Can we hear the shape of the universe?

José Figueroa-O'Farrill

School of Mathematics

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The Big Questions®

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And the answers are...

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The Newtonian universe is thus $\mathbb{R} \times \mathbb{R}^3$.

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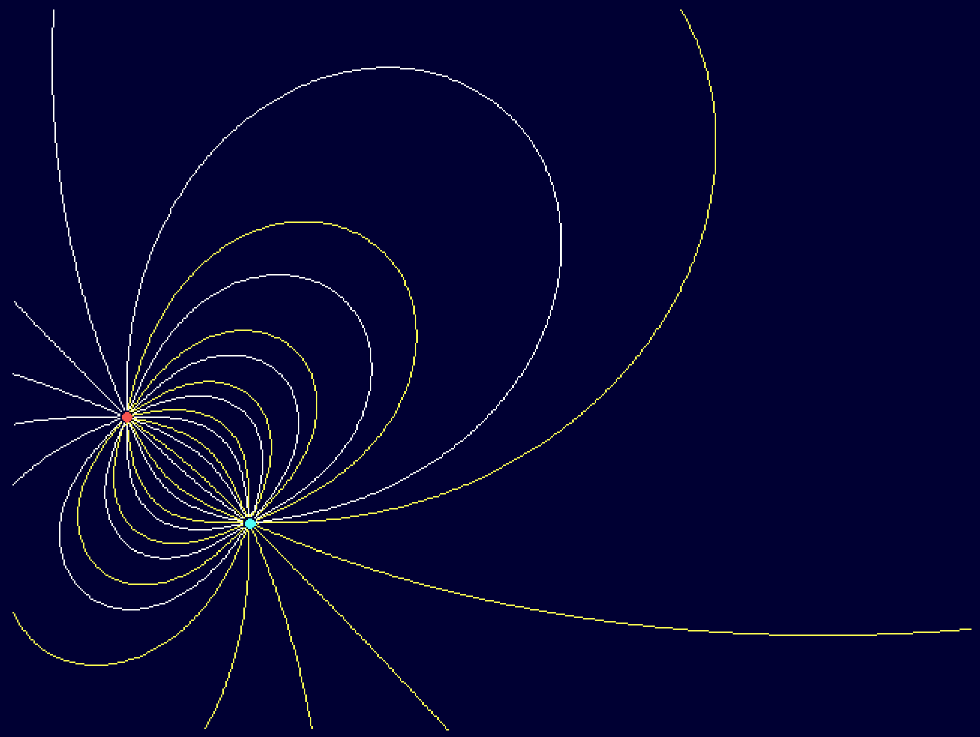

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where $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ are the electric and magnetic fields, respectively.

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Instead, Einstein found that Maxwell equations are consistent with a "special relativity."

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Instead of Galilean transformations, we have

Lorentz transformations

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Which spacetime (M, g) describes our universe?

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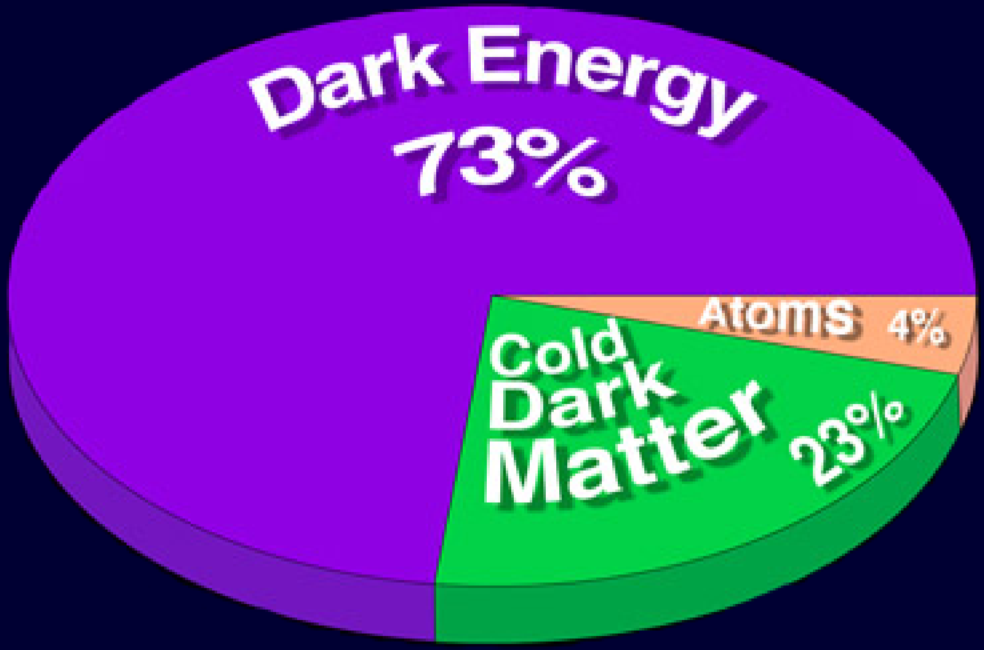
Dark Energy
73%
Atoms 4%
Cold
Dark
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23%

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Big question:

Which topology do we inhabit?

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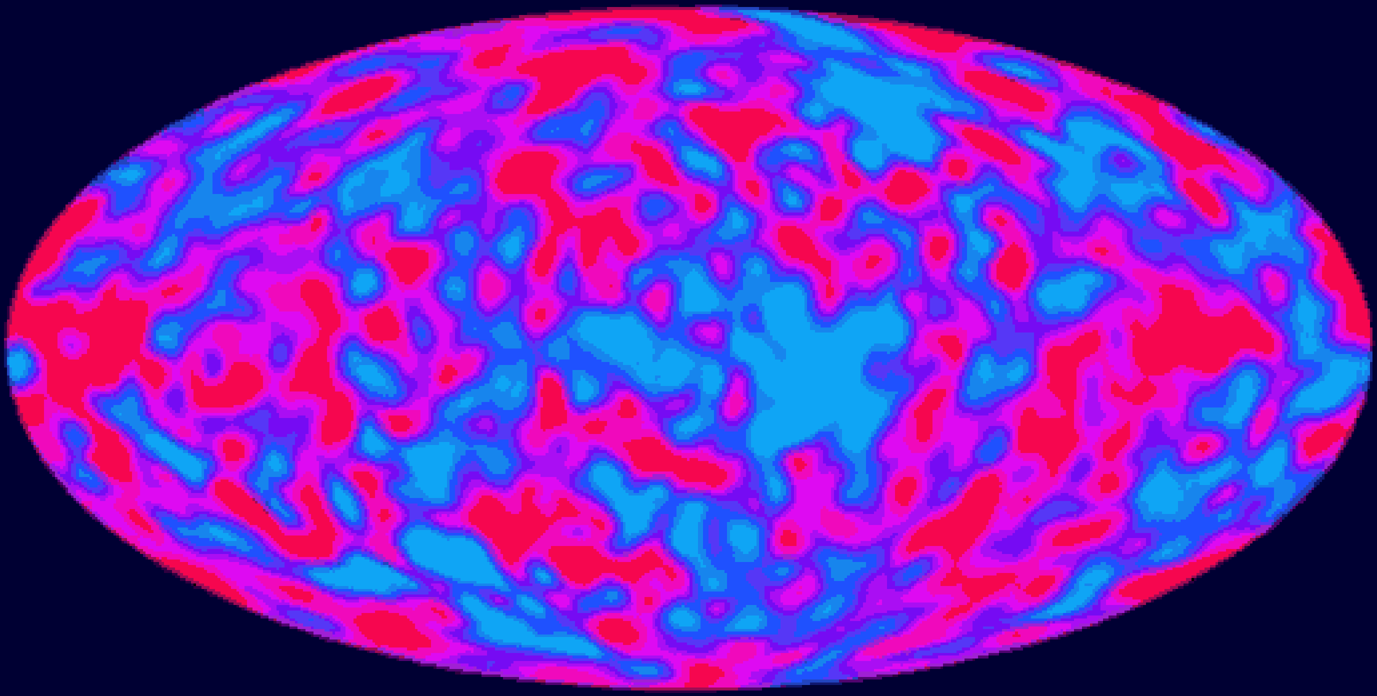

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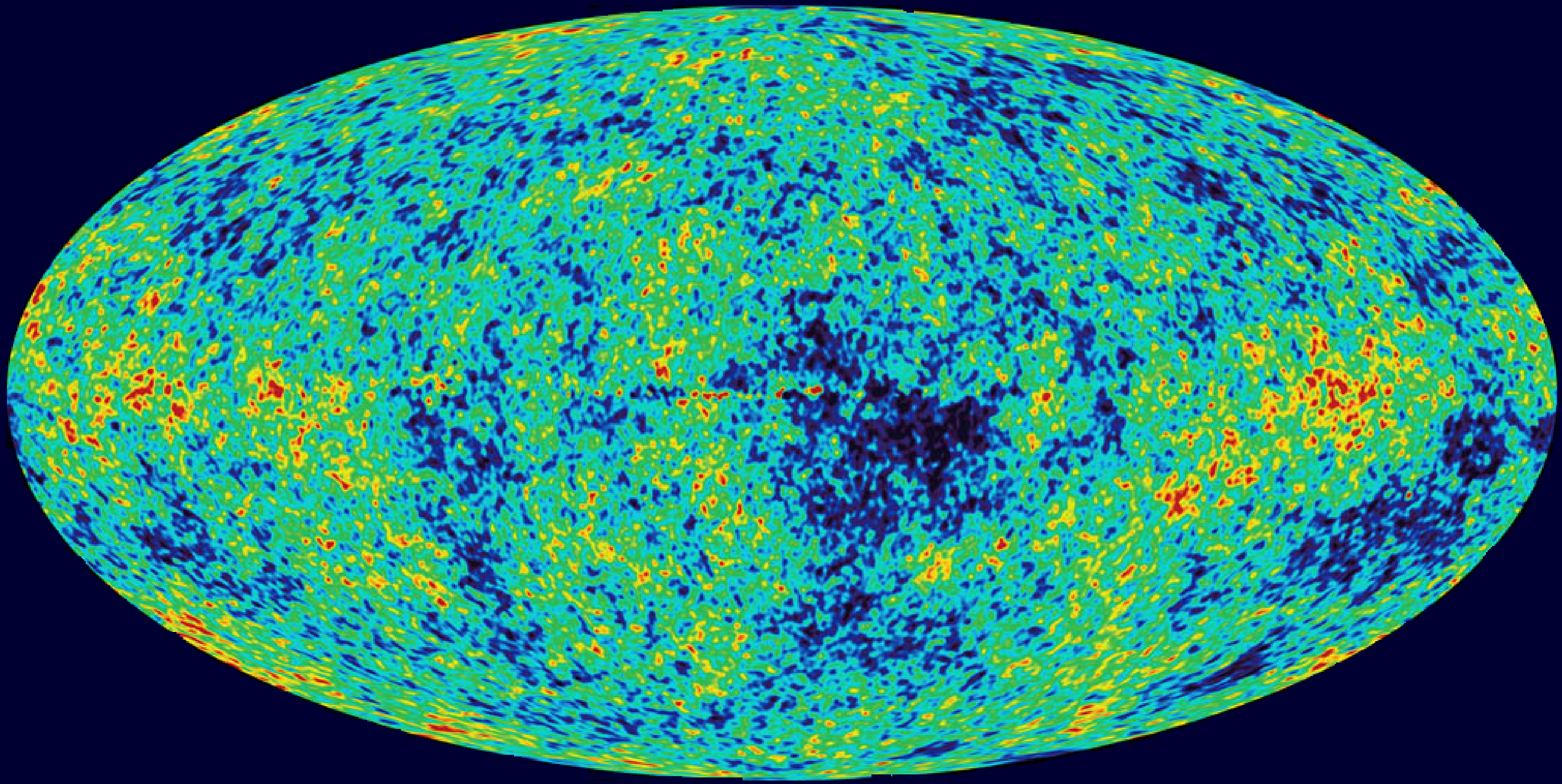

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The "Sachs–Wolfe effect" relates the temperature fluctuations in the CMB to Φ_0.

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There is a related mathematical problem...

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But it is true for three-dimensional space-forms!

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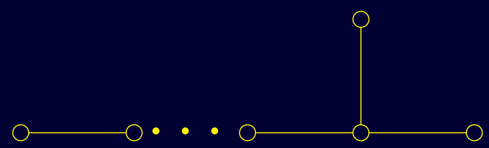
D_ℓ		binary dihedral

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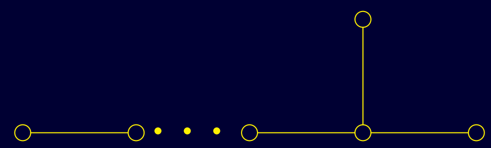
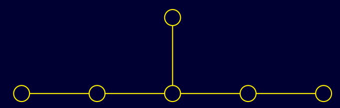
D_ℓ		binary dihedral
E_6		binary tetrahedral

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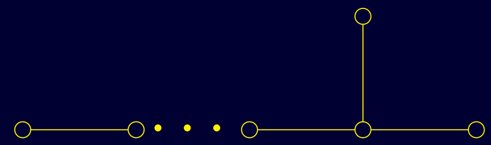
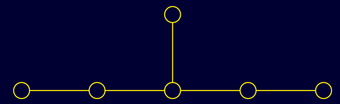
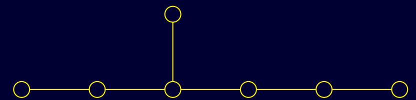
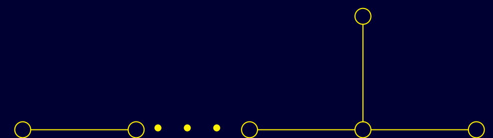
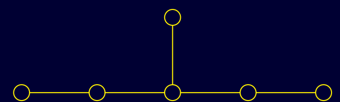
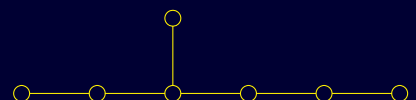
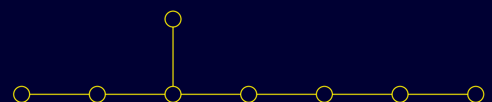
D_ℓ		binary dihedral
E_6		binary tetrahedral
E_7		binary octahedral

Finite subgroups $\Gamma \subset \mathrm{SU}(2)$

McKay correspondence:

Γ	Dynkin diagram	Name
A_ℓ		binary cyclic
D_ℓ		binary dihedral
E_6		binary tetrahedral
E_7		binary octahedral
E_8		binary dodecahedral

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- the universe is finite!
- $\pi_1(\Sigma) \cong E_8 \implies$ the universe is not simply-connected!

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- Σ is counterexample to the “wrong” Poincaré conjecture!

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- awaiting more data: WMAP, Planck surveyor,...

Watch this space.