Stringy Geometry

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String theory is changing the way we look at the geometry of spacetime, and in this lecture I'll try to explain to you what these changes consist of and how they come about. But before doing so, let's spend a little time trying to understand a priori why this should be the case at all.

With the benefit of hindsight, the fact that string theory reshapes our classical geometrical notions is not so surprising. Indeed, classical geometry arose out of the need to formalise mathematically our intuition about the concept of space; an intuition which, being mostly visual, derives from the notion of the *point*. Classical geometry is therefore a useful description of a universe whose basic constituents are point-like. In fact, the fundamental quantities of classical geometry—distances and curvatures—can be measured in (*gedanken*, perhaps) experiments involving the motion of point-like objects. On the other hand if, as string theory postulates, the fundamental objects at our disposal are not point-like but rather one-dimensional extended objects, then our ability to measure distances and curvatures is significantly altered. Intuitively speaking, it's as if the finite extension of the string wouldn't let us focus on points, so that distances smaller than a certain natural length scale cannot be resolved. Of course, this does not contradict experiment, for the stringy length scale is much smaller than anything we've been able to probe in experiments so far. It is in fact commensurate with the Planck length and from quantum mechanics we know that in trying to resolve distances that small, we would have to take into account the quantum-mechanical effects of gravity, for which a candidate point-like theory has yet to emerge.

Let us now try to understand in a little bit more detail how string theory affects our ability to measure distances. Let us assume for the moment that we put strings in a box of length L (measured in stringy units) with periodic boundary conditions and let's see if we can measure the length of the box. There are three types of string configurations. First we have the oscillator modes, in which static strings are vibrating. Since these modes don't feel the size of the box, we'll simply ignore them. Next we have modes in which strings move about the box with a centre-of-mass momentum which, just as for point-particles, will be quantised in units of the inverse length $1/L$. We call these *momentum modes.* They carry an energy which goes like the square of the momentum:

$$E_{\mathrm{M}} = \frac{m^2}{L^2} \;, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots \;. \tag{1}$$

A point-particle theory would only consist of such modes, and hence we could effectively measure the size of the box by fitting the observed energy spectrum to the formula given in (1). However, in a string theory there are also configurations in which a string wraps around the box. The energy of these so-called *winding modes*, behaves as the square of the distance that the string wraps around the box:

$$E_{\mathrm{W}} = n^2 L^2 \;, \quad \text{where } n = 0, \pm 1, \pm 2, \ldots \;. \tag{2}$$

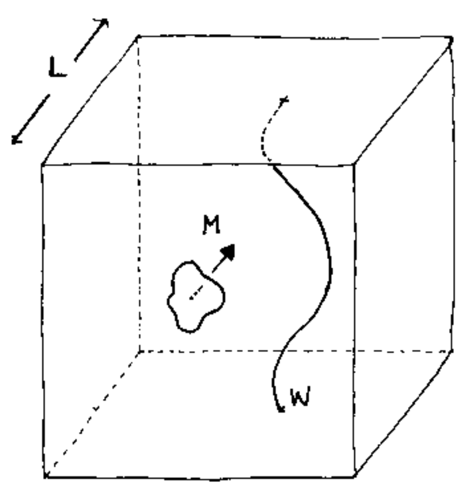


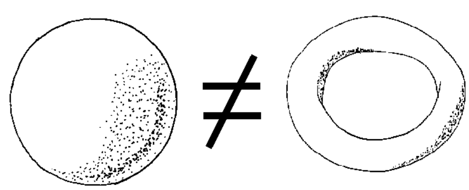
Figure 1 *Momentum and winding modes for strings in a box of length L.*

Now notice that the energy of any momentum mode in a box of length L is equal to the energy of some winding mode in a box of length $1/L$. Therefore from the energy spectrum alone we would not be able to determine whether the length of the box is L or $1/L$. Could we make other measurements to determine the length of the box? The answer is "No." The energy spectrum is not the only property of this system which is invariant under this *duality* exchanging momentum and winding modes and $L \leftrightarrow 1/L$, but indeed so are all the interactions.

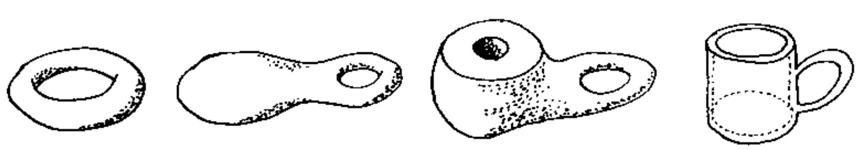
That is, the interactions between states in the box of length L also exist between dual states in the box of length $1/L$. Therefore whatever measurements a 21st century experimental physicist may try to conjure up to try and convince you that the box has length L, can be suitably re-interpreted to try and convince her that the length of the box is in fact $1/L$.

This is already quite surprising: it tells us that the microscope with which string theory provides us, cannot resolve lengths below a certain scale. Is this a limitation of the theory? Should we return the microscope and claim a refund? Let us give it another chance before answering this question. We have seen that the stringy microscope fails to differentiate between two boxes of different lengths; but after all, a box of size L and a box of size $1/L$ are qualitatively very similar. How does string theory fare when it comes to distinguishing qualitatively different spacetimes. Not much better, actually: under the stringy microscope it isn't just the geometry that gets blurred but also the *topology*.

But what is topology? Topology is that which allows us to distinguish between a scone and a bagel, but which doesn't let us distinguish between a bagel and a coffee mug. In other words, topology is "modelling clay" geometry: distances and curvatures no longer matter, what matters is whether two shapes can be continuously deformed into each other, without cutting or tearing. The scone and the bagel are clearly topologically different: there is nothing we can do to our clay-scone to turn it into a bagel short of making a hole in the clay, which isn't allowed:



But by the same token, we can easily knead our clay bagel into a coffee mug:



We will see an example later, but I ask you to believe for now that string theory tells us that there exist spacetimes of markedly different topology which are indistinguishable under the stringy microscope. Does this mean that the stringy microscope is indeed defective? On the contrary! What this tells us is that classical invariant notions like distance and curvature (and even topology) may not be invariant notions in string theory at short distances. This will be of utmost importance if we wish to make sense out of singularities present in point-particle theories. Let us simply mention two examples, both of which come from cosmology.

General relativity is beautiful, but it is incomplete. Perhaps the strongest clue of its incompleteness derives from the singularity theorems of Hawking and Penrose, which (roughly) state that realistic spacetimes are singular: they originate in a big-bang, they are fated to end in a big crunch, or else give rise to black-holes. The hope has always been that a quantum theory of gravity would somehow smooth out these singularities. Since so far the only theory that seems able to incorporate quantum gravity consistently is string theory, it is fair to ask whether string theory is also singular.

It is too early to tell, but everything points in the right direction. First of all it seems that string physics around a black-hole is nonsingular. Indeed, it turns out that a string propagating on a two-dimensional black-hole with the same singularity structure as the four-dimensional Schwarzschild black-hole:

$$ds^2 = \frac{du dv}{1 - uv} \tag{3}$$

is physically indistinguishable from a string propagating on a circle, which is certainly nonsingular. Notice that the topology of the two spaces is indeed different.

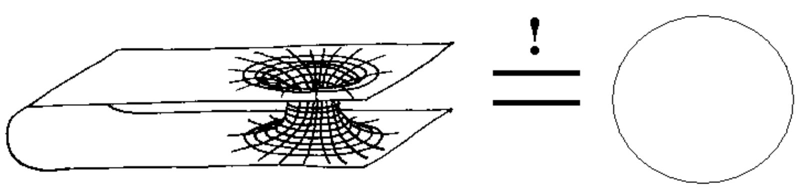


Figure 2 *A two-dimensional black hole can be physically identified with a circle.*

How about the Big Bang? Point-particle theories are certainly singular at the Big Bang. Indeed, according to general relativity[†], a short time t after the

[†] this is true in four-dimensions without cosmological constant and in a radiation dominated cosmology

Big Bang, the size of the universe goes like $L(t) \sim t^{1/2}$; whence at the Big Bang, the curvature blows up. How about in string theory? Such a calculation is not yet possible in string theory; but, naively, based on the $L \leftrightarrow 1/L$ duality one would expect that $L \to 0$ should behave like $L \to \infty$, which is known to be nonsingular.

One can actually glean some quantitative information by going to finite temperature. According to general relativity, provided the expansion is adiabatic (the entropy remains fixed), the temperature diverges as the size of the box goes to zero: $T(L) \to \infty$ as $L \to 0$. But this cannot be the case in string theory, because string theory has a maximum temperature! Notice that the asymptotic behaviour of the density of states of a string theory is exponential:

$$\rho(E) \sim E^{-p} e^{E/T_H} \; ; \tag{4}$$

whence the partition function

$$Z(T) = \int_0^\infty dE\, \rho(E)\, e^{-E/T} \tag{5}$$

only makes sense for $T < T_H$—the *Hagedorn* temperature. Since the temperature cannot grow larger than T_H, we wouldn't expect it to diverge as $L \to 0$. In fact, one can plot the string theoretic behaviour of the temperature $T(L)$ as a function of L under adiabatic expansion and one obtains qualitatively the graph in Figure 3. Notice that the graph exhibits the expected symmetry $L \leftrightarrow 1/L$, equivalently $\log L \leftrightarrow -\log L$. In particular, as L drops below the self-dual point $L = 1$, the behaviour of the temperature agrees with our classical intuition that the size is increasing again. Hence instead of a Big Bang, we have a Big Bounce! In other words, the singularity has been averted.

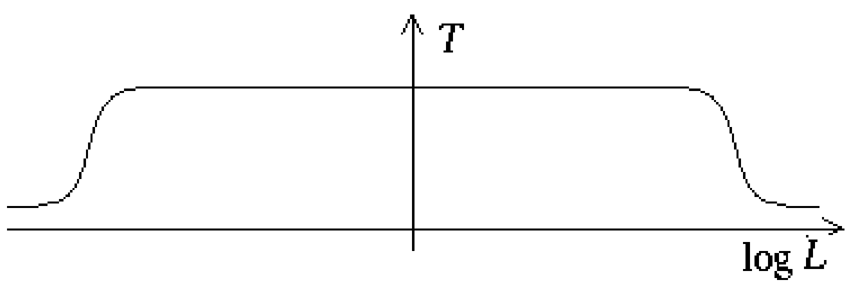


Figure 3. *The temperature as a function of the logarithm of the length L.*

What's responsible for this behaviour? For large L, the momentum modes are light and one can use them to measure distance. As L decreases, in particular when $L < 1$, the momentum modes grow heavier and the winding modes grow lighter. We can then tune our apparatus to measure distance by using the lighter winding modes. But in terms of these, the effective length of the box is $1/L > 1$. Of course this is not the typical oscillating cosmology, since when one passes the self-dual length $L = 1$ one has to change the way one measures length. But this is fine: the important is that one can continue to measure length below $L = 1$; that is, one can continue doing physics.

In summary, the seeming "myopia" of string theory isn't such. What happens is that at short distances, we simply cannot give an invariant meaning to the same quantities that we are used to from large distances.

Let us now try to understand the mechanism by which stringy geometry differs from classical geometry. From what we have seen until now, this mechanism can be traced back to the duality between momentum and winding modes. Are there other theories which possess such duality? And if so, how can we recognise the analogues of the momentum and winding modes? As we will now see, the qualitative feature that distinguishes the momentum and winding modes is that they are *topologically* distinct. What does this mean in this context? *Where's the bagel!?*

Let's look for it in the following simple mechanical analogy. Take a very long and thick piano wire, say two octaves below middle C; and fix the ends of the wire to the walls of this room. Let us now solder nails to the wire along its length in such a way that they are all pointing down. The nails are free to swing up and down, but this motion costs energy because of the torsion in the wire. In particular, the torsion makes the direction to where the nail is pointing dependent on the direction to where its nearest neighbours are pointing. Finally, let us further fix the nails at either end of the wire to the wall, to force them to

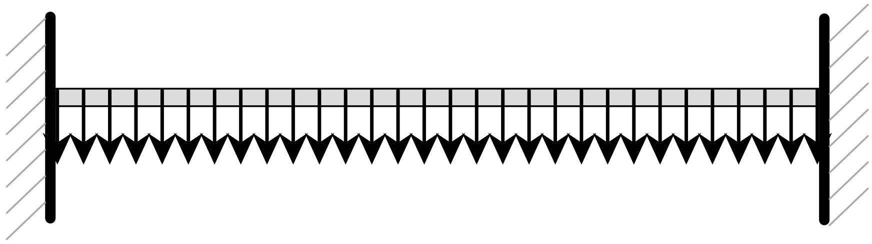


Figure 5. *The simple mechanical analogy (here shown at minimum energy).*

remain pointing down at all times.

Clearly the configuration where all nails are pointing down is the configuration of minimum energy. This is shown in Figure 5. It is the configuration to where the system would return if we were to set it in motion by displacing a few of the nails and letting go. In the absence of friction, the typical motion will be one in which the nails oscillate forever around the minimum energy configuration with a frequency which will be a harmonic of the characteristic frequency of this system. These wave-like configurations are the analogues of the momentum modes of the string.

How about the analogues of the winding modes? Consider configurations in which as we follow the directions of the nails from the rightmost to the leftmost nail, they have wound around the wire a given number of times, as shown in Figure 4. The number of times that the nails wind around is called the *winding number* and it can take, in principle, any integer value. In particular, one can easily imagine one such configuration in which all the nails are at rest. This configuration is not one of minimum energy, since there is energy stored in the twist of the wire. But it is stable. If we were to set the nails in motion starting from such a configuration, and turning friction back on, the nails would eventually slow down and return to that configuration.

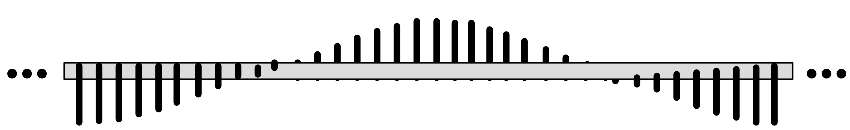


Figure 4. *A winding mode with winding number* $+1$.

These two kinds of states are *qualitatively* different: whereas the momentum states can be reached continuously from the position of minimum energy—again simply turn the friction back on and watch as the nails slow down until they remain at rest pointing down—the winding states are not. To reach the minimum energy configuration down from a winding mode, it would be necessary to unwind all the nails to the right (or to the left) of a given nail at least once around the wire. But this violates the boundary conditions at the end of the wire, since the end-nails that are fixed to the wall would have to move as well.

In other words these two kinds of configurations are *topologically* distinct, since one cannot be reached continuously from the other. In fact, the same physical reasoning used above tells us that configurations with different winding numbers are topologically distinct.

With this mechanical model in mind, we may then identify the mechanism responsible for stringy geometry as the existence of configurations which are topologically distinct, yet—in the case of the string—energetically indistinguishable. Do other theories display such characteristics?

In fact many do. Perhaps the most familiar such theory is classical electromagnetism. It is governed by Maxwell's equations, which in the absence of sources (that is, *in vacuo*) read

$$
\begin{aligned}
\vec{\nabla}\cdot\mathbf{B} &= 0 & \qquad \vec{\nabla}\cdot\mathbf{E} &= 0\\
\frac{\partial\vec{\mathbf{B}}}{\partial t} + \vec{\nabla}\times\mathbf{E} &= 0 & \qquad \frac{\partial\vec{\mathbf{E}}}{\partial t} - \vec{\nabla}\times\mathbf{B} &= 0\ .
\end{aligned}
$$

Notice that if $(\vec{\mathbf{E}},\vec{\mathbf{B}})$ obey the equations, then $(\vec{\mathbf{B}},-\vec{\mathbf{E}})$ do too. This is nothing but classical electromagnetic duality. In the presence of electric sources $(\varrho_e,\vec{\mathbf{J}}_e)$, Maxwell's equations are modified thus:

$$
\begin{aligned}
\vec{\nabla}\cdot\mathbf{B} &= 0 & \qquad \vec{\nabla}\cdot\mathbf{E} &= \varrho_e\\
\frac{\partial\vec{\mathbf{B}}}{\partial t} + \vec{\nabla}\times\mathbf{E} &= 0 & \qquad \frac{\partial\vec{\mathbf{E}}}{\partial t} - \vec{\nabla}\times\mathbf{B} &= -\vec{\mathbf{J}}_e\ ,
\end{aligned}
$$

and duality is lost. Lost, that is, unless we also introduce magnetic sources $(\varrho_m,\vec{\mathbf{J}}_m)$ and modify Maxwell's equations again:

$$
\begin{aligned}
\vec{\nabla}\cdot\mathbf{B} &= -\varrho_m & \qquad \vec{\nabla}\cdot\mathbf{E} &= \varrho_e\\
\frac{\partial\vec{\mathbf{B}}}{\partial t} + \vec{\nabla}\times\mathbf{E} &= \vec{\mathbf{J}}_m & \qquad \frac{\partial\vec{\mathbf{E}}}{\partial t} - \vec{\nabla}\times\mathbf{B} &= -\vec{\mathbf{J}}_e\ .
\end{aligned}
$$

Now the duality symmetry is restored if we also exchange the sources as follows $\varrho_e \leftrightarrow \varrho_m$ and if $(\vec{\mathbf{J}}_e, \vec{\mathbf{J}}_m)$ become $(\vec{\mathbf{J}}_m, -\vec{\mathbf{J}}_e)$.

It is by far not obvious from what I have said until now, but it is not too hard to show that whereas the electromagnetic field due to an electric charges is topologically trivial (like momentum modes), the one around a magnetic monopole is not. Therefore the duality between electric and magnetic charges, interchanges topologically distinct configurations which nevertheless are energetically indistinguishable.

But there's more! At the quantum level, electromagnetic duality has a very amazing consequence. In 1931 Dirac showed that electric and magnetic charges cannot coexist consistently in a quantum theory unless they obey the celebrated quantisation condition:

$$eg \sim 1 \tag{6}$$

where e is the unit of electric charge and g is the unit of magnetic charge; and every other electric and magnetic charges are multiples of these basic units. The quantisation condition (6) implies that if e is large then g is small (and viceversa); hence quantum electromagnetic duality implies that a strong electric interaction (e large) is physically indistinguishable from a weak magnetic interaction (g small). In other words, and taking into account that e and g are to be interpreted as running coupling constants in a quantum field theory, the nonperturbative regime in the electric theory can be identified with the perturbative regime in the magnetic theory, and viceversa. This duality is of course not realised in quantum electrodynamics, but it seems that some four-dimensional supersymmetric gauge theories are invariant under such duality—for instance, $SU(2)$ $N{=}2$ super Yang-Mills with 4 flavours.

In fact, a large body of evidence now points to the fact that such weak-strong coupling duality also holds in string theory, so that the strong coupling regime of a given kind of string propagating in a certain spacetime is indistinguishable from the weak coupling regime of maybe another kind of string propagating in another spacetime (maybe again of different geometry and topology).

This prompts us to ask the following question: *How seriously should we take duality?* I'll conclude the talk by venturing a philosophical answer to this question. Duality is a symmetry of Maxwell's equations. And if there is anything that the history of Physics teaches us, is that we could do much worse than taking the symmetries of Maxwell's equations seriously. Einstein took seriously the fact that these equations were Lorentz invariant, and not Galilean invariant like Newton's equations, and the theory of special relativity burst forth. In a similar way, the gauge invariance of Maxwell's equations is now paradigmatic in all phenomenological models in particle physics. Hence I think that the answer is "Very."

And there is a peculiar difference which makes this answer particularly exciting to contemplate. Both Lorentz invariance and gauge invariance, although they are of a very different nature, have a natural geometric description which has underpinned much of the research in quantum field theory. Duality, on the other hand, is yet a third kind of symmetry for which no appropriate geometric formalism has been found. With a bit of luck, maybe string theory will teach it to us!