the operator L. More precisely, the Green's function is given by

$$G(t) = H(t)g(t) ,$$

since it must vanish for negative t. Since the Laplace transform is only defined for non-negative t, $g(t)$ and $G(t)$ share the same Laplace transform, namely the transfer function $T(s)$.

6.3. **The Laplace transform of a distribution.** What is the inverse Laplace transform of a polynomial $P(s)$? We remarked above that for $F(s)$ to be the Laplace transform of a function $f(t)$ it is necessary that $F(s) \to 0$ as $\mathrm{Re}(s) \to \infty$. Since this is not the case for $P(s)$, we know that $P(s) \neq \mathcal{L}\{p(t)\}(s)$ for any *function* $p(t)$. It will turn out, however, that $P(s)$ is the Laplace transform of a singular distribution. Just like not every function has a Laplace transform, not every distribution will have one either. This requires a little bit of formalism, which we will simply sketch.

Suppose that T_f is a regular distribution. If f has a Laplace transform, it is natural to demand that the Laplace transform of T_f should agree with that of f:

$$\mathcal{L}\{T_f\}(s) := \mathcal{L}\{f\}(s) = \int_0^\infty f(t)\, e^{-st}\, dt ,$$

which looks like $\langle T_f, e^{-st}\rangle$ except for the minor detail of the lower limit of integration and the fact that e^{-st} is not a test function, since it does not have compact support. However, for $\mathrm{Re}(s) > 0$, the function e^{-st}

[1]The same cannot be said about the term $T(s)P(s)$, since $P(s)$, being a polynomial, cannot be the Laplace of any function; although it can be shown to be the Laplace transform of a singular distribution.

obeys the next best thing: it decays very fast. The following definition makes this notion precise.

Definition 6.9. We say that a function f is of *fast decay* if for all non-negative integers k, p there exists some positive real number $M_{k,p}$ such that

$$(1+t^2)^p \, |f^{(k)}(t)| \leq M_{k,p} \qquad \forall t \in \mathbb{R} \; .$$

The space of such functions is denoted by $\mathcal{S}$.

Proposition 6.10. *The space $\mathcal{S}$ is a vector space and contains the space $\mathcal{D}$ of test functions as a subspace.*

Proof. That $\mathcal{S}$ is a vector space is left as an exercise (see Problem 6.9). Let us simply see that $\mathcal{D} \subset \mathcal{S}$. If $\varphi \in \mathcal{D}$ is a test function, there is some $R > 0$ such that $\varphi^{(k)}(t) = 0$ for $|t| > R$. Since φ is smooth, it is bounded and so are all its derivatives. Let M_k be the maximum of $|\varphi^{(k)}(t)|$ for $|t| \leq R$. Then,

$$(1+t^2)^p \, |f^{(k)}(t)| \leq (1+R^2)^p M_k \qquad \forall t \in \mathbb{R} \; .$$

□

There is a natural notion of convergence in the space $\mathcal{S}$.

Definition 6.11. We say that a sequence $f_n \in \mathcal{S}$ of functions in $\mathcal{S}$ **converges to zero** (written $f_n \to 0$), if for any given p and k,

$$(1+t^2)^p \, |f_n^{(k)}(t)| \to 0 \qquad \text{uniformly in } t.$$

This notion of convergence agrees with the one for test functions. In other words, a sequence of test functions converging to zero in $\mathcal{D}$ also converge to zero in $\mathcal{S}$. (See Problem 6.9.)

Recall that in Section 5.2 we defined a distribution to be a continuous linear functional on $\mathcal{D}$. Some distributions will extend to linear functionals on $\mathcal{S}$.

Definition 6.12. A **tempered distribution** is a continuous linear functional on $\mathcal{S}$. The space of tempered distributions is denoted S'.

This means that $T \in \mathcal{S}'$ associates a number $\langle T, f\rangle$ with every $f \in \mathcal{S}$, in such a way that if $f, g \in \mathcal{S}$ and a, b are constants, then:

$$\langle T, a\, f + b\, g\rangle = a\, \langle T, f\rangle + b\, \langle T, g\rangle \; .$$

Continuity means that if $f_n \to 0$ then the numbers $\langle T, f_n\rangle \to 0$ as well.

Proposition 6.13. *The space of tempered distributions is a vector space. In fact, it is a vector subspace of $\mathcal{D}'$: $\mathcal{S}' \subset \mathcal{D}'$.*

Tempered distributions inherit the notion of weak convergence of distributions in Definition 5.14.

Not all distributions have a Laplace transform. Let T be a distribution with the property that "$T(t) = 0$ for $t < 0$". In other words,

$T \in \mathcal{D}'$ is such that for all test functions φ with $\operatorname{supp} \varphi \subset (-\infty, 0)$, one has $\langle T, \varphi \rangle = 0$. (Compare with the definition of Green's function in §5.4.) Assume moreover that for some $\alpha \in \mathbb{R}$, $e^{-\alpha t} T$ is tempered, that is, it belongs to $\mathcal{S}'$. Then one defines the (distributional) **Laplace transform** of T by

$$\mathcal{L}\{T\}(s) := \left\langle T, e^{-st} \right\rangle ,$$

which then exist for $\operatorname{Re}(s) > \alpha$.

Let us compute the Laplace transforms of a few singular distributions:

First, we start with $T = \delta$:

$$\mathcal{L}\{\delta\}(s) = \left\langle \delta, e^{-st} \right\rangle = e^{-st}\Big|_{t=0} = 1 .$$

Notice that this is consistent with the fact that δ is the identity under convolution (cf. Problem 5.13). Indeed, let f be a function with Laplace transform $F(s)$. Then,

$$\mathcal{L}\{\delta \star f\}(s) = \mathcal{L}\{\delta\}(s) F(s) = F(s) = \mathcal{L}\{f\}(s) .$$

We generalise to $T = \delta^{(k)}$:

$$\mathcal{L}\left\{\delta^{(k)}\right\}(s) = \left\langle \delta^{(k)}, e^{-st} \right\rangle = (-1)^k \left\langle \delta, \frac{d^k}{dt^k} e^{-st} \right\rangle = s^k e^{-st}\Big|_{t=0} = s^k .$$

These two results allows us to invert the Laplace transform of any polynomial.

Finally consider δ_τ for $\tau > 0$, defined by $\langle \delta_\tau, \varphi \rangle = \varphi(\tau)$. Its Laplace transform is given by

$$\mathcal{L}\{\delta_\tau\}(s) = \left\langle \delta_\tau, e^{-st} \right\rangle = e^{-st}\Big|_{t=\tau} = e^{-s\tau} .$$

Problems

Problem 6.1. Provide all the missing proofs of the results in Table 1.

Problem 6.2. Suppose that $f(t)$ is a periodic function with period T; that is, $f(t+T) = f(t)$ for all t. Prove that the Laplace transform $F(s)$ of $f(t)$ is given by

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt ,$$

which converges for $\operatorname{Re}(s) > 0$.
Use this to compute the Laplace transform of the function $f(t)$ which is 1 for t between 0 and 1, 2 and 3, 4 and 5, etcetera and 0 otherwise.

Problem 6.3. Solve the following integral equation:

$$f(t) = 1 - \int_0^t (t-\tau)\, f(\tau)\, d\tau \ .$$

(*Hint*: Take the Laplace transform of the equation, solve for the Laplace transform $F(s)$ of $f(t)$, and finally invert the transform.)

Problem 6.4. Show that the differential equation:

$$f''(t) + \omega^2 f(t) = u(t),$$

subject to the initial conditions $f(0) = f'(0) = 0$, has

$$f(t) = \frac{1}{\omega} \int_0^t u(\tau)\, \sin \omega(t-\tau)\, d\tau$$

as its solution.
(*Hint*: Take the Laplace transform of both sides of the equation, solve for the Laplace transform of f and invert.)

Problem 6.5. Suppose that we consider the Laplace transform of t^z, where z is a complex number with $\mathrm{Re}(z) > 0$. This is given in terms of the **Euler Γ function**, defined by

$$\Gamma(z) := \int_0^\infty t^{z-1}\, e^{-t}\, dt \ .$$

Prove that

$$\mathcal{L}\left\{t^z\right\}(s) = \frac{\Gamma(z+1)}{s^{z+1}} \ ,$$

provided $\mathrm{Re}(z) > 0$. (This is required for convergence of the integral.) Prove that

$$\Gamma(z+1) = z\,\Gamma(z) \ ,$$

for $\mathrm{Re}(z) > 0$. Compute $\Gamma(\frac{1}{2})$.

Problem 6.6. Compute the Laplace transforms of the following functions:

(a) $f(t) = 3\cos 2t - 8e^{-2t}$

(b) $f(t) = \dfrac{1}{\sqrt{t}}$

(c) $f(t) = \begin{cases} 1\ , & \text{for } t < 1, \text{ and} \\ 0\ , & \text{for } t \geq 1. \end{cases}$

(d) $f(t) = (\sin t)^2$

(e) $f(t) = \begin{cases} 0\ , & \text{for } t < 1, \\ 1\ , & \text{for } 1 \leq t \leq 2, \text{ and} \\ 0\ , & \text{for } t > 2. \end{cases}$

Make sure to specify as part of your answer the values of s for which the Laplace transform is valid.

Problem 6.7. Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{1}{s^2+4}$ **(b)** $F(s) = \dfrac{4}{(s-1)^2}$

(c) $F(s) = \dfrac{s}{s^2+4s+4}$ **(d)** $F(s) = \dfrac{1}{s^3+3s^2+2s}$

(e) $F(s) = \dfrac{s+3}{s^2+4s+7}$

Problem 6.8. Use the Laplace transform to solve the following initial value problems:

(a) $$\frac{d^2f(t)}{dt^2} - 5\frac{df(t)}{dt} + 6f(t) = 0 \ , \qquad f(0) = 1, \ f'(0) = -1 \ ,$$

(b) $$\frac{d^2f(t)}{dt^2} - \frac{df(t)}{dt} - 2f(t) = e^{-t}\sin 2t \ , \qquad f(0) = f'(0) = 0 \ ,$$

(c) $$\frac{d^2f(t)}{dt^2} - 3\frac{df(t)}{dt} + 2f(t) = \begin{cases} 0 \ , & \text{for } 0 \leq t < 3, \\ 1 \ , & \text{for } 3 \leq t \leq 6, \text{ and} \\ 0 \ , & \text{for } t > 6, \end{cases}$$

$$f(0) = f'(0) = 0 \ .$$

Problem 6.9. Prove that the space $\mathcal{S}$ of functions of fast decay is a vector space. Moreover show that vector addition and scalar multiplication are continuous operations; that is, show that if $f_n, g_n \to 0$ are sequences of functions of fast decay converging to zero, then $f_n + g_n \to 0$ and $cf_n \to 0$ for all scalars c. (This makes $\mathcal{S}$ into a **topological vector space**.)

Prove as well that convergence in $\mathcal{D}$ and in $\mathcal{S}$ are compatible. (This makes $\mathcal{D}$ is **closed subspace** of $\mathcal{S}$.)

7. ODEs with analytic coefficients

In this short section we introduce the method of power series solutions to linear ordinary differential equations with analytic coefficients. We start with a brief discussion of analytic functions and their power series solutions and then present the method of power series to solve a linear ordinary differential equation with analytic coefficients.

7.1. Analytic functions.

Power series find their most natural expression in the context of complex analysis, and you are referred to the module CAn for details. Here we will simply recall some of their basic properties.

Definition 7.1. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(t-t_0)^n = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + \cdots \ ,$$

where $a_n \in \mathbb{R}$, defines a *power series* in the real variable t about the point $t_0 \in \mathbb{R}$.

By shifting the variable if necessary, we can always assume that $t_0 = 0$. This leads to a simpler form of the power series:

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots \ . \tag{29}$$

We say that the power series (29) **converges at the point** $t \in \mathbb{R}$ if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n t^n \quad \text{exists.}$$

Otherwise we say that the power series **diverges at** t.

Clearly every power series of the form (29) converges at $t = 0$. Happily many power series also converge at other points. In fact, power series come in three varieties:

1. those which only converge for $t = 0$;
2. those which converge for all $t \in \mathbb{R}$; and
3. those which converge for all $|t| < R$ and diverge for $|t| > R$.

The real number R in the last case is called the **radius of convergence** of the power series. It is customary to assign the value $R = 0$ and $R = \infty$ respectively to the first and second cases above. This way every power series has a radius of convergence which can be any number $0 \leq R \leq \infty$. The interval $(-R, R)$ is known as the **interval of convergence**.

Remark 7.2. Convergence is undecided at either endpoint of the interval of convergence!

We now discuss a method to find the radius of convergence of a power series. Consider the series $\sum_{n=0}^{\infty} u_n$ with constant coefficients. The Ratio Test tells us that if the limit

$$L := \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

exists, then the series converges if $L < 1$ and diverges if $L > 1$. Applying this to the power series (29), we see that if the limit

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}t^{n+1}}{a_n t^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |t|$$

exists, then the power series converges for $L < 1$, which is equivalent to saying that the radius of convergence is given by

$$R^{-1} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

Even if the Ratio Test fails, it is possible to show that the radius of convergence of a power series always exists.

If the power series (29) converges for all $|t| < R$, then for those values of t it defines a function f as follows:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n .$$

This function is automatically continuous and infinitely differentiable in the interval $|t| < R$. In fact, the series can be differentiated termwise, so that

$$a_n = \tfrac{1}{n!} f^{(n)}(0) .$$

A convergent power series can also be integrated termwise, provided the limits of integration are within the interval of convergence:

$$\int_{t_1}^{t_2} f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \left(t_2^{n+1} - t_1^{n+1} \right) \quad \text{if } -R < t_1 < t_2 < R.$$

Convergent power series can be added and multiplied, the resulting power series having as radius of convergence at least as large as the smallest of the radii of convergence of the original power series. The formula for the product deserves special attention:

$$\left(\sum_{n=0}^{\infty} a_n t^n \right) \left(\sum_{n=0}^{\infty} b_n t^n \right) = \sum_{n=0}^{\infty} c_n t^n ,$$

where

$$c_n = \sum_{\ell=0}^{n} a_\ell b_{n-\ell} = \sum_{\ell=0}^{n} a_{n-\ell} b_\ell .$$

These formulae define the **Cauchy product** of the power series.

Not every function can be approximated by a power series, those which can are called "analytic." More precisely, the function $f(t)$ is said to be **analytic at** t_0 if the expression

$$f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \tag{30}$$

is valid in some interval around t_0. If this is the case,

$$a_n = \frac{1}{n!} f^{(n)}(t_0) \ ,$$

and the series (30) is the **Taylor series** of $f(t)$ at t_0. In the special case $t_0 = 0$, the series (30) is the **Maclaurin series** of $f(t)$.

Analyticity in the real line is not nearly as dramatic as in the complex plane; but it is still an important concept. Clearly there are plenty of analytic functions: polynomials, exponentials, sine and cosine are analytic at all points.

Proposition 7.3. *Analytic functions obey the following properties:*

1. *Let $f(t)$ and $g(t)$ be analytic at t_0. Then so are*

   $$f(t)g(t) \qquad af(t) + bg(t) \qquad f(t)/g(t)$$

   where $a, b \in \mathbb{R}$ and, in the last case, provided that $g(t_0) \neq 0$.
2. *Let $f(t)$ be analytic at t_0, and let f^{-1} be a continuous inverse to f. Then* if *$f'(t_0) \neq 0$, $f^{-1}(t)$ is analytic at $f(t_0)$.*
3. *Let $g(t)$ be analytic at t_0 and let $f(t)$ be analytic at $g(t_0)$. Then $f(g(t))$ is analytic at t_0.*

7.2. **ODEs with analytic coefficients.** Consider a linear differential operator in standard form:

$$L = D^n + a_{n-1}(t) D^{n-1} + \cdots a_1(t) D + a_0(t) \ .$$

If the $a_i(t)$ are analytic at t_0, we say that t_0 is an **ordinary point** for the differential operator L. There's nothing ordinary about ordinary points. In fact one has the following important result.

Theorem 7.4. *Let t_0 be an ordinary point for the differential operator L, and let c_i for $i = 0, 1, \ldots, n-1$ be arbitrary constants. Then there exists a unique function $x(t)$ analytic at t_0 which solves the initial value problem:*

$$Lx = 0 \qquad \text{subject to} \quad x^{(i)}(t_0) = c_i$$

in a certain neighbourhood of t_0. Furthermore if the $a_i(t)$ have Taylor series valid in $|t - t_0| < R$, $R > 0$, then so is the Taylor series for $x(t)$.

Proof. (Sketch) Without loss of generality, and translating the independent variable if necessary, we will assume that $t_0 = 0$. Also to simplify the discussion we will only treat the case of a second order equation

$$x'' + p(t) x' + q(t) = 0 \ . \tag{31}$$

Since analytic functions have power series approximations at the points of analyticity, we have that

$$p(t) = \sum_{n=0}^{\infty} p_n t^n \qquad \text{and} \qquad q(t) = \sum_{n=0}^{\infty} q_n t^n \ .$$

Let R denote the smallest of the radii of convergence of these two series. If an analytic solution exists, it can be approximated by a power series

$$x(t) = \sum_{n=0}^{\infty} c_n t^n \ ,$$

which converges in some interval around $t = 0$ and hence can be differentiated termwise. Doing so, we find

$$x'(t) = \sum_{n=0}^{\infty} (n+1) c_{n+1} t^n \quad \text{and} \quad x''(t) = \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} t^n \ .$$

Inserting these power series into the differential equation (31), and using the Cauchy product formulae, we can derive a **recurrence relation** for the coefficients c_n:

$$c_{n+2} = -\frac{1}{(n+1)(n+2)} \sum_{\ell=0}^{n} \left((\ell+1) c_{\ell+1} p_{n-\ell} + c_\ell q_{n-\ell} \right) \ ,$$

valid for $n \geq 0$, which allows us to solve for all the c_n $(n \geq 2)$ in terms of c_0 and c_1. Notice that $x(0) = c_0$ and $x'(0) = c_1$.

The main technical aspect of the proof (which we omit) is to prove that the resulting power series converges for $|t| < R$. This is done by majorising the series with a geometric series which converges for $|t| < R$ and using the comparison test. $\square$

There are many famous equations $Lx = 0$ with analytic solutions at $t = 0$. Among the best known are the following second order equations:

- (**Airy's equation**) $L = D^2 + t$;
- (**Hermite's equation**) $L = D^2 - 2tD + 2p$;
- (**Chebyshev's equation**) $L = (1-t^2)D^2 - tD + p^2$; and
- (**Legendre's equation**) $L = (1-t^2)D^2 - 2tD + p(p+1)$.

In the last two equations, we can divide through by $(1-t^2)$ to put the differential operators in standard form.

In the last three cases, p is a real constant which, when a non-negative integer, truncates one of the power series solutions to a polynomial. These polynomials are known, not surprisingly, as **Hermite polynomials**, **Chebyshev polynomials** and **Legendre polynomials**. On the other hand, the **Airy functions**, which solve Airy's equation, are not polynomials.

Problems

Problem 7.1. Let $f(t)$ and $g(t)$ be analytic at $t = 0$, with power series expansions

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad g(t) = \sum_{n=0}^{\infty} b_n t^n \ .$$

Supposing that $g(0) = b_0 \neq 0$, then their quotient $h(t) := f(t)/g(t)$ is analytic at $t = 0$, whence it will have a power series expansion

$$h(t) = \sum_{n=0}^{\infty} c_n t^n \ .$$

Write a recurrence relation for the coefficients c_n and solve for the first three coefficients c_0, c_1, c_2 in terms of the coefficients $\{a_n, b_n\}$.

Problem 7.2. Let $f(t)$ be analytic at $t = 0$, with power series expansion

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \ .$$

Let f^{-1} be the inverse function (assumed continuous), so that $f^{-1}(f(t)) = t$. Assuming that $f'(0) = a_1 \neq 0$, then f^{-1} has a power series expansion about $f(0) = a_0$ given by

$$f^{-1}(t) = \sum_{n=0}^{\infty} b_n (t - a_0)^n \ .$$

Solve for the first four coefficients $b_0, \ldots, b_3$ in terms of the coefficients $\{a_n\}$.

Problem 7.3. For each of the following equations, verify that $t = 0$ is an ordinary point, write down a recurrence relation for the coefficients of the power series solutions, and solve for the first few terms. Can you recognise the power series in any of the cases?

(a) $x'' + x' - tx = 0$
(b) $(1 + t^2)x'' + 2tx' - 2x = 0$
(c) $x'' + tx' + x = 0$

Problem 7.4. Consider *Airy's equation*:

$$x'' + tx = 0 \ .$$

(a) Is $t = 0$ an ordinary point? Why? What is the radius of convergence of analytic solutions of Airy's equation?
(b) Derive a recurrence relation for the coefficients of the power series solutions around $t = 0$, and compute the first 6 coefficients.

Problem 7.5. Consider *Chebyshev's equation*:

$$(1-t^2)x'' - tx' + p^2 x = 0 \ , \tag{32}$$

where p is a constant.

(a) Is $t=0$ an ordinary point? Why? What is the radius of convergence of analytic solutions of equation (32)?

(b) Derive a recurrence relation for the coefficients of power series solutions of (32) at $t=0$.

(c) Show that when p is a non-negative integer, (32) admits a polynomial solution. More concretely show that when $p = 0, 2, 4, \ldots$ (32) admits a solution which is an even polynomial of degree p; and that when $p = 1, 3, 5, \ldots$ it admits a solution which is an odd polynomial of degree p.

(d) Define the *Chebyshev polynomials* T_p, as the polynomial found above, normalised so that $T_p(0) = 1$ for p even and $T_p'(0) = 1$ for p odd. Calculate the first six Chebyshev polynomials: T_0, T_1, ..., T_5 using the recurrence relation found above.
(**Note:** The standard Chebyshev polynomials in the literature are normalised in a different way.)

(e) Show that Chebyshev polynomials are orthogonal with respect to the following inner product on the space of polynomials:

$$\langle f, g\rangle := \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t) g(t)\, dt \ ,$$

using the following method:

(i) Show that the operator $\mathcal{T}$, defined on a polynomial function $f(t)$ as $\mathcal{T}f(t) := tf'(t) - (1-t^2)f''(t)$, is self-adjoint relative to the above inner product.

(ii) Show that T_p is an eigenfunction of $\mathcal{T}$ with eigenvalue p^2: $\mathcal{T}T_p = p^2\, H_p$.

(iii) Deduce that if $p \neq q$, then $\langle T_p, T_q\rangle = 0$.