M342 PDE: THE DIVERGENCE THEOREM

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1. Statement of the divergence theorem

Let R be a bounded open subset of $\mathbb{R}^n$ with smooth (or piecewise smooth) boundary ∂R. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $R \cup \partial R$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂R. Then the divergence theorem states:

(1) $$\int_R \operatorname{div} \mathbf{X}\, dV = \int_{\partial R} \mathbf{X} \cdot \mathbf{n}\, dA$$

where

$$\operatorname{div} \mathbf{X} = \nabla \cdot \mathbf{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂R.

1.1. Suitable domains.

Examples of suitable bounded domains R include: if $n = 1$, intervals (a, b); if $n = 2$, rectangles $\{a_1 < x < b_1, a_2 < y < b_2\}$, discs, and pieces of discs such as half-discs, quarter-discs etc.; if $n = 3$, boxes $\{a_1 < x < b_1, a_2 < y < b_2, a_3 < z < b_3\}$, balls, half-balls, etc. We shall seldom go beyond 3 dimensions in this course.

1.2. Construction of n and n dA.

If $n = 1$ and $R = (a, b)$, then vectors are just real numbers and $\mathbf{n} = -1$ at $x = a$ and $= +1$ at $x = b$.

If $n = 2$, the normal is got by rotating the tangent vector through 90° (in the correct direction so that it points out!). The quantity $\mathbf{t}\, ds$ can be written (dx, dy) along the surface, so that

(2) $$\mathbf{n}\, dA := \mathbf{n}\, ds = (dy, -dx).$$

Here $\mathbf{t}$ is the tangent vector along the boundary curve and ds is the element of arc-length.

If $n = 3$, then we have to decide how the boundary of R is to be described. You may recall that if ∂R is described as a level-set of a function of 3 variables (i.e. $\partial R = \{\mathbf{x} : F(\mathbf{x}) = 0\}$), then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} F$. We shall use the case where $F = z - f(x, y)$ and R corresponds to the inequality $z < f(x, y)$. Then

(3) $$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{(1 + f_x^2 + f_y^2)^{1/2}}, \quad dA = (1 + f_x^2 + f_y^2)^{1/2}\, dx dy.$$

Hence the quantity $\mathbf{n}\, dA$ is *simpler* than either $\mathbf{n}$ or dA separately:

(4) $$\mathbf{n}\, dA = (-f_x, -f_y, 1)\, dx dy.$$

2. The divergence theorem in 1 dimension

In this case, vectors are just numbers and so a vector field is just a function $f(x)$. Moreover, div $= d/dx$ and the divergence theorem (if $R = [a,b]$) is just the fundamental theorem of calculus:

$$\int_a^b (df/dx)dx = f(b) - f(a)$$

3. The divergence theorem in 2 dimensions

Let R be a 2-dimensional bounded domain with smooth boundary and let $C = \partial R$ be its boundary curve. Recall Green's theorem states:

$$\int_R (\partial_x Q - \partial_y P)\, dxdy = \int_C P\, dx + Q\, dy.$$

This is the same as the two dimensional divergence theorem if we take the vector field (X_1, X_2) with $X_1 = Q$ and $X_2 = -P$. For then it reads

$$\int_R \operatorname{div} \mathbf{X}\, dxdy = \int_R (\partial_x X_1 + \partial_y X_2)\, dxdy = \int_C -X_2\, dx + X_1\, dy$$

$$= \int_C (X_1, X_2) \cdot (dy, -dx) = \int_{\partial R} \mathbf{X} \cdot \mathbf{n}\, ds$$

where we have used (2).

4. The divergence theorem in 3 dimensions

We shall give a 'proof' of this theorem in stages.

4.1. **The divergence theorem for a box.** Consider the box $R = \{a_1 < x < b_1, a_2 < y < b_2, a_3 < z < b_3\}$. Let u be a function of $\mathbf{x} = (x,y,z)$. For each fixed (y,z) the fundamental theorem of calculus gives

$$\int_{a_1}^{b_1} u_x(\mathbf{x})\, dx = u(b_1,y,z) - u(a_1,y,z)$$

Now integrating with respect to y and z,

$$\int_R f_x(\mathbf{x})\, dV = \int_{S_1} [f(b_1,y,z) - f(a_1,y,z)] dxdy, \tag{5}$$

where $S_1 = \{a_2 < y < b_2,\ a_3 < z < b_3\}$. This is just the divergence theorem for the vector field $\mathbf{X} = (u,0,0)$! To see this, note $\operatorname{div}\mathbf{X} = \partial_1 f$ for this vector field, so the LHS of (5) is certainly $\int_R \operatorname{div}\mathbf{X}\, dV$. Now ∂R is a union of six rectangles in parallel pairs

$$S_{11} = \{x = a_1, a_2 < y < b_2,\ a_3 < z < b_3\},\ S_{12} = \{x = b_1, a_2 < y < b_2,\ a_3 < z < b_3\},$$

parallel to the (y,z)-plane,

$$S_{21} = \{a_1 < x < b_1, y = a_2,\ a_3 < z < b_3\},\ S_{22} = \{a_1 < x < b_1, y = b_2,\ a_3 < z < b_3\},$$

parallel to the (x,z)-plane, and

$$S_{31} = \{a_1 < x < b_1, a_2 < y < b_2,\ z = a_3\},\ S_{32} = \{a_1 < x < b_1, a_2 < y < b_2,\ z = b_3\},$$

parallel to the (x,y)-plane. It looks complicated, and a diagram would tell the story much better. Draw one for yourself. Moreover we have that

$$\mathbf{n} = -\mathbf{i} \text{ on } S_{11}, \mathbf{n} = \mathbf{i} \text{ on } S_{12},$$

$$\mathbf{n} = -\mathbf{j} \text{ on } S_{21}, \mathbf{n} = \mathbf{j} \text{ on } S_{22},$$
$$\mathbf{n} = -\mathbf{k} \text{ on } S_{31}, \mathbf{n} = \mathbf{k} \text{ on } S_{32}.$$

So for $\mathbf{X} = (u,0,0)$, $\mathbf{X}\cdot\mathbf{n} = 0$ on the four faces $S_{21}, S_{22}, S_{31}, S_{32}$, whereas

$$\mathbf{X}\cdot\mathbf{n} = -u(a_1,y,z) \text{ on } S_{11}, \ \ \mathbf{X}\cdot\mathbf{n} = u(b_1,y,z) \text{ on } S_{12}.$$

This is precisely the combination of signs on the RHS of (5), so that this really is the divergence theorem for $\mathbf{X} = (u,0,0)$ and this R.

In precisely analogous fashion, the divergence theorem for $\mathbf{X} = (0,v,0)$ and for $\mathbf{X} = (0,0,w)$ is verified. Adding these results, we obtain the divergence theorem for the box, with any vector field $\mathbf{X} = (u,v,w)$.

4.2. **Cutting lemma.** Consider now a bounded domain R decomposed as a union of 2 subdomains R_1 and R_2, with a common interface S_0. Typical example: an apple cut in half. Let $\partial R = S$ and write $S = S_1 \cup S_2$, so that

$$\partial R_1 = S_1 \cup S_0, \ \ \partial R_2 = S_2 \cup S_0.$$

Let the normal of S_1 be denoted $\mathbf{n}_1$, the normal of S_2 be denoted $\mathbf{n}_2$ and the normal of S_0, *pointing into* R_2 be denoted $\mathbf{n}_0$. (Draw a picture.) In particular, the *outward drawn* normal of R_1 is equal to $\mathbf{n}_0$ along S_0 and the *outward drawn* normal of R_2 is equal to $-\mathbf{n}_0$ along S_0.

We claim that if the divergence theorem holds for the pieces R_1 and R_2, then it holds for R. To see this, let $\mathbf{X}$ be a smooth vector field, and apply the divergence theorem for R_1 and R_2, taking careful note of the sign of $\mathbf{n}_0$ as in the previous paragraph. We get

$$\int_{R_1} \operatorname{div}\mathbf{X}\,dV = \int_{S_1} \mathbf{X}\cdot\mathbf{n}\,dA + \int_{S_0} \mathbf{X}\cdot\mathbf{n}_0\,dA, \ \ \int_{R_2} \operatorname{div}\mathbf{X}\,dV = \int_{S_2} \mathbf{X}\cdot\mathbf{n}\,dA - \int_{S_0} \mathbf{X}\cdot\mathbf{n}_0\,dA.$$

Adding, the contributions from S_0 cancel out and so

$$\int_{R} \operatorname{div}\mathbf{X}\,dV = \int_{R_1} \operatorname{div}\mathbf{X}\,dV + \int_{R_2} \operatorname{div}\mathbf{X}\,dV = \int_{S_1} \mathbf{X}\cdot\mathbf{n}\,dA + \int_{S_2} \mathbf{X}\cdot\mathbf{n}\,dA = \int_{S} \mathbf{X}\cdot\mathbf{n}\,dA,$$

just as required.

4.3. **Dissection argument.** With the aid of the divergence theorem for boxes and the cutting lemma, one can imagine proving the divergence theorem by slicing a given domain R into small boxes. We know the divergence theorem for boxes, so by the cutting lemma, we know it for any domain that can be cut up into boxes. But most domains have a curved boundary, so the whole of R is unlikely to be a union of boxes. It is not uncommon to argue that by taking the boxes to be smaller and smaller you can approximate any reasonable domain R better and better, and hence taking some sort of limit, the divergence theorem follows for any such domain.

If you are not satisfied with this argument, read on.

4.4. **Divergence theorem for regions with a curved boundary.** Let $D \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary ∂D, and consider the region

$$R = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D, 0 < z < f(x,y)\} \tag{6}$$

where f is a smooth function in D that is continuous up to ∂D. We shall prove the divergence theorem for this region R.

The motivation for considering this kind of R is that it is intuitively plausible that any reasonable domain in $\mathbb{R}^3$ can be split up as a union of subdomains each of which is either a box or one like (6). By 'like' here, I mean that you may have to permute the roles of x, y and z in the definition. For example, if D were itself a rectangle, then R would be a box with 5 flat sides and one curved side. The flat sides are given by the vertical planes through the sides of D, plus the bottom face $z = 0$. The curved side corresponds to the surface $z = f(x,y)$.

In general the boundary of R consists of 3 pieces, S_0, S_1 and S_2, say, where the bottom face

$$S_0 = \{(x,y,0) : (x,y) \in D\},\ \mathbf{n}\,dA = -\mathbf{k}\,dxdy \tag{7}$$

the 'vertical' side

$$S_1 = \{(x,y,z) : (x,y) \in \partial D, 0 \le z \le f(x,y)\},\ \mathbf{n}\,dA = (dydz, -dxdz, 0) \tag{8}$$

and the top face

$$S_2 = \{(x,y,f(x,y)) : (x,y) \in D\},\ \mathbf{n}\,dA = (-f_x, -f_y, 1)\,dxdy. \tag{9}$$

In (8) and (9) we have used (2) and (4). S_1 may naturally consist of several pieces, but for the purposes of the proof it is enough to think of ∂R as consisting of S_0, S_1 and S_2.

We shall now prove the divergence theorem for R. We shall do it for vector fields $\mathbf{X} = (0,0,u)$ and $\mathbf{X} = (v,0,0)$. The argument for a vector field with x- and z-coordinates zero is very similar to that for $(v,0,0)$ and will be omitted. The general result follows by addition, just as for the box.

The easiest case is $\mathbf{X} = (0,0,u)$. Then $\operatorname{div}\mathbf{X} = u_z$, and

$$\int_R \operatorname{div}\mathbf{X}\,dV = \int_D \left[\int_{z=0}^{f(x,y)} u_z\,dz\right] dx\,dy = \int_D u(x,y,f(x,y))\,dxdy - \int_D u(x,y,0)\,dxdy.$$

Now $\mathbf{X}\cdot\mathbf{n} = 0$ in this case over S_1. So, taking into account (7) and (9), this equation can be rewritten as

$$\int_R \operatorname{div}\mathbf{X}\,dV = \int_{S_0} \mathbf{X}\cdot\mathbf{n}\,dA + \int_{S_2} \mathbf{X}\cdot\mathbf{n}\,dA = \int_{\partial R} \mathbf{X}\cdot\mathbf{n}\,dA.$$

Now we consider the case $\mathbf{X} = (v,0,0)$. Pick w so that

$$\partial_z w(x,y,z) = v(x,y,z). \tag{10}$$

We have $\operatorname{div}\mathbf{X} = v_x = \partial_x\partial_z w = \partial_z\partial_x w$. Hence

$$\begin{aligned}\int_R \operatorname{div}\mathbf{X}\,dV &= \int_D \left[\int_{z=0}^{f(x,y)} \partial_z\partial_x w dz\right] dxdy \\ &= \int_D [w_x(x,y,f(x,y)) - w_x(x,y,0)]dxdy.\end{aligned} \tag{11}$$

We will use Green's theorem to turn this into a boundary integral, but note first that $w_x(x,y,f(x,y))$ the partial derivative of w with respect to x, evaluated at the point $(x,y,f(x,y))$, is *not* the same as $[w(x,y,f(x,y))]_x$, the partial derivative of $w(x,y,f(x,y))$ with respect to x! In fact, using the chain rule and (10),

$$\begin{aligned}\partial_x[w(x,y,f(x,y))] &= w_x(x,y,f(x,y)) + w_z(x,y,f(x,y))\,f_x \\ &= w_x(x,y,f(x,y)) + v(x,y,f(x,y))\,f_x.\end{aligned}$$

Substituting this into (11), we get

$$\int_D [w_x(x,y,f(x,y)) - w_x(x,y,0)]\,dxdy \tag{12}$$

$$= \int_D [\partial_x[w(x,y,f(x,y)) - w(x,y,0)] - v(x,y,f(x,y))\,f_x]dxdy.$$

The second term here is just $\int_{S_2} \mathbf{X}\cdot\mathbf{n}\,dA$, by (9). We apply Green's theorem to the other term, getting

$$\int_D \partial_x[w(x,y,f(x,y)) - w(x,y,0)]\,dxdy = \int_{\partial D}[w(x,y,f(x,y)) - w(x,y,0)]dy$$

$$= \int_{\partial D}\int_{z=0}^{f(x,y)} v(x,y,z)dydz.$$

We recognize this as $\int_{S_1} \mathbf{X}\cdot\mathbf{n}\,dA$. Putting all the pieces together we find at last:

$$\int_R \operatorname{div}\mathbf{X}\,dV = \int_{\partial R}\mathbf{X}\cdot\mathbf{n}\,dA$$

for $\mathbf{X} = (v,0,0)$. Here we have also used the fact that $\int_{S_0}\mathbf{X}\cdot\mathbf{n}dA = 0$ since $\mathbf{X}\cdot\mathbf{n}$ is identically zero on S_0.

The method for a vector field of the form $\mathbf{X} = (0,w,0)$ is exactly analogous to the argument we've just seen, with ∂_y replacing ∂_x at the appropriate places, and the corresponding modification of Green's theorem. As indicated, the theorem now follows by considering a general vector field $\mathbf{X} = (X_1,X_2,X_3)$ as the sum

$$\mathbf{X} = (X_1,0,0) + (0,X_2,0) + (0,0,X_3).$$

5. Consequences: Green's Identities

The divergence theorem is important in PDE because it allows one to integrate by parts. To state the fundamental result, let R be a bounded domain with piecewise smooth boundary as before, and let u be a smooth function and $\mathbf{X}$ a smooth vector field in R (continuous up to ∂R). By Exercise (6.5),

$$\operatorname{div}(u\mathbf{X}) = \operatorname{grad} u\cdot\mathbf{X} + u\operatorname{div}\mathbf{X}.$$

We integrate this over R, applying the divergence theorem to the LHS:

$$\int_{\partial R} u\mathbf{X}\cdot\mathbf{n}\,dA = \int_R(\operatorname{grad} u\cdot\mathbf{X} + u\operatorname{div}\mathbf{X})\,dV. \tag{13}$$

Although this does not have a fancy name, it is every bit as important as Green's first and second identities, (14) and (16) below.

5.1. **Green's first identity.**

Taking $\mathbf{X} = \operatorname{grad} v$ in (13), where v is another suitable function in R, we obtain

$$\int_{\partial R} u\frac{\partial v}{\partial n}dA = \int_R \operatorname{grad} u\cdot\operatorname{grad} v\,dV + \int_R v\Delta u\,dV \tag{14}$$

where

$$\frac{\partial v}{\partial n} = v_n = \text{directional derivative of } v \text{ in direction } \mathbf{n} = \mathbf{n}\cdot\operatorname{grad} v. \tag{15}$$

Equation (14) is known as *Green's first identity*.

5.2. **Green's second identity.** If we swap u and v around in (14), then the first term on the RHS does not change. Subtracting these two versions of (14), we obtain

$$\int_R (u\Delta v - v\Delta u)\, dV = \int_{\partial R} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dA. \tag{16}$$

(The process by which (16) was derived from (14) is an example of *symmetrization*.) This is *Green's second identity* and is a basic tool in the study of Δ.

5.3. **Inner-product-space interpretation.** Let $C^\infty(R)$ stand for the space of smooth (infinitely differentiable) functions on R, such that all derivatives are continuous up to ∂R. Let $C^\infty(R,\mathbb{R}^3)$ stand for the space of all smooth vector fields on R, again with all derivatives continuous up to ∂R. Make these into infinite-dimensional inner-product spaces by

$$\langle u, v\rangle = \int_R uv\, dV, \;\; (u, v \in C^\infty(R))$$

and

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \int_R \mathbf{X}\cdot\mathbf{Y}\, dV, \;\; (\mathbf{X}, \mathbf{Y} \in C^\infty(R,\mathbb{R}^3)).$$

Then grad, div and Δ define linear operators

$$\text{grad} : C^\infty(R) \to C^\infty(R,\mathbb{R}^3),\; \text{div} : C^\infty(R,\mathbb{R}^3) \to C^\infty(R),\; \Delta : C^\infty(R) \to C^\infty(R).$$

The identity (13) becomes

$$\langle \text{grad}\, u, \mathbf{X}\rangle + \langle u, \text{div}\,\mathbf{X}\rangle = \int_{\partial R} u\mathbf{X}\cdot\mathbf{n}\, dA. \tag{17}$$

In particular, grad and $-$div are adjoint to each other on any subspace which guarantees the vanishing of the boundary term. For example, the subspaces of functions vanishing on ∂R, or the subspace of vector fields such that $\mathbf{X}\cdot\mathbf{n} = 0$ on ∂R.

Similarly, (14) becomes

$$\langle \text{grad}\, u, \text{grad}\, v\rangle + \langle \Delta u, v\rangle = \int_{\partial R} v u_n\, dA \tag{18}$$

and (16) becomes

$$\langle \Delta u, v\rangle - \langle u, \Delta v\rangle = \int_{\partial R} (uv_n - u_n v)\, dA. \tag{19}$$

From this we obtain the self-adjointness of Δ on suitable subspaces of $C^\infty(R)$, for example the subspace of functions which satisfy Dirichlet or Neumann boundary conditions.

6. Exercises on the divergence theorem

6.1. Write down $\mathbf{n}$ when

1. $R = \{x^2 + y^2 < a^2\} \subset \mathbb{R}^2\}$,
2. $R = \{(x-p)^2 + (y-q)^2 < a^2\} \subset \mathbb{R}^2\}$,
3. $R = \{x^2 + y^2 + z^2 < a^2\} \subset \mathbb{R}^3\}$,
4. $R = \{(x-p)^2 + (y-q)^2 + (z-r)^2 < a^2\} \subset \mathbb{R}^3\}$.

[Hint: sketch these sets and think geometrically. Use general formulae only if all else fails.]

6.2. Consider the piece S of the plane $x+y+z=1$ cut off by the coordinate planes, so $S=\{(x,y,z):x+y+z=1, x\geq 0, y\geq 0, z\geq 0\}$. Let R be the 3-dimensional region bounded by the coordinate planes and S, $R=\{(x,y,z):x+y+z\leq 1, x\geq 0, y\geq 0, z\geq 0\}$.

1. Sketch (or get maple to sketch?) S and R.
2. Show how to parameterize S as a graph $z=f(x,y)$ where the real-valued function f is defined in some region D of the (x,y)-plane. Don't forget to specify D as well as f.
3. Write down the area element dA in terms of $dx\,dy$. Write down also the two unit normal vectors to S.
4. Calculate $\int_S dA$. What is the interpretation of this integral? Can you check it using elementary geometry?
5. Calculate $\int_S \mathbf{i}\cdot\mathbf{n}\,dA$, where $\mathbf{n}$ is the choice of normal that points away from the origin. Can you give the values of $\int_S \mathbf{j}\cdot\mathbf{n}\,dA$ and $\int_S \mathbf{k}\cdot\mathbf{n}\,dA$ without any further detailed calculation?
6. Calculate $\int_R dV$ and check your answer using the formula for the volume of a pyramid. Use the divergence theorem to deduce the value of

$$\int_{\partial R} x\mathbf{i}\cdot\mathbf{n}\,dA.$$

6.3. Let $B=\{(x,y,z):-1\leq x\leq 1, -1\leq y\leq 1, -1\leq z\leq 1\}$ be the cube with centre at the origin and of side 2. Calculate directly, and using the divergence theorem,

$$\int_{\partial B}\mathbf{i}\cdot\mathbf{n}\,dA,\quad \int_{\partial B} x\mathbf{i}\cdot\mathbf{n}\,dA,\quad \int_{\partial B} x^2\mathbf{i}\cdot\mathbf{n}\,dA.$$

6.4. Let

$$\mathbf{X}(x,y,z)=(\sin(yz), e^{x^2}\cos z+y, ye^{x^4-y^5}-z).$$

What is $\int_S \mathbf{X}\cdot\mathbf{n}\,dA$ if S is the unit sphere $x^2+y^2+z^2=1$ and $\mathbf{n}$ is the normal pointing away from the origin? [Hint: $\mathbf{X}$ is very complicated, but very little work is needed to answer this question.]

6.5. Let u be a smooth function and $\mathbf{X}$ be a smooth vector field in $\mathbb{R}^3$. Show that

$$\operatorname{div}(u\mathbf{X})=\operatorname{grad}u\cdot\mathbf{X}+u\operatorname{div}\mathbf{X}.$$

6.6. Let $\mathbf{x}=(x,y,z)$ and $r=(x^2+y^2+z^2)^{1/2}$. Compute:

1. $\operatorname{div}\mathbf{x}$;
2. $\operatorname{grad}\phi(r)$;
3. $\operatorname{div}(\mathbf{x}/r^n)$. (6.5).

6.7. Let $a>0$ and let $S=\{x^2+y^2+z^2=a^2\}$ be the sphere of radius a, centre the origin. Let $\mathbf{n}=\mathbf{x}/a$ be the outward-pointing normal at S. Show that if $\mathbf{X}=\mathbf{x}/r^3$, then

$$\int_S \mathbf{X}\cdot\mathbf{n}\,dA=4\pi.$$

What is $\operatorname{div}(\mathbf{x}/r^3)$? Why do these results not contradict the divergence theorem?