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IRREDUCIBLE UNITARY REPRESENTATIONS OF THE LORENTZ GROUP

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PART I

Introduction

It is the purpose of this paper to construct and to analyze the irreducible unitary representations of the Lorentz group which satisfy certain regularity conditions stated below. More specifically, we deal with the proper Lorentz group, i.e., the group of all homogeneous linear transformations in four variables x^0, x^1,x^2, x^3 which leave the quadratic form $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ invariant, have the determinant 1, and do not reverse the direction of time (the variable x^0). This group will be denoted by $\mathfrak{L}_4$. It is known that $\mathfrak{L}_4$ (as well as the group $\mathfrak{L}_3$ defined below) has only *infinite-dimensional* unitary representations (by operators in Hilbert space) except for the trivial one-dimensional case, where every group element is represented by 1 [Wigner, p. 165].[1] In addition to $\mathfrak{L}_4$ we shall also investigate the corresponding group $\mathfrak{L}_3$ of all homogeneous linear transformations in the three variables x^0, x^1, x^2 which leave the form $(x_0)^2 - (x^1)^2 - (x^2)^2$ invariant—with the same restrictions as above.

Apart from possible applications in Mathematical Physics [cf. Dirac 2] this investigation has an intrinsic mathematical interest as a detailed analysis of the unitary representations of a non-compact group. This holds in particular in the case of $\mathfrak{L}_3$ where the results are fairly explicit and complete. Moreover, the representations of both $\mathfrak{L}_3$ and $\mathfrak{L}_4$ play a part in Wigner's classification of the unitary representations of the *inhomogeneous* Lorentz group [Wigner, p. 192], $\mathfrak{L}_3$ being the "little group" in the case $P < 0$, and $\mathfrak{L}_4$ being the little group in the case 0_0 (the representations of these two groups are not classified in Wigner's paper). It should be noted however that the conditions which we impose on the representations are more stringent than Wigner's.

PLAN OF THE INVESTIGATION. We shall discuss both single- and double-valued representations and hence deal with the corresponding spinor groups $\mathfrak{S}_3$ and $\mathfrak{S}_4$ rather than with $\mathfrak{L}_3$ and $\mathfrak{L}_4$. If $\mathfrak{S}$ (which stands here for either $\mathfrak{S}_3$ or $\mathfrak{S}_4$) is represented on a Hilbert space $\mathfrak{H}$ by unitary operators $U(a)$ which are continuous in a (a is a group element of $\mathfrak{S}$), every one parameter subgroup may be expressed, by Stone's theorem, in the form $U_t = \exp(-itH)$ where H is a self-adjoint operator on $\mathfrak{H}$. Since on the other hand, every one parameter subgroup is generated by an infinitesimal transformation (an element of the Lie algebra $\mathfrak{s}$) of the group $\mathfrak{S}$, there is a correspondence between the operators H and the elements of $\mathfrak{s}$. Our main assumption will be that the operators H define a representation of $\mathfrak{s}$ which will be called an *infinitesimal representation* of

[1] See bibliography at the end of the paper.

$\mathfrak{S}$. It is sufficient to require that the sum of two elements of $\mathfrak{g}$ is mapped into the sum of the corresponding operators H. The H being unbounded, their sum can only be properly defined if the common part of the domains of the different H is large enough. We need, therefore, a condition about these domains, which will be stated in §5. If these requirements are satisfied the possible irreducible infinitesimal representations of $\mathfrak{S}$ may be classified. By an explicit construction it is shown that to every infinitesimal representation corresponds an irreducible unitary representation of $\mathfrak{S}$ itself. In each case the Hilbert space $\mathfrak{H}$ is defined as a function space over a properly chosen manifold $\mathfrak{M}$ on which $\mathfrak{S}$ acts as a transformation group (to every element a of $\mathfrak{S}$ corresponds a homeomorphism of $\mathfrak{M}$ into itself denoted by $y = ax$, where x, y are points on $\mathfrak{M}$. The group property requires that $a(bx) = (ab)x$. The operators $U(a)$ are obtained as follows. If $f(x)$ is an element of $\mathfrak{H}$, then $U(a)f(x) = \mu(a, a^{-1}x)\cdot f(a^{-1}x)$, where $\mu(a, x)$ is a fixed non-vanishing function of the group element a and the point x which satisfies the condition $\mu(ab, x) = \mu(a, bx)\cdot\mu(b, x)$, and is called a *multiplier* of the transformation group. It is easily seen that $U(ab) = U(a)U(b)$, and with a suitable definition of the inner product the operators $U(a)$ are unitary. We turn now to a brief summary of the results obtained.

I. *The group* $\mathfrak{S}_3$. The infinitesimal representation contains three linearly independent elements H_{12}, H_{20}, and H_{01}, where H_{kl} corresponds to a transformation of the $(k - l)$ plane into itself. For any irreducible representation the operator

$$Q = (H_{01})^2 + (H_{20})^2 - (H_{12})^2$$

is a *scalar*, i.e., it has the form $Q = q\cdot 1$, q being a real number. Moreover, H_{12}, whose spectrum is always discrete, has simple proper values m which are either all *integral* or all *half-integral* and which characterize the representations of the rotations in the $(1 - 2)$ plane. In the half-integral case we obtain a double-valued representation of $\mathfrak{L}_3$. The representations may be classified according to the value of q and the values of m. The following possibilities are found: (1) C_q^0 : q may be any positive number while m assumes all integral values $0, \pm 1, \cdots$. (2) $C_q^{\frac{1}{2}}$: q may be any number in the interval $\frac{1}{4} < q < \infty$, while m assumes all half integral values $\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots$. (3) D_k^+: k may be one of the numbers $\frac{1}{2}, 1, \frac{3}{2}, \cdots$; q has the value $k(1 - k)$, and m assumes all values $k, k + 1, k + 2, \cdots$. (4) D_k^-: k may be one of the numbers $\frac{1}{2}, 1, \frac{3}{2}, \cdots$; q is equal to $k(1 - k)$, and m assumes all values $-k, -(k + 1), -(k + 2), \cdots$. The two classes C_q^0 and $C_q^{\frac{1}{2}}$ are termed *continuous* because in each case the possible values of q fill an interval. By contrast, the two classes $D_k^{\pm}$ are termed *discrete*, because q may only assume the values $k(1 - k)$.

The unitary representations of $\mathfrak{S}_3$ corresponding to the two *continuous* classes C_q^0 and $C_q^{\frac{1}{2}}$ may be realized on a function space over the unit circle, which is the manifold $\mathfrak{M}$ mentioned above, with suitably chosen multipliers. The group $\mathfrak{L}_3$—and hence also the group $\mathfrak{S}_3$—acts on $\mathfrak{M}$ as the group of the projective transformations of $\mathfrak{M}$ into itself. As long as $q \geqq \frac{1}{4}$, the Hilbert space $\mathfrak{H}$ consists of all square integrable functions over $\mathfrak{M}$, while the inner product in $\mathfrak{H}$ is defined by a positive definite integral form depending on q if q is in the "*exceptional interval*" $0 < q < \frac{1}{4}$.

The representations of the *discrete* class are realized on a space of analytic functions of a complex variable z which are regular on the open unit circle $|z| < 1$, with suitably chosen multipliers and a suitable definition of the inner product in $\mathfrak{H}$. $\mathfrak{M}$ is the open unit circle, and the transformation group is the group of all conformal transformations of $\mathfrak{M}$ into itself.

If the proper vectors f_m of H_{12} are chosen as basic vectors of $\mathfrak{H}$, the matrix elements $u_{mn}(a) = (f_m, U(a)f_n)$ may be explicitly determined for every representation, and may be studied as functions of a. They are always analytic in a, and if the f_m are multiplied with suitably chosen complex numbers of absolute value one they are also analytic in q for each of the classes C_q^0 and $C_q^{\frac{1}{2}}$ (including the exceptional interval $0 < q < \frac{1}{4}$ for C_q^0). It is particularly interesting to study the *asymptotic* behavior of the matrix elements u (we omit the indices m, n) on the group manifold $\mathfrak{S}_3$. For this purpose we introduce the non-negative parameter τ defined by the relation $\cosh \tau = (1 - v^2)^{-\frac{1}{2}}$, where v is the relative velocity of two frames of reference connected by a Lorentz transformation a. We thus obtain the following asymptotic expressions for large τ: For the classes C_q^0 and $C_q^{\frac{1}{2}}$ $(q > \frac{1}{4})$ $u \sim e^{-\tau/2}e^{\pm is\tau}$ $(s = (q - \frac{1}{4})^{\frac{1}{2}})$. For C_q^0 $(0 < q < \frac{1}{4})$ $u \sim e^{-\tau/2}e^{\sigma\tau}$ $(\sigma = +(\frac{1}{4} - q)^{\frac{1}{2}})$. For $D_k^{\pm}$ $u \sim e^{-k\tau}$.

It is evident that the matrix elements of C_q^0 where q is in the exceptional interval $0 < q < \frac{1}{4}$ exhibit an asymptotic behavior which differs markedly from that of the other matrix elements of the continuous class: They are not oscillatory, and they decrease less rapidly. It is reasonable to relate the asymptotic behavior of the matrix elements to the invariant measure of $\mathfrak{S}_3$, which, incidentally, is both right- and left-invariant. The portion of the group manifold between τ and $\tau + d\tau$ has the volume const. $\sinh \tau \, d\tau$. Therefore only the matrix elements of $D_k^{\pm}$ with $k > \frac{1}{2}$ are *square integrable* over $\mathfrak{S}_3$. Square integrable functions may also be formed by integrating the matrix elements of C_q^0 and $C_q^{\frac{1}{2}}$ (where $q > \frac{1}{4}$) with respect to s. These results may be generalized so as to be independent of the basis in the representation space $\mathfrak{H}$.

The matrix elements of the representations C_q^0 $(q > \frac{1}{4})$, $C_q^{\frac{1}{2}}$, and $D_k^{\pm}$ $(k > \frac{1}{2})$ satisfy *orthogonality relations* similar to those which hold for irreducible representations of a *compact* group. Finally, it may be shown that the matrix elements of $D_k^{\pm}$ $(k > \frac{1}{2})$ and integrals (with respect to s) of the matrix elements of C_q^0 $(q > \frac{1}{4})$ and $C_q^{\frac{1}{2}}$ are dense in the Hilbert space of square integrable functions over $\mathfrak{S}_3$. (It is interesting that this holds for the matrix elements of only part of all unitary representations.)

II. *The group* $\mathfrak{S}_4$. We may choose the following linearly independent elements of an infinitesimal representation: H_{12}, H_{23}, H_{31}, H_{01}, H_{02}, H_{03}. For an irreducible representation the two operators

$$Q = (H_{01})^2 + (H_{02})^2 + (H_{03})^2 - (H_{12})^2 - (H_{23})^2 - (H_{31})^2$$

$$R = H_{01}H_{23} + H_{02}H_{31} + H_{03}H_{12}$$

are scalars, i.e., $Q = q \cdot 1$, $R = r \cdot 1$, where q and r are real numbers. The spectrum of the operator $(H_{12})^2 + (H_{23})^2 + (H_{31})^2$ is always *discrete* and consists of numbers of the form $j(j + 1)$ related to the $(2j + 1)$-dimensional irreducible representations of the three-dimensional rotation group, every one of which occurs at most once. The values j are either all *integral* or all *half-integral*, the latter case corresponding to a double-valued representation of the group $\mathfrak{L}_4$. The possible representatons may be classified as follows: (1) $C_q^0 : q$ may be any *positive* number, while j assumes all values, 0, 1, 2, $\cdots$, and $r = 0$. (2) $C_{k,r} : r$ may be any *real* number, k may have one of the values $\frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$. In this case $q = 1 - k^2 + (r/k)^2$, and j assumes all values k, $k + 1$, $k + 2$, $\cdots$. There is no counterpart of the discrete class which we found for the group $\mathfrak{S}_3$. For the representations C_q^0 there exists an "exceptional interval," viz., $0 < q < 1$.

The realization of the unitary representations of $\mathfrak{S}_4$ is quite similar to the one described above. In all cases, the manifold $\mathfrak{M}$ is the unit sphere, and the transformation group operating on it is the group of projective transformations of the sphere into itself. With suitably chosen multipliers we obtain unitary transformations $U(a)$, the Hilbert space $\mathfrak{H}$ being defined by all square integrable functions over the unit sphere except for the representations C_q^0 with q in the exceptional interval $0 < q < 1$, in which case the inner product in $\mathfrak{H}$ is again defined by a positive definite integral form depending on q.

With respect to the matrix elements of the representations, the results are not as simple and complete as they are for the group $\mathfrak{S}_3$. However, the analytic nature as well as the asymptotic behavior of the matrix elements are easily determined. As a basis we use vectors f_{jm} in $\mathfrak{H}$ which are proper vectors of both operators $(H_{12})^2 + (H_{23})^2 + (H_{31})^2$ and H_{12}. Omitting the indices, and using the same parameter τ as above, we have for large τ: For C_q^0 $(q > 1)$ $u \sim e^{-\tau}e^{\pm is\tau}$ $(s = (q - 1)^{\frac{1}{2}})$. For C_q^0 $(0 < q < 1)$ $u \sim e^{-\tau}e^{\sigma\tau}$ $(\sigma = +(1 - q)^{\frac{1}{2}})$. For $C_{k,r}$ $u \sim e^{-\tau}e^{\pm ir\tau/k}$. The same remarks as above apply to the matrix elements of C_q^0 where q is in the exceptional interval $0 < q < 1$.

In the case of $\mathfrak{S}_4$ the portion of the group manifold between τ and $\tau + d\tau$ has the volume const$\cdot(\sinh \tau)^2 \, d\tau$, and it follows that none of the matrix elements is square integrable, and that by integrating the matrix elements of C_q^0 $(q > 1)$ over s and the matrix elements of $C_{k,r}$ over r we may obtain square integrable functions on the group manifold. Orthogonality relations may again be derived excepting the representations C_q^0 for the exceptional interval.

In an appendix to Part II, Dirac's *expansor representations* [Dirac 2] are analyzed. They are reducible, and it is shown that they contain all representations C_q^0 with $q > 1$ and the representations $C_{k,0}$. Moreover, the equivalence of different homogeneous expansor representations is demonstrated.

Contents of Part I. Part I of this paper deals mainly with the group $\mathfrak{S}_3$, Part II deals with $\mathfrak{S}_4$. It should be noted, though, that §§1–3 as well as part of §5 form the basis for the discussion of both groups and that, generally speaking, the method applied to analyze $\mathfrak{S}_4$ is a straight-forward generalization of the method used in Part I. In §1, Lie groups and the multipliers associated with them are surveyed to the extent necessary for the following investigation. §§2 and 3 contain a discussion of the Lorentz groups and the spinor groups, $\mathfrak{S}_3$, $\mathfrak{S}_4$. §4 treats the group $\mathfrak{S}_3$ and its manifold in greater detail. In §5 the infinitesimal representations of $\mathfrak{S}_3$ are classified, and the construction of the corresponding unitary representations of $\mathfrak{S}_3$ is carried out in §§6–9. §§10–11 deal with the matrix elements as functions on the group manifold, the orthogonality relations are derived in §12, and the completeness of the matrix elements is demonstrated in §13. The appendix contains some remarks on the spectra of the infinitesimal operators.

At a later occasion, the writer intends to discuss several questions not treated in this paper, in particular the connection with the representations of the inhomogeneous Lorentz group, and with the field equations for particles of higher spin ([Dirac 1], [Fierz, 1, 2]).

The main results of this paper were worked out during the years 1940–1942. In the intervening years the writer did not have the opportunity to complete the manuscript.

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§1. Preliminary remarks on Lie groups and multipliers

In this section we shall collect a number of formulas and introduce the notations which will be used in the present paper. In particular, we shall be concerned with a discussion of the multipliers mentioned in the introduction.

1a. *Introduction.* Let $\mathfrak{G}$ be an n-dimensional connected Lie group. Its elements will be denoted by $a, b, \cdots$, its unit element by e. We assume that the group manifold, which we also denote by $\mathfrak{G}$, can be covered by one single coordinate system, so that every group element a is described by n real variables $a^1, \cdots, a^n$, the parameters of the group.[2] (The range of the a^i will be specified in each particular case.)

The parameters of the product ab are defined by n analytic functions Φ^i

$$(ab)^i = \Phi^i(a^1, \cdots, a^n, b^1, \cdots, b^n) \equiv \Phi^i(a, b) \qquad 1 \leqq i \leqq n$$

and the parameters of the inverse a^{-1} of the group element a are given by n analytic functions θ^i

$$(a^{-1})^i = \theta^i(a^1, \cdots, a^n) \equiv \theta^i(a) \qquad 1 \leqq i \leqq n.$$

[2] In the second part of the paper we shall have to use what amounts to polar coordinates on a sphere, i.e., coordinates which are not everywhere regular. However, this will not lead to any difficulties, and we disregard this complication in the present discussion.

It is known that the partial derivatives of the functions Φ^i with respect to the variables a^k may be expressed as follows:

$$\frac{\partial (ab)^i}{\partial a^k} = \frac{\partial \Phi^i}{\partial a^k} = \chi^i_s(ab)\psi^s_k(a). \tag{1.1}$$

We use the convention of tensor calculus that a summation is to be carried out with respect to repeated indices. The χ^i_s are n^2 analytic functions, and the ψ^s_k are related to them by the equations $\chi^i_s(a)\psi^s_k(a) = \delta^i_k$, i.e., the matrix (ψ^s_k) is the inverse of the matrix (χ^s_k). The functions $\chi^s_k(a)$ are uniquely determined by the values which they assume for the unit element e; these values $\chi^s_k(e)$ may be arbitrarily chosen, provided that their determinant be different from zero.[3]

Finally, the χ^i_s, which determine the infinitesimal transformations of the group, satisfy the commutation rules

$$\frac{\partial \chi^i_k(a)}{\partial a^s}\chi^s_l(a) - \frac{\partial \chi^i_l(a)}{\partial a^s}\chi^s_k(a) = c^r_{kl}\chi^i_r(a) \tag{1.2}$$

where c^r_{kl} denote the structure constants of the group. We have $c^i_{kl} = -c^i_{lk}$.

If we derive the functions $\Phi^i = (ab)^i$ with respect to the parameters of b, we obtain a set of equations similar to (1.1), viz.:

$$\frac{\partial (ab)^i}{\partial b^k} = \frac{\partial \Phi^i}{\partial b^k} = \hat{\chi}^i_s(ab)\hat{\psi}^s_k(b) \tag{1.3}$$

where $\hat{\chi}^i_s(a)\hat{\psi}^s_k(a) = \delta^i_k$. Again, the $\hat{\chi}^i_s(a)$ are determined by the values $\hat{\chi}^i_s(e)$, and we choose them such that

$$\hat{\chi}^i_s(e) = \chi^i_s(e). \tag{1.3a}$$

The commutation rules for the $\hat{\chi}^i_s$ are given by

$$\frac{\partial \hat{\chi}^i_k(a)}{\partial a^s}\hat{\chi}^s_l(a) - \frac{\partial \hat{\chi}^i_l(a)}{\partial a^s}\hat{\chi}^s_k(a) = c^r_{lk}\hat{\chi}^i_r(a) \tag{1.4}$$

with the same constants c. (Other relations between the χ^i_s and the $\hat{\chi}^i_s$ will not be discussed here.)

1b. *One parameter subgroups. The Lie algebra of* $\mathfrak{G}$, *and the adjoint group.* A one-parameter subgroup of $\mathfrak{G}$ is defined by an analytic curve $C: a(t)$ on the group manifold $(-\infty < t < \infty)$ on which the relations[4]

$$a(s)a(t) = a(s+t) \tag{1.5}$$

hold for any two values s and t. Clearly $a(0) = e$, and $a(-s) = (a(s))^{-1}$. It follows from (1.5) that the n expressions $\kappa^i = \psi^i_r(a)\dfrac{da^r}{dt}$ $(i = 1, 2, \cdots, n)$ are

[3] If the equations (1.1) hold for two systems $\chi'^i_k(a)$ and $\chi^i_k(a)$, then $\chi'^i_k(a) = \chi^i_s(a)\gamma^s_k$, with constants γ^s_k whose determinant is different from zero.

[4] We include the degenerate case that C reduces to one point, so that $a(t) = e$, for all t.

constant along C, so that we have

(1.6) $$\frac{da^i}{dt} = \kappa^r \chi_r^i(a)$$

with constant κ^r. Conversely, every set of differential equations of the form (1.6)—with arbitrarily chosen constants κ^r—gives rise to a one-parameter subgroup. It should be noted that on C we have likewise

(1.6a) $$\frac{da^i}{dt} = \kappa^r \hat{\chi}_r^i(a)$$

with the same constants κ^r (in consequence of the normalization (1.3a)).

Set $\chi^i(a) = \kappa^r \chi_r^i(a)$, and denote by χ the vector field $(\chi^1(a), \cdots, \chi^n(a))$ on $\mathfrak{G}$. We say, then, that the one-dimensional subgroup $a(t)$ is generated by the *infinitesimal transformation* χ, and we write

(1.7) $$a(t) = \exp(t\chi)$$

If α is a constant different from zero, it is readily seen that

(1.7a) $$\exp(t'\chi') = \exp(t\chi), \quad \chi' = \alpha\chi, \quad t' = \alpha^{-1}t$$

which shows that χ and $\alpha\chi$ generate the same subgroup. If we denote by χ_r the vector field $(\chi_r^1(a), \cdots, \chi_r^n(a))$, we may represent χ as a linear combination (with constant coefficients) of the χ_r, viz.:

(1.8) $$\chi = \kappa^1\chi_1 + \cdots + \kappa^n\chi_n = \kappa^r\chi_r .$$

Therefore, the infinitesimal transformations χ of our group $\mathfrak{G}$ form an n-dimensional vector space $\mathfrak{g}$, the Lie algebra of $\mathfrak{G}$. In fact, if χ and χ' are any two elements of $\mathfrak{g}$, every linear combination $\alpha\chi + \alpha'\chi'$ (with constant coefficients α and α') belongs to $\mathfrak{g}$, and by (1.8) $\mathfrak{g}$ contains precisely n linearly independent elements, e.g., the χ_r ($1 \leqq r \leqq n$). (The independence of the χ_r follows from the fact that the determinant of the χ_r^i does not vanish.)

Moreover, the product, or bracket, $[\chi\chi']$ of two elements χ, χ' of $\mathfrak{g}$ is defined as follows: Let χ^i, χ'^i, and χ''^i be the components of χ, χ', and $\chi'' = [\chi\chi']$ respectively. Then

(1.9) $$\chi''^i = \chi^s \frac{\partial \chi'^i}{\partial a^s} - \chi'^s \frac{\partial \chi^i}{\partial a^s}.$$

The commutation rules (1.2) show that the bracket of any two elements χ and χ' of $\mathfrak{g}$ belongs to $\mathfrak{g}$. More precisely, if $\chi = \kappa^r\chi_r$, $\chi' = \kappa'^r\chi_r$, then $\chi'' = [\chi\chi'] = \kappa''^r\chi_r$, where

$$\kappa''^r = c_{ij}^r \chi'^i \chi^j .$$

In terms of the n basic elements χ_r the commutation rules may be written as follows:

(1.10) $$[\chi_l \chi_k] = c_{kl}^r \chi_r .$$

They characterize the Lie algebra $\mathfrak{g}$.

The bracket $[\chi\chi']$ is antisymmetric in χ and χ', i.e. $[\chi'\chi] = -[\chi\chi']$, it is linear in each factor, and for any three elements χ, χ', χ'' we have Jacobi's identity: $[\chi[\chi'\chi'']] + [\chi'[\chi''\chi]] + [\chi''[\chi\chi']] = 0$.

Let $a(t) = \exp(t\chi)$ $(\chi \in \mathfrak{g})$ be a one parameter subgroup of $\mathfrak{G}$, and b a fixed element of $\mathfrak{G}$. If we set $a'(t) = ba(t)b^{-1}$, then $a'(s)a'(t) = a'(s + t)$. Hence $a'(t)$, too, is a one parameter subgroup of $\mathfrak{G}$, conjugate to $a(t)$, which is generated by some infinitesimal transformation $\chi' \in \mathfrak{g}$. We shall denote χ' by $b\chi b^{-1}$, so that we have

$$b \exp(t\chi)b^{-1} = \exp t(b\chi b^{-1}). \tag{1.11}$$

This equation may be considered as the definition of the symbol $b\chi b^{-1}$. It is known that $b\chi b^{-1}$ is linear in χ, i.e., for any two infinitesimal transformations χ, χ', and any two constants α, α', we have $b(\alpha\chi + \alpha'\chi')b^{-1} = \alpha b\chi b^{-1} + \alpha' b\chi' b^{-1}$. Consequently, for a fixed group element b of $\mathfrak{G}$, the operation $b\chi b^{-1}$ defines a linear transformation of the Lie algebra $\mathfrak{g}$ into itself. Let $\chi = \kappa^r\chi_r$ be an element of $\mathfrak{g}$; then $\chi' = b\chi b^{-1} = \kappa'^r\chi_r$, where $\kappa'^r = s_l^r(b)\kappa^l$. The coefficients $s_l^r(b)$ of this linear transformation may be defined by the equations

$$b\chi_l b^{-1} = s_l^r(b)\chi_r, \qquad 1 \leqq l \leqq n. \tag{1.12}$$

From the linearity of the transformation $b\chi b^{-1}$, and from the relations $e\chi e^{-1} = \chi$, $b'(b\chi b^{-1})b'^{-1} = (b'b)\chi(b'b)^{-1}$ we infer that

$$s_l^r(e) = \delta_l^r, \qquad s_i^r(b')s_l^i(b) = s_l^r(b'b) \tag{1.13}$$

for any two group elements b, b' of $\mathfrak{G}$. The equations (1.13) show that the linear transformations $s_i^r(b)$ form a group, the *adjoint group* of $\mathfrak{G}$, which is a linear representation of $\mathfrak{G}$.

REMARK. As was pointed out in §1a, the functions $\chi_s^i(a)$ are determined by the choice of the $\chi_s^i(e)$ subject to the condition that their determinant be different from zero. It is seen from the foregoing discussion that this amounts to the choice of n linearly independent basic elements $\chi_1, \cdots, \chi_n$ of $\mathfrak{g}$.

1c. *Group invariant integration.* Let $\nu_r(a)$ be a positive continuous function defined on $\mathfrak{G}$. It defines a right invariant group integration if for every continuous function $f(a)$ and every fixed group element b of $\mathfrak{G}$ we have

$$\int_{\mathfrak{G}} \cdots \int f(a)\nu_r(a)\, da^1 \cdots da^n = \int_{\mathfrak{G}} \cdots \int f(ab)\nu_r(a)\, da^1 \cdots da^n, \tag{1.14}$$

provided these two integrals exist. They are extended over the whole group manifold. Once such a function ν_r is found, a theory of right invariant Lebesque integration can be developed, and the equation (1.14) may be extended to measurable functions [cf. Weil]. It is easily inferred from the equations (1.1) that the condition on ν_r is equivalent to the equation $\nu_r(a)\Delta(a) = \text{const}$, where $\Delta(a)$ is the determinant of the $\chi_s^i(a)$. Consequently,

$$\nu_r(a) = \gamma\cdot(\Delta(a))^{-1}, \qquad \Delta(a) = \operatorname{Det}(\chi_s^i(a)) \tag{1.15}$$

where the constant γ is so chosen that $\nu_r(a)$ be positive.

Correspondingly, left invariant group integration is defined by

$$(1.16) \qquad \int_{\mathfrak{G}} \cdots \int f(a)\nu_l(a)\, da^1 \cdots da^n = \int_{\mathfrak{G}} \cdots \int f(ba)\nu_l(a)\, da^1 \cdots da^n$$

with the same conditions on ν and f as above. For $\nu_l(a)$ we find, with a constant γ,

$$(1.17) \qquad \nu_l(a) = \gamma \cdot (\hat{\Delta}(a))^{-1}, \qquad \hat{\Delta}(a) = \text{Det}\,(\hat{\chi}^i_s(a)).$$

The two constants in (1.15) and (1.17) may be chosen equal to each other, since the normalization is arbitrary.

The functions $\nu_r(a)$ and $\nu_l(a)$ coincide if and only if

$$(1.18) \qquad c^i_{ir} = 0 \qquad 1 \leqq r \leqq n.$$

In fact, it may be shown that $\chi^i_r\, \partial/\partial a^i\, (\log \Delta(a) - \log \hat{\Delta}(a)) = c^i_{ir}$. If the n conditions (1.18) are satisfied, the integral $\int_{\mathfrak{G}} \cdots \int f(a)\nu(a)da^1 \cdots da^n$, with $\nu(a) = \nu_r(a) = \nu_l(a)$, is both right and left invariant and it will be shortly denoted by $\int_{\mathfrak{G}} f(a)da$. In what follows we shall only deal with groups for which $\nu_r(a) = \nu_l(a)$.

1d. *Realization of the group* $\mathfrak{G}$ *by transformations of a manifold* $\mathfrak{M}$. Let $\mathfrak{M}$ be a real (or complex) m-dimensional manifold described by the m real (or complex) coordinates $x^1, \cdots, x^m$. Its points will be denoted by $x, y, \cdots$. The group $\mathfrak{G}$ is said to act on $\mathfrak{M}$ if to every $a \in \mathfrak{G}$ a transformation (homeomorphism) $y = ax$ of $\mathfrak{M}$ onto itself is defined, such that $ex = x$, and $b(ax) = (ba)x$ for any two elements a, b of $\mathfrak{G}$. In particular it follows that $a^{-1}(ax) = a(a^{-1}x) = x$.

More specifically, we assume the transformation to be *analytic*, i.e., defined by m analytic functions

$$(1.19) \qquad (ax)^i = Z^i(a^1, \cdots, a^n, x^1, \cdots, x^m) \equiv Z^i(a, x) \qquad 1 \leqq i \leqq m.$$

The partial derivatives of the Z^i with respect to the a^k may be expressed in the form

$$(1.20) \qquad \frac{\partial (ax)^i}{\partial a^k} = \frac{\partial Z^i}{\partial a^k} = \lambda^i_r(ax)\psi^r_k(a) \qquad 1 \leqq i \leqq m, \quad 1 \leqq k \leqq n.$$

The $m \cdot n$ quantities $\lambda^i_r(x)$ are analytic functions of the $x^1, \cdots, x^m$. They characterize the infinitesimal transformations, and they satisfy the commutation rules

$$(1.21) \qquad \frac{\partial \lambda^i_k(x)}{\partial x^j}\, \lambda^j_l(x) - \frac{\partial \lambda^i_l(x)}{\partial x^j}\, \lambda^j_k(x) = c^r_{kl}\lambda^i_r(x) \qquad 1 \leqq i \leqq m, \quad 1 \leqq k, l \leqq n.$$

The transformations $y = ax$ give rise to linear transformations of the functions $f(x)$ over $\mathfrak{M}$. In fact, with a function $f(x)$ we may associate a function $g(y)$

defined by $g(y) = f(x)$, where $y = ax$, i.e., $g(y) = f(a^{-1}y)$. Since g depends on a we write, more specifically,

$$g(a, y) = T(a)f(y) = f(a^{-1}y). \tag{1.22}$$

The operators $T(a)$ defined by this equation are clearly linear in f. Moreover we have

$$T(a)(T(b)f) = T(ab)f, \text{ or } T(a)T(b) = T(ab), \tag{1.23}$$

and $T(e)$ is the unit operator, i.e., $T(e)f = f$. Therefore the $T(a)$ furnish a *linear representation* of the group $\mathfrak{G}$ which will be called the *standard representation*.

If f has continuous second derivatives (or is analytic, in the case of a complex manifold $\mathfrak{M}$) we may obtain the infinitesimal operators related to $T(a)$. By (1.22), $g(a,ay) = f(y)$ is independent of a, and hence we obtain from (1.20)

$$\frac{\partial g(a, x)}{\partial a^k} + \psi_k^r(a)\lambda_r^i(x) \frac{\partial g(a, x)}{\partial x^i} = 0 \qquad 1 \leqq k \leqq n. \tag{1.24}$$

We now introduce the differential operators

$$\Lambda_r \equiv -\lambda_r^j(x) \frac{\partial}{\partial x^j} \qquad 1 \leqq r \leqq n \tag{1.25}$$

and

$$\chi_r \equiv \chi_r^i(a) \frac{\partial}{\partial a^i} \qquad 1 \leqq r \leqq n. \tag{1.26}$$

(Notice that we use for the differential operator (1.26) the same symbol as for the corresponding element of the Lie algebra $\mathfrak{g}$.) More generally, if $\chi = \kappa^r\chi_r$ is an element of $\mathfrak{g}$, we set

$$\chi \equiv \chi^i(a) \frac{\partial}{\partial a^i}, \qquad \chi^i(a) = \kappa^r \chi_r^i(a) \tag{1.27}$$

and

$$\Lambda_\chi \equiv -\lambda_\chi^j(x) \frac{\partial}{\partial x^j}, \qquad \lambda_\chi^j(x) = \kappa^r \lambda_r^j(x). \tag{1.28}$$

We, then, derive from (1.24) the relation

$$\chi g(a, x) = \chi(T(a)f(x)) = \Lambda_\chi g(a, x) = \Lambda_\chi(T(a)f(x)) \tag{1.29}$$

for every $\chi \in \mathfrak{g}$, where χ is the differential operator (1.27).

With the help of the operator χ, the equations (1.20) may be replaced by

$$\chi(ax)^i = \lambda_\chi^i(ax). \tag{1.29a}$$

Let $a(t) = \exp(t\chi)$ be a one parameter subgroup, and denote the operator $T(a(t))$ by T_t. Since $da^i/dt = \chi^i(a)$, it follows from (1.24) that

$$\frac{\partial}{\partial t}(T_t f(x)) = \Lambda_\chi(T_t f(x)). \tag{1.30}$$

Consider the operators (1.27) corresponding to the elements χ, χ' of $\mathfrak{g}$. The linear operator $\chi\chi' - \chi'\chi$ then corresponds to the bracket $[\chi\chi']$ as is seen from (1.9). We may, therefore, define in a consistent way the bracket of any two linear operators A, B as their commutator, i.e.

$$[AB] = AB - BA. \tag{1.31}$$

With this definition we obtain from (1.21), if the operators Λ_k and Λ_l are applied to a function f which has continuous second derivatives (we use the fact that $\partial/\partial x^i \cdot \partial f/\partial x^k = \partial/\partial x^k \cdot \partial f/\partial x^i$),

$$[\Lambda_k \Lambda_l] = c^r_{kl} \Lambda_r . \tag{1.32}$$

By (1.28) it follows that $\Lambda_{(\alpha\chi + \alpha'\chi')} = \alpha\Lambda_\chi + \alpha'\Lambda_{\chi'}$ for any two constants α, α' and any two elements χ, χ' of $\mathfrak{g}$. Comparing (1.32) with (1.10) we may add to this that

$$[\Lambda_\chi \Lambda_{\chi'}] = \Lambda_{[\chi'\chi]} . \tag{1.33}$$

(This relation may also be proved directly from (1.29), cf. §1g, in particular (1.48).)

1e. *Remarks on linear representations of the group* $\mathfrak{G}$. Let the manifold $\mathfrak{M}$ be a real or complex m-dimensional vector space, and the transformations $y = ax$ linear transformations on M. Then $ax = U(a)x$, where $U(a)$ is a matrix depending on the group element a. We have $U(e) = 1$ (unit matrix), $U(a)U(b) = U(ab)$, and $U(a^{-1}) = (U(a))^{-1}$. The $\lambda^i_r(x)$ introduced in §1d must be linear in x, so that $\lambda^i_r(x) = l^i_{jr} x^j$. If L_r denotes the constant matrix with the elements l^i_{jr} we obtain from (1.20) and (1.21)

$$\frac{\partial U(a)}{\partial a^k} = \psi^r_k(a) L_r U(a) \tag{1.34}$$

and

$$[L_k L_l] = L_k L_l - L_l L_k = c^r_{kl} L_r . \tag{1.35}$$

We mention here that the derivatives of $U(a)$ may be expressed in the equivalent form

$$\frac{\partial U(a)}{\partial a^k} = \hat{\psi}^r_k(a) U(a) L_r \tag{1.34a}$$

with the same matrices L_r.

With any element $\chi = \kappa^r \chi_r$ of the Lie algebra $\mathfrak{g}$ we associate the matrix $L_\chi = \kappa^r L_r$. We then obtain a representation of $\mathfrak{g}$ by matrices, i.e., we have

$$L_{(\alpha\chi + \alpha'\chi')} = \alpha L_\chi + \alpha' L_{\chi'} , \quad L_{[\chi\chi']} = [L_{\chi'} L_\chi]. \tag{1.36}$$

The equations (1.34) may be replaced by

$$\chi U(a) = L_\chi U(a) \tag{1.37}$$

where on the left hand side χ denotes the differential operator (1.27). For a one-parameter subgroup $a(t) = \exp(t\chi)$ we find $(dU_t)/dt = L_\chi U_t$, with $U_t = U(a(t))$. Since $U_0 = 1$, the solution of this differential equation is $U_t = \exp(tL_\chi)$, the exponential function being defined by the power series[5] $\exp(tL_\chi) = \sum_{k=0}^{\infty} t^k/k!\, L_\chi^k$. Consider, now, for a fixed element b of G, the subgroup $a'(t) = ba(t)b^{-1} = \exp(t\chi')$, with $\chi' = b\chi b^{-1}$, and the corresponding matrices $U'_t = U(a'(t)) = U(b)U_tU(b)^{-1}$. Then $dU'_t/dt = L_{\chi'}U'_t$, and we immediately obtain

$$L_{\chi'} = U(b)L_\chi U(b)^{-1}, \qquad \chi' = b\chi b^{-1}. \tag{1.38}$$

Applying this equation to χ_l, and using our previous results concerning the adjoint group (cf. (1.12)) we find

$$U(b)L_lU(b)^{-1} = s_l^r(b)L_r \qquad 1 \leqq l \leqq n. \tag{1.39}$$

Later, in constructing the unitary representations of the Lorentz group (cf. §5), we shall extend the equations of this subsection to the case of infinite matrices (operators in Hilbert space). The matrices $U(a)$ and L_χ will be replaced by operators of the type $T(a)$ and Λ_χ respectively. The close analogy of the equations (1.29), (1.33) with (1.37) and (1.36) is evident.

1f. *Multipliers.* In what follows we shall need a certain generalization of the standard representation introduced above (cf. (1.22)). Define, for every function $f(x)$ over the manifold $\mathfrak{M}$ and the group element $a \in \mathfrak{G}$, a tranformation $T(a)f$ by the equation

$$g(a, x) = T(a)f(x) = \mu(a, a^{-1}x)\cdot f(a^{-1}x) \tag{1.40}$$

where $\mu(a, x)$ is a fixed real or complex function of $a \in \mathfrak{G}$ and $x \in \mathfrak{M}$ (i.e. μ is the same for all functions f considered). The $T(a)$ are linear operators, and the question arises under which conditions they furnish a representation of the group $\mathfrak{G}$ such that (1) $T(e)f = f$,(2) $T(a)(T(b)f) = T(ab)f$ for every function f. If $\mu(a, x) = 1$, we are led back to the standard representation, which, in the remainder of this section, will be designated by the superscript 0 ($T^0(a)f = f(a^{-1}x)$). For the function μ we have

$$\mu(e, x) = 1, \qquad \mu(ab, x) = \mu(a, bx)\cdot\mu(b, x). \tag{1.41}$$

The first condition follows from $T(e)f = f$. To obtain the second, set $f_1 = T(b)f$, and $f_2 = T(a)f_1$, for an arbitrary function $f(x)$. From (1.40) we find $f_1(x) = \mu(b, b^{-1}x)\cdot f(b^{-1}x)$, $f_2(y) = \mu(a, a^{-1}y)\cdot f_1(a^{-1}y) = \mu(a, a^{-1}y)\cdot\mu(b, b^{-1}a^{-1}y)\cdot f(b^{-1}a^{-1}y)$. The function $f_2(y) = T(a)(T(b)f)$ coincides with $f_3(y) = T(ab)f = \mu(ab, b^{-1}a^{-1}y)\cdot f(b^{-1}a^{-1}y)$ if and only if the second equation (1.41) holds as is seen by inserting $x = b^{-1}a^{-1}y$, $bx = a^{-1}y$.

DEFINITION 1. *Let the group $\mathfrak{G}$ act on a manifold $\mathfrak{M}$, and denote the transformations of $\mathfrak{M}$ by $y = ax$ ($a \in \mathfrak{G}$, $x, y \in \mathfrak{M}$). A function $\mu(a, x)$ which satisfies the equations $\mu(e, x) = 1$, $\mu(ab, x) = \mu(a, bx)\cdot\mu(b, x)$ is called a multiplier associated*

[5] In this expression the superscript k denotes of course the kth power.

with this group of transformations. The transformations $T(a)f(x) = \mu(a, a^{-1}x) \cdot f(a^{-1}x)$ *of the functions* $f(x)$ *over* $\mathfrak{M}$ *define the corresponding multiplier representation of* $\mathfrak{G}$.

The following properties of multipliers are evident. $\mu(a, x) = 1$ is a multiplier; with any multiplier $\mu(a, x)$ its reciprocal, also, is a multiplier, and with any two multipliers $\mu_1(a, x)$, $\mu_2(a, x)$, their product is a multiplier. This shows that the multipliers themselves form a group with respect to multiplication. It should also be mentioned that any (not necessarily integral) power of a multiplier is again a multiplier provided that the power may be properly defined (which is not always the case for complex multipliers).

On setting $b = a^{-1}$ we find from (1.41)

$$\mu(a, a^{-1}x) \cdot \mu(a^{-1}, x) = 1. \tag{1.42}$$

A special class of multipliers can be constructed in a very simple way. Let $\rho(x)$ be a non-vanishing function over $\mathfrak{M}$. Then

$$\mu(a, x) = \frac{\rho(ax)}{\rho(x)} \tag{1.43}$$

satisfies the conditions (1.41). It is evident that, for all a,

$$T(a)\rho(x) = \rho(x). \tag{1.43a}$$

Conversely, if there exists a non-vanishing function $\rho(x)$ which is invariant under all transformations $T(a)$ of a given multiplier representation, the corresponding multiplier $\mu(a, x)$ is given by (1.43). Moreover, $\mu(a, x) = 1$ whenever $ax = x$.[6]

1g. *Infinitesimal multipliers.* Assuming $\mu(a, x)$ to be analytic in all its variables we next compute $\chi\mu(a, x) = \chi^i(a)\partial/\partial a^i \cdot \mu(a, x)$, where $\chi^i(a) = \kappa^r \chi_r^i(a)$. On replacing in (1.41) a and b by ab and b^{-1} respectively, we obtain $\mu(a, x) = \mu(ab, b^{-1}x) \cdot \mu(b^{-1}, x)$. We now apply the differential operator χ to both sides of this equation. If we denote $\partial\mu(a, x)/\partial a^k$ by $\mu_k(a, x)$ we find

$$\chi\mu(a, x) = \chi^i(a) \frac{\partial(ab)^k}{\partial a^i} \mu_k(ab, b^{-1}x)\mu(b^{-1}, x).$$

By (1.1) $\chi^i(a)\partial(ab)^k/\partial a^i = \chi^k(ab)$. Consequently,

$$\chi\mu(a, x) = \{\chi^k(ab)\mu_k(ab, b^{-1}x)\}\mu(b^{-1}, x). \tag{1.44}$$

In this equation we set $b = a^{-1}$. If we denote $\{\chi\mu(a, x)\}_{a=e}$ by $\tau_\chi(x)$, which for a given element χ of the Lie algebra $\mathfrak{g}$ is a function of x, the curly bracket in (1.44) assumes the value $\tau_\chi(ax)$ for $b = a^{-1}$, and we finally obtain

$$\chi\mu(a, x) = \tau_\chi(ax) \cdot \mu(a, x) \qquad \tau_\chi = \{\chi\mu(a, x)\}_{a=e}. \tag{1.45}$$

The function $\tau_\chi(x)$ is called the *infinitesimal multiplier* associated with $\mu(a, x)$, and corresponding to the element χ of $\mathfrak{g}$. The infinitesimal multiplier corresponding to χ_k will be denoted by $\tau_k(x)$.

[6] Multipliers will be further discussed in Part II of this paper.

Infinitesimal transformations of the multiplier representation. We assume here that the functions $f(x)$ under consideration have continuous second derivatives (or are analytic in the case of a complex manifold $\mathfrak{M}$). To obtain the expression $\chi(T(a)f(x))$ we set $T(a)f(x) = \mu(a, a^{-1}x)\cdot T^0(a)f(x)$, so that $\chi(T(a)f(x)) = \{\chi\mu(a, a^{-1}x)\}\cdot T^0(a)f(x) + \mu(a, a^{-1}x)\cdot(T^0(a)f(x))$. If the infinitesimal operator of the standard representation $-\lambda_\chi^i(x)\partial/\partial x^i$ (cf. (1.28)) is denoted by Λ_χ^0, we have $\chi(T^0(a)f(x)) = \Lambda_\chi^0(T^0(a)f(x)$, and hence

$$\chi(T(a)f(x)) = \{\chi\mu(a, a^{-1}x)\}\cdot T^0(a)f(x) + \mu(a, a^{-1}x)\Lambda_\chi^0(T^0(a)f(x)).$$

Furthermore, by (1.45) and (1.29) $\chi\mu(a, a^{-1}x) = \tau_\chi(x)\cdot\mu(a, a^{-1}x) + \Lambda_\chi^0\mu(a, a^{-1}x)$. Consequently,

$$\chi(T(a)f(x)) = \tau_\chi(x)\cdot T(a)f(x) + \Lambda_\chi^0\{\mu(a, a^{-1}x)T^0(a)f(x)\}$$

or

$$\chi(T(a)f(x) = \Lambda_\chi(T(a)f(x)). \tag{1.46}$$

Λ_χ is an *infinitesimal operator* of the multiplier representation. It only involves the coordinates of the point x, and it is defined by

$$\Lambda_\chi f(x) = \tau_\chi(x)\cdot f(x) + \Lambda_\chi^0\cdot f(x), \qquad \Lambda_\chi^0 = -\lambda_\chi^i\frac{\partial}{\partial x^i}. \tag{1.47}$$

The operator corresponding to χ_k will be denoted by $\Lambda_k = \tau_k + \Lambda_k^0$.

Clearly $\Lambda_{(\alpha\chi+\alpha'\chi')} = \alpha\Lambda_\chi + \alpha'\Lambda_{\chi'}$ for any two constants α, α' and any two elements χ, χ' of $\mathfrak{g}$. To obtain the operator corresponding to the bracket $[\chi'\chi]$ we first apply χ' to the equation (1.46). The operators χ' and Λ_χ commute, since χ' operates on the a^i only, and hence $\chi'\chi(T(a)f) = \chi'\{\Lambda_\chi(T(a)f)\} = \Lambda_\chi\{\chi'(T(a)f)\} = \Lambda_\chi\Lambda_{\chi'}(T(a)f)$. Similarly, $\chi\chi'(T(a)f) = \Lambda_{\chi'}\Lambda_\chi(T(a)f)$. Therefore we have

$$(\chi'\chi - \chi\chi')(T(a)f) = (\Lambda_\chi\Lambda_{\chi'} - \Lambda_{\chi'}\Lambda_\chi)(T(a)f)$$

which may be written as

$$[\Lambda_\chi\Lambda_{\chi'}] = \Lambda_{[\chi'\chi]}. \tag{1.48}$$

This equation has the same form as the equation (1.33) previously derived for the standard representation. On setting $\chi = \chi_k$, $\chi' = \chi_l$, we obtain the analogue of (1.32).

By a straightforward computation we find from (1.47)

$$\Lambda_{[\chi'\chi]}f \equiv (\Lambda_{[\chi'\chi]}^0 + \tau_{[\chi'\chi]})f = [\Lambda_\chi^0\Lambda_{\chi'}^0]f + (\Lambda_\chi^0\tau_{\chi'} - \Lambda_{\chi'}^0\tau_\chi)\cdot f.$$

Since $\Lambda_{[\chi'\chi]}^0 = [\Lambda_\chi^0\Lambda_{\chi'}^0]$ (cf. (1.33)), $\tau_{[\chi'\chi]}$ is given by

$$\tau_{[\chi'\chi]} = \Lambda_\chi^0\tau_{\chi'} - \Lambda_{\chi'}^0\tau_\chi \tag{1.49}$$

where $\Lambda_\chi^0\tau_{\chi'}$ equals $-\lambda_\chi^i\partial\tau_{\chi'}(x)/\partial x^i$ etc. In particular, for $\chi = \chi_k$, $\chi' = \chi_l$

$$\Lambda_k^0\tau_l - \Lambda_l^0\tau_k = c_{kl}^r\tau_r. \tag{1.49a}$$

Additional remarks. (1) If μ is a multiplier, and τ_X the associated infinitesimal multiplier, then any power of μ, say μ^h, has the infinitesimal multiplier $h\tau_X$ (cf. (1.45)).

(2) Let μ, μ' be any two multipliers and let τ_X, τ'_X be the associated infinitesimal multipliers. The product $\mu'' = \mu\cdot\mu'$ has the infinitesimal multiplier $\tau''_X = \tau_X + \tau'_X$ (cf. (1.45)).

(3) Let $\mu(a, x)$ be a multiplier of the special form $\rho(ax)/\rho(x)$ (cf. (1.43)), where $\rho(x)$ is an analytic function. Then $T(a)\rho(x) = \rho(x)$, and hence $\chi(T(a)\rho(x)) = 0$. This implies that $\Lambda_X(T(a)\rho(x)) = \Lambda_X\rho(x) = 0$. Consequently, by (1.47) the infinitesimal multiplier τ_X may be expressed as

$$\tau_X(x) = -(\rho(x))^{-1}\Lambda_X^0\rho(x). \tag{1.50}$$

Moreover, for any differentiable function $g(x)$, $\Lambda_X(\rho\cdot g) = (\Lambda_X\rho)\cdot g + \rho\cdot\Lambda_X^0 g = \rho\Lambda_X^0 g$, so that, with $f = \rho\cdot g$,

$$\Lambda_X f(x) = \rho(x)\Lambda_X^0(\rho(x)^{-1}f(x)). \tag{1.51}$$

This is the infinitesimal analogue of the relation

$$T(a)f(x) = \rho(x)T^0(a)(\rho(x)^{-1}f(x)). \tag{1.52}$$

1h. *A method of constructing multipliers.* Let $\mathfrak{G}$ act on a real or a complex m-dimensional manifold $\mathfrak{M}$, where $m > 1$, and assume that coordinates $x^1, \cdots, x^m$ may be so chosen on $\mathfrak{M}$ that the (real or complex) variable x^m does not assume the value zero and that furthermore the transformation $y = ax$ is described by equations of the following form (cf. (1.19))

$$\begin{cases} (ax)^i = Z^i(a, x) = Z^i(a, x^1, \cdots, x^{m-1}) & 1 \leqq i \leqq m-1 \\ (ax)^m = Z^m(a, x) = x^m\cdot\mu(a, x^1, \cdots, x^{m-1}). \end{cases} \tag{1.53}$$

The first $(m - 1)$ equations (1.53) define a group of transformations of the variables x^i $(1 < i < m - 1)$ among themselves, i.e., a group of transformations of a manifold $\mathfrak{M}^*$ defined by points x^* with the coordinates $x^1, \cdots, x^{m-1}$. The function $\mu(a, x^*)$ in the last equations (1.53) is a multipier for this transformation group, because the group properties $ex = x$, and $a(bx) = (ab)x$ imply the relations $\mu(e, x^*) = 1$, and $\mu(ab, x^*) = \mu(a, bx^*)\cdot\mu(b, x^*)$ as is readily verified. (Conversely, if a multiplier $\mu(a, x^*)$ is defined on a manifold $\mathfrak{M}^*$, we may construct a manifold $\mathfrak{M}$ (a fiber space over $\mathfrak{M}^*$) by including a new variable x^m, and define a transformation group on $\mathfrak{M}$ by the equations (1.53).)

If $y = ax$, we may write

$$y^* = ax^*, \quad y^m = x^m\cdot\mu(a, x^*). \tag{1.54}$$

Since the functions $\lambda_r^i(y)$ (cf. (1.20)) may be defined as $\chi_r y^i = \chi_r(ax)^i$ $(1 \leqq i \leqq m)$ the following may be inferred from (1.54): (1) If $1 \leqq i \leqq m - 1$, λ_r^i is independent of y^m, and hence a function of y^*. (2) $\lambda_r^m = \chi_r y^m = x^m\cdot\chi_r\mu(a, x^*) = y^m\cdot(\mu(a, x^*))^{-1}\cdot\chi_r\mu(a, x^*)$. Since λ_r^m is a function of y^m and y^*, it is therefore of the form $y^m \cdot \tau_r(y^*)$, and we obtain $\chi_r\mu(a, x^*) = \tau_r(ax^*)\cdot\mu(a, x^*)$ in accordance with (1.45). For any element $\chi = \kappa^r\chi_r$ of $\mathfrak{g}$, we finally get

$$\chi y^i = \lambda^i_\chi(y^*)\ (1 \leqq i \leqq m-1); \qquad \chi y^m = y^m\cdot\tau_\chi(y^*). \tag{1.55}$$

with $\lambda^i = \kappa^r\lambda^i_r$, $\tau_\chi = \kappa^r\tau_r$.

Multiplier representations on $\mathfrak{M}^*$. We at once construct the multiplier representation on $\mathfrak{M}^*$ which corresponds to the h^{th} power of $\mu(a, x^*)$. Its transformations will be denoted by $T(a)$, while the transformations of the standard representation on the manifold $\mathfrak{M}$ will be denoted by $T^0(a)$. On applying $T^0(a)$ to a function $F(x)$ over $\mathfrak{M}$ of the particular form

$$F(x) = (x^m)^{-h}f(x^1, \cdots, x^{m-1}) = (x^m)^{-h}f(x^*) \tag{1.56}$$

we obtain

$$T^0(a)F(x) = F(a^{-1}x) = \{(a^{-1}x)^m\}^{-h}\cdot f(a^{-1}x^*) = (x^m)^{-h}(\mu(a^{-1}, x^*))^{-h}f(a^{-1}x^*).$$

From (1.42) we therefore find

$$T^0(a)F(x) = (x^m)^{-h}(\mu(a, a^{-1}x^*))^h\cdot f(a^{-1}x^*) = (x^m)^{-h}T(a)f(x^*). \tag{1.57}$$

The application of the differential operator χ to the preceding equation leads to

$$\chi(T^0(a)F(x)) = (x^m)^{-h}\chi(T(a)f(x^*)).$$

On the other hand, we have $\chi(T^0(a)F(x)) = \Lambda^0_\chi(T^0(a)F(x))$ and $\chi(T(a)f(x^*)) = \Lambda_\chi(T(a)f(x^*))$, which implies $\Lambda^0_\chi(T^0(a)F(x)) = (x^m)^{-h}\Lambda_\chi(T(a)f(x^*))$. For $a = e$ we have

$$\Lambda_\chi f(x^*) = (x^m)^h\cdot\Lambda^0_\chi((x^m)^{-h}f(x^*)). \tag{1.58}$$

Here $\Lambda^0_\chi \equiv -(\sum_{j=1}^{m-1}\lambda^j_\chi(x^*)\partial/\partial x^j) - x^m\cdot\tau_\chi(x^*)\partial/\partial x^m$ (cf. (1.55)) is an infinitesimal transformation of the standard representation on $\mathfrak{M}$, and consequently $\Lambda_\chi \equiv -(\sum_{j=1}^{m-1}\lambda^j_\chi(x^*)\partial/\partial x^j) + h\tau_\chi(x^*)$ in accordance with (1.47). It is also seen that

$$\Lambda_{\chi'}\Lambda_\chi f(x^*) = (x^m)^h\Lambda^0_{\chi'}\Lambda^0_\chi((x^m)^{-h}f(x^*)). \tag{1.58a}$$

These relations will be used in the sequel.

Remark on projective transformations. Denote the coordinates on $\mathfrak{M}$ by ξ^i $(1 \leqq i \leqq m)$ and assume that $y = ax$ is a linear transformation of $\mathfrak{M}$ into itself given by $\eta^i = w^i_j(a)\xi^j$, η^i being the coordinates of y. If, on $\mathfrak{M}$, ξ^m is different from zero, we may introduce new variables

$$x^i = \xi^i/\xi^m\ (1 \leqq i \leqq m-1), \qquad x^m = \xi^m.$$

With these variables we have

$$\begin{cases} (ax)^i = \{\sum_{j=1}^{m-1} w^i_j(a)x^j + w^i_m(a)\}\mu(a, x^*)^{-1} & 1 \leqq i \leqq m-1 \\ (ax)^m = x^m\cdot\mu(a, x^*), & \mu(a, x^*) = \sum_{j=1}^{m-1} w^m_j(a)x^j + w^m_m(a). \end{cases} \tag{1.59}$$

The first $(m - 1)$ equations (1.59) define a group of projective transformations of the $(m - 1)$ variables $x^1, \cdots, x^{m-1}$, and $\mu(a, x^*)$ is evidently a multiplier.

1i. *Invariant densities.* Let $\mathfrak{G}$ act on a real m-dimensional manifold $\mathfrak{M}$, let $\mu(a, x)$ be a multiplier, and let $T(a)$ be the transformations of the corresponding multiplier representation. Denote, for a continuous function $f(x)$, by $I(a)$ the integral

$$I(a)[f] = \int_{\mathfrak{M}} (T(a)f(x))\omega(x)\, dX$$

extended over the whole manifold $\mathfrak{M}$, where $dX = dx^1 \cdots dx^m$, and where ω is a fixed positive continuous function. If for every f for which all $I(a)[f]$ exist $I(a)[f]$ is independent of the group element a, $\omega(x)$ will be called an *invariant density* with respect to the multiplier representation under consideration. (Clearly such a function $\omega(x)$ also defines an invariant Lebesque integration on $\mathfrak{M}$.)

We introduce the variables $y^i = (a^{-1}x)^i$, so that $T(a)f(x) = \mu(a, y)f(y)$, and we denote by $J_a(y)$ the Jacobian $\partial(x^1, \cdots, x^m)/\partial(y^1, \cdots, y^m)$. Then

$$I(a)[f] = \int_{\mathfrak{M}} f(y)\mu(a, y)\omega(ay)J_a(y)\, dY.$$

Consequently, $\omega(x)$ is an invariant density if

$$\eta(a, y) \equiv \mu(a, y)\omega(ay)J_a(y) = \omega(y) \tag{1.60}$$

for all a and all y.

If ω is a differentiable function (1.60) may be replaced by the condition that the partial derivatives of $\eta(a, y)$ with respect to all a^i vanish, or by the condition that $\chi_r\eta(a, y) = 0$ $(1 \leqq r \leqq n)$. To compute $\chi_r\eta(a, y)$ we remember that $\chi_r\, \mu(a, y) = \tau_r(ay)\cdot\mu(a, y) = \tau_r(x)\cdot\mu(a, y)$, and that $\chi_r\omega(ay) = \partial\omega(x)/\partial x^i\, \lambda_r^i(x)$ (cf. (1.29a)). Finally,

$$\chi_r J_a(y) = J_a(y)\left\{\frac{\partial y^i}{\partial x^j}\left(\chi_r \frac{\partial x^j}{\partial y^i}\right)\right\} = J_a(y)\left\{\frac{\partial y^i}{\partial x^j}\cdot\frac{\partial}{\partial y^i}(\chi_r\, x^j)\right\} = J_a(y)\cdot\frac{\partial}{\partial x^j}(\lambda_r^j(x)).$$

(Cf. (1.29a))[7]. Collecting terms, we have $\chi_r\eta(a, y) = \mu(a, y)\cdot J_a(y)\cdot \{\partial/\partial x^i(\omega(x)\lambda_r^i(x)) + \tau_r(x)\omega(x)\}$. Therefore a positive differentiable function $\omega(x)$ is an invariant density if and only if

$$\mathbf{M}_r\omega(x) = 0 \qquad 1 \leqq r \leqq n \tag{1.61}$$

where $\mathbf{M}_rf(x) = \partial/\partial x^i(\lambda_r^i(x)f(x)) + \tau_r(x)f(x)$. It may be shown that

$$\mathbf{M}_l\mathbf{M}_k - \mathbf{M}_k\mathbf{M}_l = c^r_{k\,l}\mathbf{M}_r \tag{1.62}$$

if these operators are applied to a function with continuous second derivatives.

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[7] To apply (1.29a) the variables x and y must be interchanged.

§2. The infinitesimal transformations of the Lorentz and rotation groups

2a. *Linear transformations which leave a quadratic form invariant.* We consider here the group of linear tranformations on an m-dimensional real vector space $\mathfrak{M}$ which leave a non-singular quadratic form invariant. (By specializing the dimension m and the quadratic form in question we shall later on obtain relations for the Lorentz group as well as the orthogonal group in three variables.)

More specifically, we discuss the (connected) component of the group which contains the identity. It will be denoted by $\mathfrak{G}$. $\mathfrak{G}$ is an n-dimensional connected Lie group, where $n = \frac{1}{2}m(m - 1)$. (In the case of the Lorentz group, $\mathfrak{G}$ consists of all Lorentz transformations of determinant 1 which do not reverse the direction of time (proper Lorentz transformations). In the case of the orthogonal group, $\mathfrak{G}$ consists of all transformations of determinant 1, i.e., of all rotations.)

The elements of $\mathfrak{G}$ will be called $a, b, \cdots$. We shall not yet introduce any specific parameters on the group manifold (this will be done in later sections). The following notation is used. A point x of $\mathfrak{M}$ has the coordinates $x^1, \cdots, x^m$. The quadratic form is given by $g_{ij}x^ix^j$, where $g_{ij} = g_{ji}$. The contravariant symmetric tensor g^{ij} is defined by the equations $g^{ij}g_{kj} = \delta^i_k$; g_{ij} and g^{ij} are used to lower and to raise indices. In a suitable coordinate system g_{ij} is diagonal, with elements ± 1, i.e.

$$(2.1) \qquad g_{11} = \eta_1, \; g_{22} = \eta_2, \; \cdots, \; g_{mm} = \eta_m ; \; g_{ij} = 0 \text{ if } i \neq j, \; \eta_i = \pm 1.$$

We apply here the results of §1e. To every group element a of $\mathfrak{G}$ corresponds a transformation $y = ax = U(a)x$, where $U(a)$ is a matrix with the elements $w^i{}_j(a)$. We then have

$$(2.2) \qquad y^i = w^i{}_j(a)x^j.$$

The conditions for $g_{ij}x^ix^j$ to remain invariant under the transformation (2.2) may be written in the two equivalent forms

$$(2.3) \qquad w^i{}_j(a)w_k{}^j(a) = \delta^i_k ; \qquad w_j{}^i(a)w^j{}_k(a) = \delta^i_k$$

where $w_k{}^j = g_{ki}g^{jl}w^i{}_l$. Every element χ of the Lie algebra $\mathfrak{g}$ of the group $\mathfrak{G}$ gives rise to an infinitesimal transformation L_χ with the constant matrix elements $l^i{}_j$ (cf. (1.37))[8] so that

$$(2.4) \qquad \chi y = L_\chi y, \qquad \chi y^i = l^i{}_j y^j.$$

Since, for every a, $g_{ij}y^iy^j = g_{ij}x^ix^j$, it follows that $\chi(g_{ij}y^iy^j) = 2g_{ij}(\chi y^j)y^i = 2g_{ij}l^i{}_ky^ky^j = 2l_{jk}y^ky^j = 0$. Consequently,

$$(2.5) \qquad l_{ij} + l_{ji} = 0.$$

Conversely, it is readily seen that every matrix L whose elements $l^i{}_j$ satisfy the relations (2.5) corresponds to an element χ of $\mathfrak{g}$.

[8] For convenience we suppress here the subscript χ.

Every L_X may be expressed as a linear combination of the transformations L_{kl} defined as follows:

$$(L_{kl}y)^i = (\delta^i_k g_{lj} - \delta^i_l g_{kj})y^j = \delta^i_k y_l - \delta^i_l y_k. \tag{2.6}$$

(1) L_{kl} is of the required form, because its covariant matrix element (i, j) is equal to $g_{ik}g_{lj} - g_{il}g_{kj}$, and consequently (2.5) is satisfied. (2) If a matrix L has antisymmetric covariant matrix elements l_{kl}, it may be expressed as $L = \frac{1}{2}l^{kl}L_{kl}$. Since, by the definition (2.6), $L_{lk} = -L_{kl}$, only $\frac{1}{2}m(m - 1)$ of the L_{kl} are linearly independent—for example those for which $k < l$. The corresponding χ_{kl} may be chosen as a basis of the Lie algebra $\mathfrak{g}$. The use of double indices for characterizing this basis seems to be more appropriate, although this does not quite correspond with the notation used in §1 (cf., however, §2e below). In what follows we shall mostly use all L_{kl} without restricting the indices k, l in any way.

From (2.6) we immediately obtain the commutation rules

$$[L_{ij}L_{kl}] = g_{jk}L_{il} - g_{ik}L_{jl} + g_{il}L_{jk} - g_{jl}L_{ik}. \tag{2.7}$$

These equations define (implicitly) the structure constants of $\mathfrak{G}$. A simple discussion shows that the analogue of (1.18) holds, i.e., that on $\mathfrak{G}$ right and left invariant group integration coincide.

Operators of the standard representation. Let $T^0(a)f(x) = f(a^{-1}x)$, then $\chi(T^0(a)f(x)) = \Lambda^0_\chi(T^0(a)f(x))$, where

$$\Lambda^0_\chi = -l^i{}_j\, x^j \frac{\partial}{\partial x^i} \tag{2.8}$$

if L is defined by (2.4). (Cf. (1.28). For χ_{kl} we have therefore

$$\Lambda^0_{kl} = x_k \frac{\partial}{\partial x^l} - x_l \frac{\partial}{\partial x^k}. \tag{2.8a}$$

One parameter subgroups. If the tensor g_{ij} is chosen in diagonal form it is very simple to obtain explicit expressions for the transformations of the one parameter subgroup $y(t) = \exp(tL_{kl})x$. We distinguish the two cases $\eta_k = \eta_l = \eta$, and $\eta_k = -\eta_l = \eta$.

(1) For $\eta_k = \eta_l = \eta$, we find

$$y^k(t) = \cos t\, x^k + \eta \sin t\, x^l,\quad y^l(t) = -\eta \sin t\, x^k + \cos t\, x^l, \qquad y^i(t) = x^i \quad (i \neq k, l). \tag{2.9}$$

(2) For $\eta_k = -\eta_l = \eta$, we find

$$y^k(t) = \cosh t\, x^k - \eta \sinh t\, x^l,\quad y^l(t) = -\eta \sinh t\, x^k + \cosh t\, x^l, \qquad y^i(t) = x^i \; (i \neq k, l). \tag{2.9a}$$

It is seen that L_{kl} generates transformations of the $(k - l)$ plane into itself.

2b. *The adjoint group of* $\mathfrak{G}$. By (1.38), $L_{\chi'} = U(b)L_\chi U(b)^{-1}$, if $\chi' = b\chi b^{-1}$.

Denoting the matrix elements of $L_{\chi'}$ and L_χ by $l'^i{}_j$ and $l^i{}_j$ respectively, we therefore have

$$l'^i{}_j = w^i{}_k(b)l^k{}_r w_j{}^r(b)$$

or

$$l'^{ij} = w^i{}_k(b)w^j{}_r(b)l^{kr} \tag{2.10}$$

i.e., the l^{kr} transform as the components of a skewsymmetric tensor. The corresponding equations for the χ_{ij} are

$$b\chi_{ij}b^{-1} = w^k{}_i(b)w^l{}_j(b)\chi_{kl}. \tag{2.10a}$$

2c. *The operators Q and R.* Let L_{kl} be a set of linear operators so chosen that $L_{kl} = -L_{lk}$ and that the commutation rules (2.7) hold, and let

$$Q = \tfrac{1}{2}L_{kl}L^{kl} = \tfrac{1}{2}g^{ki}g^{lj}L_{kl}L_{ij}. \tag{2.11}$$

(The precise nature of these operators is immaterial, provided that they can be multiplied without restrictions.) Then Q commutes with all L_{ij}, as is readily verified. (Q is Casimir's operator which can be constructed for every semi-simple group [Casimir].)

In the case of a four-dimensional manifold $\mathfrak{M}$ a second *invariant* operator (i.e., an operator commuting with all L_{ij}) exists, viz.

$$R = \tfrac{1}{4}\eta^{ijkl}L_{ij}L_{kl} \tag{2.11a}$$

where η^{jikl} is antisymmetric in any two indices, and $\eta^{1234} = 1$.

The two operators Q and R (the latter in the case of $\mathfrak{L}_4$) will later serve to characterize the irreducible representations of $\mathfrak{L}_3$ and $\mathfrak{L}_4$.

2d. *The Lorentz group* $\mathfrak{L}_4$ ($m = 4$). The coordinates will be denoted by x^0, x^1, x^2, x^3; the metric tensor g_{ij} has the components $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, $g_{ij} = 0$ ($i \neq j$). Since we consider only *proper* Lorentz transformations the determinant of the $w^i{}_j$ is equal to 1, and $w^0{}_0 > 0$. The operators Q and R are given by

$$Q = (L_{23})^2 + (L_{31})^2 + (L_{12})^2 - (L_{01})^2 - (L_{02})^2 - (L_{03})^2 \tag{2.12}$$

$$R = L_{23}L_{10} + L_{31}L_{20} + L_{12}L_{30}. \tag{2.12a}$$

2e. *The case* $m = 3$. We now turn to a more detailed discussion of the case $m = 3$ which will be treated in the first part of this paper. The well known equivalence of vectors and antisymmetric tensors in three-dimensional vector algebra (with respect to the transformations of the group $\mathfrak{G}$) may be used for the following simplification. We define the infinitesimal tranformations χ_r and L_r by the equations

$$\chi_r = \tfrac{1}{2}\eta_{rkl}g^{kj}g^{li}\chi_{ij}, \quad L_r = \tfrac{1}{2}\eta_{rkl}L^{kl} \tag{2.13}$$

where η_{rkl} is antisymmetric in any two indices, $\eta_{123} = 1$, and $L^{kl} = g^{ki}g^{lj}L_{ij}$.

Hence

$$L_1 = L^{23}, L_2 = L^{31}, L_3 = L^{12} \tag{2.14}$$

and similarly for the χ_r. Correspondingly we write

$$\chi = \kappa^r \chi_r, \ L_\chi = \kappa^r L_r$$

with $\kappa^1 = l_{23}$, $\kappa^2 = l_{31}$, $\kappa^3 = l_{12}$. (This is in accordance with the notation used in §1. The χ_r form a base for the Lie algebra $\mathfrak{g}$.) Instead of (2.4) we now may write

$$(Lx)_1 = \kappa^3 x^2 - \kappa^2 x^3, \ (Lx)_2 = \kappa^1 x^3 - \kappa^3 x^1, \ (Lx)_3 = \kappa^2 x^1 - \kappa^1 x^2. \tag{2.15}$$

The transformations of the adjoint group may be expressed as follows. Let $\chi' = b\chi b^{-1}$, where $\chi = \kappa^r \chi_r$, and $\chi' = \kappa'^r \chi_r$. Then

$$\kappa'^i = w^i{}_j(b)\kappa^j. \tag{2.16}$$

Furthermore,

$$b\chi_j b^{-1} = w^i{}_j(b)\chi_i. \tag{2.17}$$

This is true because the transformations $w^i{}_j$ considered have the determinant 1.

Finally, we shall specify these relations for the two groups which we have to discuss, viz., the group $\mathfrak{R}$ of rotations, and the Lorentz group $\mathfrak{L}_3$.

The group $\mathfrak{R}$. The metric tensor will be chosen as $g_{11} = g_{22} = g_{33} = -1$, $g_{ij} = 0$ (if $i \neq j$). (The negative sign is chosen because we consider here $\mathfrak{R}$ as a subgroup of $\mathfrak{L}_4$ for which the choice $g_{00} = 1$ is the most convenient.) From (2.7) we obtain

$$[L_1 L_2] = L_3, \ [L_2 L_3] = L_1, \ [L_3 L_1] = L_2. \tag{2.18}$$

By (2.16) any two one-parameter subgroups of $\mathfrak{R}$ are conjugate if they are generated by non-vanishing χ. Since χ and $\alpha\chi$ ($\alpha \neq 0$) generate the same subgroup (cf. (1.7a)), only the *direction* of the vector (κ^r) matters, and by a suitable rotation any direction may be transformed into a given direction.

For the operator Q we find

$$Q = (L_1)^2 + (L_2)^2 + (L_3)^2 = -g^{ij}L_i L_j \tag{2.19}$$

(except for a constant factor, the well known expression for the angular momentum in Quantum Mechanics.)

The group $\mathfrak{L}_3$. We use the coordinates x^0, x^1, x^2, replacing the index 3 by the index 0. The metric tensor is then given by $g_{00} = 1$, $g_{11} = g_{22} = -1$, $g_{ij} = 0$ (if $i \neq j$). The equations (2.14) and (2.15) are then replaced by

$$L_0 = L^{12}, \ L_1 = L^{20}, \ L_2 = L^{01} \tag{2.20}$$

and

$$(Lx)_0 = \kappa^2 x^1 - \kappa^1 x^2, \ (Lx)_1 = \kappa^0 x^2 - \kappa^2 x^0, \ (Lx)_2 = \kappa^1 x^0 - \kappa^0 x^1. \tag{2.21}$$

The commutation rules are

$$[L_0L_1] = L_2\,,\ [L_1L_2] = -L_0\,,\ [L_2L_0] = L_1\,. \tag{2.22}$$

(Note the minus sign in the second equation which is due to the indefinite metric.)

Since the transformation (2.16) preserves the expression $g_{ij}\kappa^i\kappa^j = \kappa^i\kappa_i$, we now have *three* different classes of one-parameter subgroups for non-vanishing χ corresponding to the three cases $\kappa^i\kappa_i > 0$, $\kappa^i\kappa_i < 0$, and $\kappa^i\kappa_i = 0$. Within each class any two subgroups are conjugate, because with a suitably chosen α, any space-like (time-like or null) vector (κ^r) can be transformed into a given space-like (time-like or null) vector $(\alpha\kappa'^r)$ by some transformation of the group $\mathfrak{L}_3$. These three classes will be respectively called *elliptic* ($\kappa^i\kappa_i > 0$), *hyperbolic* ($\kappa^i\kappa_i < 0$), and *parabolic* ($\kappa^i\kappa_i = 0$) subgroups. (Cf. §4f.)

The operator Q is given by

$$Q = (L_0)^2 - (L_1)^2 - (L_2)^2 = g^{ij}L_iL_j\,. \tag{2.23}$$

§3. The spinor groups

In this section we briefly discuss the spinor groups which correspond to $\mathfrak{L}_4$, $\mathfrak{L}_3$ and $\mathfrak{R}$, and which will be denoted by $\mathfrak{S}_4$, $\mathfrak{S}_3$, and $\mathfrak{S}_R$ respectively. (For this section c.f. [v.d. Waerden, §16, §20].)

3a. *The group* $\mathfrak{S}_4$. Let x be a point of a four-dimensional vector space with the coordinates x^0, x^1, x^2, x^3. With x we associate the Hermitian second order matrix (where $i = \sqrt{-1}$.

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{3.1}$$

Conversely, every Hermitian matrix may be written in this form (with *real* x^i). The determinant of X, which we denote by $D(X)$, has the value $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = g_k{}^l x^k x^l$. If the matrix

$$W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.2}$$

with complex elements $\alpha, \beta, \gamma, \delta$, has the determinant 1, and if

$$W^* = \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}$$

is its Hermitian adjoint, then

$$Y = WXW^*, \qquad Y = \begin{pmatrix} y^0 + y^3 & y^1 - iy^2 \\ y^1 + iy^2 & y^0 - y^3 \end{pmatrix} \tag{3.3}$$

defines a linear transformation, $y^k = w^k{}_l x^l$, with *real* coefficients $w^i{}_j$, for which $D(Y) = D(X)$, i.e., a *real* Lorentz transformation. Moreover, the determinant of the $w^i{}_j$ is equal to one, and $w^0{}_0 = \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}) \geqq 1$ (since $\alpha\delta - \beta\gamma = 1$). Hence this transformation is a *proper* Lorentz transformation, i.e., an element of the group $\mathfrak{L}_4$.

The matrices W (3.2) with determinant 1 form a six-dimensional connected Lie group, $\mathfrak{S}_4$, (we count real dimensions), whose elements will be denoted by $a, b, \cdots$. We have seen that to every element $a \in \mathfrak{S}_4$ corresponds an element, say a', of $\mathfrak{L}_4$. It is readily verified that the mapping $a \rightarrow a'$ has the following properties: (1) $e \rightarrow e$, (2) $a^{-1} \rightarrow (a')^{-1}$, (3) $ab \rightarrow a'b'$. This mapping is *two-to-one*. Every proper Lorentz transformation may be represented in the form (3.3), however W and $(-W)$ (but no other matrix) give rise to the same Lorentz transformation. Let us denote the elements of $\mathfrak{S}_4$ which correspond to W and $(-W)$ by a and $(-a)$ respectively. Then a and $(-a)$ are mapped into the same element a' of $\mathfrak{L}_4$. More specifically, $\mathfrak{L}_4$ is isomorphic to the factor group $\mathfrak{S}_4/\mathfrak{N}$, where $\mathfrak{N}$ is the invariant subgroup of $\mathfrak{S}_4$ consisting of the two elements e and $(-e)$.

The two groups $\mathfrak{L}_4$ and $\mathfrak{S}_4$ are *locally isomorphic* (in a suitably chosen neighborhood of the unit element), and therefore their Lie algebras are the same. (It is easily seen that this isomorphism is *analytic*.)

3b. *The group* $\mathfrak{S}_3$. The group $\mathfrak{L}_3$ may be considered as the subgroup of all transformations of $\mathfrak{L}_4$ which leave the variable x^3 invariant (so that $y^3 = x^3$). The group of all matrices W which correspond to these is the group $\mathfrak{S}_3$. By (3.3) we have to require that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}.$$

Since $\alpha\delta - \beta\gamma = 1$, this is equivalent to $\delta = \bar{\alpha}$, $\gamma = \bar{\beta}$. We are, then, dealing with matrices of the form

$$W = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha\bar{\alpha} - \beta\bar{\beta} = 1. \tag{3.4}$$

The variable x^3 may now be omitted, and we may set

$$X = \begin{pmatrix} x^0 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 \end{pmatrix},$$

the equations (3.3) remaining unchanged if W is of the form (3.4).

The relation between $\mathfrak{L}_3$ and $\mathfrak{S}_3$ is the same as that between $\mathfrak{L}_4$ and $\mathfrak{S}_4$, i.e., $\mathfrak{L}_3$ is isomorphic with $\mathfrak{S}_3/\mathfrak{N}$, and locally isomorphic with $\mathfrak{S}_3$.

Clearly, $\mathfrak{L}_3$ might as well be defined by those transformations which leave the variable x^2 invariant. Thereby we are led to matrices $\tilde{W}$ which have *real* elements (and the determinant 1).[9] If we omit x^2 from X (cf. (3.1)) X is a

[9] They leave $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ invariant.

real symmetric matrix, and the transformations (3.3) contain only real matrices. ($\tilde{W}^*$ is the transposed of $\tilde{W}$.) The equivalence of the two realizations of $\mathfrak{S}_3$ may be put in evidence by

$$(3.5) \qquad \tilde{W} = TWT^{-1} \qquad T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

where $\tilde{W}$ is real, and W of the form (3.4). T is a unitary matrix which transforms $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ into $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let W have the elements $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ (cf. (3.4.)). Then $\tilde{W}$ is the real matrix

$$(3.6) \qquad \tilde{W} = \begin{pmatrix} \alpha_1 - \beta_2 & -\alpha_2 + \beta_1 \\ \alpha_2 + \beta_1 & \alpha_1 + \beta_2 \end{pmatrix}.$$

3c. *The group* $\mathfrak{S}_R$. $\mathfrak{R}$ is the subgroup of all transformations of $\mathfrak{L}_4$ which leave x^0 invariant. The corresponding matrices W, which define the group $\mathfrak{S}_R$, are *unitary* matrices of the form

$$(3.7) \qquad W = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha\bar{\alpha} + \beta\bar{\beta} = 1.$$

Again, $\mathfrak{R}$ is isomorphic with $\mathfrak{S}_R/\mathfrak{N}$, and locally isomorphic with $\mathfrak{S}_R$.

3d. *Linear transformations of a complex variable.* With the matrices W we may correlate the linear transformations of a *complex* variable

$$(3.8) \qquad z' = \frac{\delta z + \gamma}{\beta z + \alpha}.$$

The group (3.8) is, however, isomorphic to $\mathfrak{L}_4$, and not to $\mathfrak{S}_4$, since both a and $(-a)$ lead to the same transformation (3.8).

The two realizations of $\mathfrak{S}_3$ which we have discussed have a simple geometric meaning in terms of the transformations (3.8). To the matrices W of the form (3.4) correspond the conformal transformations of the interior of the unit circle onto itself, to the real matrices $\tilde{W}$ (3.6) correspond the conformal transformations of the upper half of the complex plane onto itself (and also the projective transformations of the real line).

In our later discussion we shall make use of the transformations (3.8).

3e. *Single- and double-valued representations.* A representation of any one of the groups $\mathfrak{L}_4$, $\mathfrak{L}_3$, or $\mathfrak{R}$ is also a representation of the corresponding spinor group. A representation of the spinor group, however, may be a single- or a double-valued representation of $\mathfrak{L}_4$, $\mathfrak{L}_3$, or $\mathfrak{R}$, depending on whether a and $(-a)$ are represented by the same transformation (which in turn depends on whether the two elements e and $(-e)$ are represented by the same transformation). In this paper we are concerned with the spinor groups $\mathfrak{S}_4$ and $\mathfrak{S}_3$ rather than with the groups $\mathfrak{L}_4$ and $\mathfrak{L}_3$, both for their intrinsic mathematical interest and for their significance in Mathematical Physics.

The manifold of the matrices W of the form (3.2), i.e., the group manifold

$\mathfrak{S}_4$, is *simply connected*, and hence $\mathfrak{S}_4$ is the universal covering group of the proper Lorentz group $\mathfrak{L}_4$. Consequently, any continuous representation of $\mathfrak{S}_4$ must be *single-valued.*

By contrast, the group manifold $\mathfrak{S}_3$ has *infinite connectivity* (cf. §4), and there exist many-valued unitary representations of $\mathfrak{S}_3$. On the whole we shall restrict ourselves to the single-valued representations of $\mathfrak{S}_3$, but occasionally we shall also discuss some many-valued representations.

3f. *Infinitesimal transformations of the spinor group.* We have mentioned before that $\mathfrak{L}_4$ and $\mathfrak{S}_4$ are locally isomorphic and that they have therefore the same Lie algebra. To derive the infinitesimal transformations we may restrict ourselves to a sufficiently small neighborhood of the unit element and proceed as follows. Let $y = ax$, and correspondingly, $Y = W(a)XW^*(a)$. For an element χ of the Lie algebra, we have $\chi y^k = l^k{}_j y^j$ (cf. (2.4)), with $l_{kj} + l_{jk} = 0$. Likewise $\chi W(a) = M_\chi W(a)$ with a matrix M_χ to be determined, and $\chi W^*(a) = W^*(a)M^*_\chi$. It follows that

$$\chi Y = (\chi W(a))XW^*(a) + W(a)X(\chi W^*(a)) = M_\chi Y + YM^*_\chi . \tag{3.9}$$

Since the determinant of $W(a)$ is constant, the trace of M_χ must vanish. Inserting the expressions for χy^k in

$$\chi Y = \begin{pmatrix} \chi y^0 + \chi y^3 & \chi y^1 - i\chi y^2 \\ \chi y^1 + i\chi y^2 & \chi y^0 - \chi y^3 \end{pmatrix}$$

we obtain from (3.9) by a straightforward computation

$$\begin{cases} M_\chi = \frac{1}{2}\begin{pmatrix} \mu^1_1 & \mu^1_2 \\ \mu^2_1 & \mu^2_2 \end{pmatrix} \\ \mu^1_1 = l^{30} - il^{12}, \qquad \mu^1_2 = (l^{10} - l^{31}) - i(l^{20} + l^{23}) \\ \mu^2_1 = (l^{10} + l^{31}) + i(l^{20} - l^{23}), \qquad \mu^2_2 = -(l^{30} - il^{12}). \end{cases} \tag{3.10}$$

By specializing the coefficients $l^k{}_j$, we may find the infinitesimal transformations of any subgroup of $\mathfrak{S}_4$.

From the local isomorphism of $\mathfrak{L}_4$, $\mathfrak{L}_3$, $\mathfrak{R}$, with their spinor groups, it follows that the equations (2.10) as well as (2.17) concerning the adjoint group remain valid for the spinor group.

3g. *Some remarks on spinors.* We conclude this section with a few remarks which will be of use later on. Denote the matrix elements of X (cf. (3.1)) by $x^{\rho\sigma}$, those of W by α^ρ_σ ($\rho, \sigma = 1, 2$), so that

$$X = \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix}, \qquad W = \begin{pmatrix} \alpha^1_1 & \alpha^1_2 \\ \alpha^2_1 & \alpha^2_2 \end{pmatrix}. \tag{3.11}$$

Because of the Hermitian character of X, $\overline{x^{\rho\sigma}} = x^{\sigma\rho}$. The transformation (3.3) may be written as

$$y^{\lambda\mu} = \alpha^\lambda_\rho \overline{\alpha^\mu_\sigma} x^{\rho\sigma} \qquad (\lambda, \mu, \rho, \sigma = 1, 2) \tag{3.12}$$

where again the summation convention is used.

Null vectors. For a null vector x the determinant of X vanishes. One may therefore choose a two-component spinor ξ^λ such that

$$x^{\lambda\mu} = \xi^\lambda \overline{\xi^\mu} \text{ if } x^0 > 0, \qquad x^{\lambda\mu} = -\xi^\lambda \overline{\xi^\mu} \text{ if } x^0 < 0. \tag{3.13}$$

The two components ξ^λ are determined up to a common factor of absolute value 1.

Assume $x^0 > 0$, so that x^{11} is a non-negative real number. If it is different from zero, we may set

$$\xi^\lambda = \frac{x^{\lambda 1}}{(x^{11})^{\frac{1}{2}}} \qquad \lambda = 1, 2. \tag{3.14}$$

For null vectors the transformations (3.12) assume the form

$$y^{\lambda\mu} = \frac{1}{x^{11}} (\alpha^\lambda_\rho x^{\rho 1})\overline{(\alpha^\mu_\sigma x^{\sigma 1})} \tag{3.15}$$

provided that $x^{11} > 0$.

If we express the components of a null vector by polar coordinates ($x^0 = r$, $x^3 = r \cos \theta$, $x^1 + ix^2 = r \sin \theta e^{i\phi}$), we have (cf. (3.1))

$$x^{11} = r(1 + \cos \theta) = 2r \cos^2 (\tfrac{1}{2}\theta), \qquad x^{21} = r \sin \theta e^{i\phi} \tag{3.16}$$

and hence by (3.14)

$$\xi^1 = (2r)^{\frac{1}{2}} \cos (\tfrac{1}{2}\theta), \qquad \xi^2 = (2r)^{\frac{1}{2}} \sin (\tfrac{1}{2}\theta) e^{i\phi}. \tag{3.16a}$$

For the group $\mathfrak{L}_3$ (and $\mathfrak{S}_3$), W has to be chosen in the form (3.4), and x^3 set equal to zero ($\theta = \frac{1}{2}\pi$). Instead of (3.16) and (3.16a) we have then

$$x^{11} = r, \ x^{21} = re^{i\phi}; \ \xi^1 = r^{\frac{1}{2}}, \ \xi^2 = r^{\frac{1}{2}} e^{i\phi}. \tag{3.17}$$

Denoting the polar coordinates of y by r' and ϕ', and introducing the matrix elements α, β from (3.4) we find from (3.15)

$$r' = y^{11} = r \,|\, \alpha + \beta e^{i\phi} \,|^2 \tag{3.18}$$

and

$$e^{i\phi'} = \frac{y^{21}}{y^{11}} = \frac{\alpha^2_\rho x^{\rho 1}}{\alpha^1_\sigma x^{\sigma 1}} = \frac{\bar\beta + \bar\alpha e^{i\phi}}{\alpha + \beta e^{i\phi}} = e^{i\phi} \frac{\bar\alpha}{\alpha} \frac{1 + (\bar\beta/\bar\alpha) e^{-i\phi}}{1 + (\beta/\alpha) e^{i\phi}}. \tag{3.19}$$

§4. Detailed study of the group $\mathfrak{S}_3$

In the remaining sections of Part I, we shall only be concerned with the groups $\mathfrak{S}_3$ and $\mathfrak{L}_3$ (mainly with $\mathfrak{S}_3$). We shall therefore omit the subscripts 3 so that $\mathfrak{S}$ and $\mathfrak{L}$ will be understood to mean $\mathfrak{S}_3$ and $\mathfrak{L}_3$ respectively. The Lie algebra of $\mathfrak{S}$ will be denoted by $\mathfrak{s}$.

4a. *Introduction of parameters. The universal covering group of* $\mathfrak{S}$. The topological nature of the group manifold $\mathfrak{S}$ may be inferred from the *polar* *decomposition* of the matrices $\tilde{W}$ (cf. (3.6)). In fact, every real matrix $\tilde{W}$ of determinant 1 may be written in the form $\tilde{W} = OA$, where O is an orthogonal matrix and A a positive definite matrix, both of determinant 1. The matrices O and A depend analytically on $\tilde{W}$. For our two dimensional matrices the set of the O is homeomorphic to a circle, and the set of the A to a Euclidean plane, so that the group manifold $\mathfrak{S}$ is the topological product of a circle and a plane. It follows that the *universal covering group* of $\mathfrak{S}$, which will be denoted by $\mathfrak{C}$, is homeomorphic to a three-dimensional Euclidean space, and that $\mathfrak{C}$ covers $\mathfrak{S}$ infinitely often.

To introduce parameters on $\mathfrak{S}$ we start from the matrices (cf. (3.4))

$$W = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \tag{4.1}$$

and set $\beta/\alpha = \gamma = \gamma_1 + i\gamma_2$. Since $\alpha\bar{\alpha}(1 - \gamma\bar{\gamma}) = 1$, we may choose another real variable ω so that

$$\alpha = e^{i\omega}(1 - \gamma\bar{\gamma})^{-\frac{1}{2}}, \qquad \beta = e^{i\omega}\gamma(1 - \gamma\bar{\gamma})^{-\frac{1}{2}} \qquad (\gamma = \gamma_1 + i\gamma_2). \tag{4.2}$$

By the elements of $\mathfrak{S}$, ω is only defined mod 2π, by the elements of $\mathfrak{L}$, ω is defined mod π. To any two different values of ω correspond, however, different elements of the covering group $\mathfrak{C}$. The three parameters a^1, a^2, a^3 of a group element a will be chosen to be γ_1, γ_2, ω. For convenience we shall often use $\gamma = \gamma_1 + i\gamma_2$ and ω instead. Their range is determined as follows

$$\begin{cases} \mathfrak{S}: -\pi \leqq \omega < \pi & (\gamma_1)^2 + (\gamma_2)^2 = \gamma\bar{\gamma} < 1 \\ \mathfrak{C}: -\infty < \omega < \infty & (\gamma_1)^2 + (\gamma_2)^2 = \gamma\bar{\gamma} < 1. \end{cases} \tag{4.3}$$

The unit element e has the parameters $\gamma = \omega = 0$. (It should be noted that these parameters are adjusted to the polar decomposition of $\tilde{W}$ which we discussed above. In particular O is the rotation by the angle ω.)

The group operations are easily expressed in terms of γ and ω. Let a and a' be any two group elements, let $a'' = a'a$, and denote the parameters of a, a' and a'' by (γ, ω), (γ', ω') and (γ'', ω''). Then

$$\gamma'' = (\gamma + \gamma' e^{-2i\omega}) \cdot (1 + \bar{\gamma}\gamma' e^{-2i\omega})^{-1} \tag{4.4}$$

$$\omega'' = \omega + \omega' + \frac{1}{2i} \log \{(1 + \bar{\gamma}\gamma' e^{-2i\omega}) \cdot (1 + \gamma\bar{\gamma}' e^{2i\omega})^{-1}\}. \tag{4.5}$$

These equations are obtained by the multiplication of the corresponding matrices W (cf. (4.1)). Both equations (4.4) and (4.5) hold on $\mathfrak{S}$ and $\mathfrak{C}$, with the difference, however, that on $\mathfrak{S}$ the equation (4.5) is to be understood mod 2π. (4.4) and (4.5) are *analytic* in the real variables $(\gamma_1, \gamma_2, \omega)$ and $(\gamma_1', \gamma_2', \omega')$, the logarithm being defined by its power series. By (4.4), γ'' has, as it should, an absolute value less than one since

$$1 - \gamma''\bar{\gamma}'' = (1 - \gamma\bar{\gamma}) \cdot (1 - \gamma'\bar{\gamma}') \cdot |1 + \bar{\gamma}\gamma' e^{-2i\omega}|^{-2} > 0.$$

Consider, next, $a' = a^{-1}$. Then

$$\gamma' = -\gamma e^{2i\omega}, \qquad \omega' = -\omega. \tag{4.6}$$

Any analytic function of the real variables γ_1, γ_2, ω ($(\gamma_1)^2 + (\gamma_2)^2 < 1$) is an analytic function on $\mathfrak{C}$, any analytic function of these variables with the period 2π in ω is an analytic function on $\mathfrak{S}$ (in particular, any function which may be expressed as an analytic function of the real and imaginary parts of α and β, as long as $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$).

4b. *Infinitesimal transformations. Invariant integration.* For any element $\chi = \kappa^r \chi_r$ ($r = 0, 1, 2$) of the Lie algebra we have

$$(4.7)\quad \begin{cases} \chi W(a) \equiv \chi^i(a) \dfrac{\partial W(a)}{\partial a^i} = M_\chi W(a) \qquad \chi^i(a) = \kappa^r \chi_r^i(a) \qquad (i = 1, 2, 3) \\ M_\chi = -\dfrac{i}{2}\begin{pmatrix} \kappa^0 & -(\kappa^1 - i\kappa^2) \\ \kappa^1 + i\kappa^2 & -\kappa^0 \end{pmatrix} = \kappa^r M_r. \end{cases}$$

The expression for M_χ is obtained from (3.10) if we set $l_{30} = l_{31} = l_{32} = 0$, and $\kappa^0 = l_{12}$, $\kappa^1 = l_{20}$, $\kappa^2 = l_{01}$. Since the three matrices $\partial W(a)/\partial a^i$ are linearly independent, the functions $\chi_r^i(a)$ are uniquely determined by (4.7). Using complex notation, we may express the differential operators χ_r in the somewhat condensed form

$$(4.8)\qquad \chi_0 = -\tfrac{1}{2}\frac{\partial}{\partial \omega}, \qquad \chi_1 + i\chi_2 = e^{-2i\omega}\left\{i(1 - \gamma\bar{\gamma})\frac{\partial}{\partial \gamma} + \tfrac{1}{2}\bar{\gamma}\frac{\partial}{\partial \omega}\right\}$$

where $\partial/\partial\gamma = \frac{1}{2}(\partial/\partial\gamma_1 - i\partial/\partial\gamma_2)$. (The functions $\hat{\chi}_r^i(a)$ may be computed in a similar way from (1.34a), with U, L replaced by W, M).

The determinant $\Delta(a)$ of the $\chi_r^i(a)$ has by (4.8) the value $(1/8)(1 - \gamma\bar{\gamma})^2$. As we have seen in §2a the group integration on $\mathfrak{S}$ is both right and left invariant, and hence $da = \text{const.}\,(\Delta(a))^{-1} d\gamma_1 d\gamma_2 d\omega$. (cf. §1c). We choose the constant equal to $1/16\pi^2$, and define

$$(4.9)\qquad da = \frac{d\gamma_1\, d\gamma_2\, d\omega}{2\pi^2(1 - \gamma\bar{\gamma})^2}.$$

The total volume of the group $\mathfrak{S}$ is obviously infinite.

4c. *The parameters μ, ζ, ν.* In the subsequent discussion it will be more convenient to use a different set of parameters analogous to the Euler angles of a rotation. Since $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$, it is possible to choose for each pair (α, β) three real numbers μ, ζ, ν such that

$$(4.10)\qquad \alpha = \cosh \zeta e^{-i(\mu+\nu)}, \qquad \beta = \sinh \zeta e^{i(\nu-\mu)}.$$

The corresponding matrix W is decomposed as follows:

$$(4.11)\qquad W = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{-i\mu} & 0 \\ 0 & e^{i\mu} \end{pmatrix}\begin{pmatrix} \cosh \zeta & \sinh \zeta \\ \sinh \zeta & \cosh \zeta \end{pmatrix}\begin{pmatrix} e^{-i\nu} & 0 \\ 0 & e^{i\nu} \end{pmatrix}.$$

By (4.7) the three matrices on the right hand side of this equation are respectively equal to $\exp(2\mu M_0)$, $\exp(2\zeta M_2)$, and $\exp(2\nu M_0)$. We conclude that every element a of the group $\mathfrak{S}$ may be expressed as a product

$$(4.12)\qquad a = \exp(2\mu\chi_0)\exp(2\zeta\chi_2)\exp(2\nu\chi_0)$$

with suitably chosen μ, ζ, ν.

By (3.3) the factors of the product (4.12) give rise to the following transformations of the variables x^k:

(4.13a) $\exp(2\mu\chi_0): y^0 = x^0, y^1 = \cos 2\mu\, x^1 - \sin 2\mu\, x^2, y^2 = \sin 2\mu\, x^1 + \cos 2\mu\, x^2$

(4.13b) $\exp(2\zeta\chi_2): y^0 = \cosh 2\zeta\, x^0 + \sinh 2\zeta\, x^1, y^1 = \sinh 2\zeta\, x^0 + \cosh 2\zeta\, x^1,$
$$y^2 = x^2$$

(cf. also (2.9) and (2.9a)). If we write (4.13b) in the familiar form

(4.14) $$y^0 = (1 - v^2)^{-\frac{1}{2}}(x^0 + vx^1), \quad y^1 = (1 - v^2)^{-\frac{1}{2}}(vx^0 + x^1), \quad y^2 = x^2,$$

we have for the velocity v defined by the Lorentz transformation (4.12)

(4.14a) $$v = \tanh 2\zeta, \ (1 + v)^{\frac{1}{2}}(1 - v)^{-\frac{1}{2}} = e^{2\zeta}.$$

Range of the parameters μ, ζ, ν. It is evident that the equations (4.10) do not determine the parameters μ, ζ, ν in a unique way. For reasons of symmetry, the range of these parameters will be defined by the inequalities

(4.15) $$0 \leqq \zeta < \infty, \qquad -\pi \leqq \mu, \nu < \pi.$$

Then $\sinh \zeta = |\beta|$, and as long as $\zeta \neq 0$, there are two sets which correspond to the same group element, viz. (μ, ζ, ν) and $(\mu \pm \pi, \zeta, \nu \pm \pi)$ (where the signs are so chosen that the inequalities (4.15) hold).[10] If $\zeta = 0$, only the sum $\mu + \nu$ is defined. We are free to use any values (μ, ζ, ν) in the equation (4.12) while the inequalities (4.15) will be mainly applied to integrations over the whole group manifold $\mathfrak{S}$. Our parameters cover $\mathfrak{S}$ *twice* except for the set $\zeta = 0$, which may be neglected since its measure is zero.

The parameters γ and ω are obtained as follows.

(4.16) $$\gamma = \tanh \zeta e^{2i\nu}, \qquad \omega = -(\mu + \nu) \pmod{2\pi}.$$

Infinitesimal transformations; the volume element on $\mathfrak{S}$. In terms of the parameters (μ, ζ, ν) the equations (4.8) take the form

(4.17) $$\chi_0 = \frac{1}{2}\frac{\partial}{\partial \mu}, \qquad \chi_1 + i\chi_2 = \frac{1}{2}e^{2i\mu}\left\{i\frac{\partial}{\partial \zeta} + \frac{1}{\sinh 2\zeta}\left(\cosh 2\zeta \frac{\partial}{\partial \mu} - \frac{\partial}{\partial \nu}\right)\right\}$$

((4.17) may be directly obtained from (4.7)).

We shall later use the second order differential operator Ω corresponding to Q (cf. (2.23)), viz.

(4.18) $$\Omega = (\chi_0)^2 - (\chi_1)^2 - (\chi_2)^2.$$

Writing $\Omega = (\chi_0)^2 - \frac{1}{2}(\chi_1 + i\chi_2)(\chi_1 - i\chi_2) - \frac{1}{2}(\chi_1 - i\chi_2)(\chi_1 + i\chi_2)$ we readily find from (4.17)

(4.19) $$-4\Omega = (\sinh 2\zeta)^{-1}\frac{\partial}{\partial \zeta}\left(\sinh 2\zeta \frac{\partial}{\partial \zeta}\right) + (\sinh 2\zeta)^{-2}\left(\frac{\partial^2}{\partial \mu^2} - 2\cosh 2\zeta \frac{\partial^2}{\partial \mu \partial \nu} + \frac{\partial^2}{\partial \nu^2}\right).$$

[10] Any function defined on $\mathfrak{S}$ remains unchanged if (μ, ν) are replaced by $(\mu \pm \pi, \nu \pm \pi)$.

Finally, the volume element da is given by

$$(4.20)\qquad da = (2\pi)^{-2} \sinh 2\zeta \, d\zeta \, d\mu \, d\nu.$$

Since the parameters (μ, ζ, ν) as defined by (4.15) cover the group manifold twice an additional factor $\frac{1}{2}$ is included in (4.20)).

4d. *Invariant metric on* $\mathfrak{S}$. The group operations on $\mathfrak{S}$ are linear transformations of the real and imaginary parts of α and β which leave the form $\alpha\bar{\alpha} - \beta\bar{\beta}$ invariant. Clearly the indefinite metric $ds^2 = d\alpha \, d\bar{\alpha} - d\beta \, d\bar{\beta} = -(d\zeta)^2 + (d\mu)^2 + (d\nu)^2 + 2\cosh 2\zeta \, d\mu \, d\nu$ is also invariant under these transformations. We may write $ds^2 = h_{kl} d\rho^k d\rho^l$, where $(\rho^1, \rho^2, \rho^3) = (\mu, \zeta, \nu)$. It follows that the volume element $h^{\frac{1}{2}} d\rho^1 d\rho^2 d\rho^3$ (where h is the determinant of the h_{kl}) defines a right and left invariant volume element on the group manifold. Since $h^{\frac{1}{2}} = \sinh 2\zeta$ it coincides—up to a constant factor—with (4.20). It should also be noted that the *Laplacian* related to the metric ds^2, viz., $h^{-\frac{1}{2}}\partial/\partial\rho^k(h^{kl}h^{\frac{1}{2}}\partial/\partial\rho^l)$, equals the operator 4Ω defined above.

4e. *Conformal transformations of the unit circle onto itself.* Consider the two-dimensional[11] complex manifold $\mathfrak{M}$ described by two complex variables ξ^1, ξ^2 subject to the inequality $\xi^1\bar{\xi}^1 > \xi^2\bar{\xi}^2$, and let the group $\mathfrak{S}$ act on $\mathfrak{M}$ such that, for any $a \in \mathfrak{S}$,

$$(4.21)\qquad \eta^1 = (a\xi)^1 = \alpha\xi^1 + \beta\xi^2, \qquad \eta^2 = (a\xi)^2 = \bar{\beta}\xi^1 + \bar{\alpha}\xi^2, \qquad \alpha\bar{\alpha} - \beta\bar{\beta} = 1.$$

Under these transformations the form $\xi^1\bar{\xi}^1 - \xi^2\bar{\xi}^2$ is invariant, in particular $\eta^1 \neq 0$. By (4.7)

$$(4.22)\qquad \chi\eta^1 = -\frac{i}{2}(\kappa^0\eta^1 - (\kappa^1 - i\kappa^2)\eta^2), \qquad \chi\eta^2 = -\frac{i}{2}((\kappa^1 + i\kappa^2)\eta^1 - \kappa^0\eta^2)$$

if $\chi = \kappa^r\chi_r$.

The variable $z = \xi^2/\xi^1$ defines a manifold $\mathfrak{M}^*$, the open unit circle $z\bar{z} < 1$. On $\mathfrak{M}^*$ the transformation (4.21) induces the conformal mapping

$$(4.23)\qquad y = az = \frac{\bar{\alpha}z + \bar{\beta}}{\beta z + \alpha}$$

of $\mathfrak{M}^*$ onto itself. The infinitesimal transformation obtained from (4.22) is[12]

$$(4.24)\qquad \begin{aligned} \chi y = \lambda_\chi(y) = \chi\left(\frac{\eta^2}{\eta^1}\right) &= \frac{\eta^1(\chi\eta^2) - \eta^2(\chi\eta^1)}{(\eta^1)^2} \\ &= -\frac{i}{2}\{(\kappa^1 + i\kappa^2) - 2\kappa^0 y + (\kappa^1 - i\kappa^2)y^2\}. \end{aligned}$$

As we have seen at the end of §1h, the denominator in (4.23) is a *multiplier* of this transformation group,

$$(4.25)\qquad \mu(a, z) = \alpha + \beta z.$$

[11] We refer here to complex dimensions.

[12] Here y^2 is the second power of y.

Consequently, the *infinitesimal multiplier* $\tau_\chi(z)$ is given by $\{\chi(\alpha + \beta z)\}_{a=e} = \{(\chi\alpha) + (\chi\beta)z\}_{a=e}$ (cf. (1.45)), i.e.,

$$\tau_\chi(z) = \frac{i}{2}(-\kappa^0 + (\kappa^1 - i\kappa^2)z). \tag{4.26}$$

Incidentally, $\tau_\chi(z) = -\frac{1}{2}d\lambda_\chi(z)/dz$.

With the help of the parameters γ and ω the transformation (4.23) is written in the more familiar form

$$y = e^{-2i\omega}\frac{z + \bar{\gamma}}{1 + \gamma z}.$$

4f. *Remarks on the one-parameter subgroups of* $\mathfrak{S}$. The three classes of one parameter subgroups of $\mathfrak{S}$ which were defined at the end of §2, viz., the elliptic, hyperbolic, and parabolic subgroups, may be easily characterized by the fixed points of the transformations (4.23). A fixed point u of the one-parameter subgroup $a(t) = \exp(t\chi)$ is determined by the quadratic equation $\lambda_\chi(u) = 0$ (cf. (4.24)). Depending on the sign of $\kappa^r\kappa_r$ we have the following three cases.

I. *Elliptic subgroup* ($\kappa^r\kappa_r > 0$). We may assume $\kappa^0 > 0$. The equation $\lambda_\chi(u) = 0$ has two different roots u and $1/\bar{u}$ one of which, say u, has an absolute value less than 1. For the one fixed point in the interior of the unit circle we find

$$u = (\kappa^1 + i\kappa^2)/(\kappa^0 + \sqrt{\kappa^r\kappa_r}), \qquad \sqrt{\kappa^r\kappa_r} > 0.$$

If $\chi = \chi_0$, this fixed point is the origin.

II. *Hyperbolic subgroup* ($\kappa^r\kappa_r < 0$). Here we have two fixed points, u_+ an u_-, both on the circle $|z| = 1$. They are explicitly given by

$$\left.\begin{matrix} u_+ \\ u_- \end{matrix}\right\} = (\kappa^0 \pm i\sqrt{-\kappa^r\kappa_r})/(\kappa^1 - i\kappa^2).$$

For χ_1 and χ_2 the fixed points are $\pm i$ and ± 1 respectively.

III. *Parabolic subgroup* ($\kappa^r\kappa_r = 0$). $\lambda_\chi(u) = 0$ has a double root u of absolute value 1. This is a limiting case of both I ($|u| \to 1$) and II (u_+ and u_- coincide). We have $u = (\kappa^1 + i\kappa^2)/\kappa^0$. For $\chi_0 + \chi_1$, and $\chi_0 + \chi_2$, $u = 1$ and $u = i$ respectively.

§5. The Infinitesimal Representations of $\mathfrak{S}$

We now proceed with the construction of irreducible unitary representations of the group $\mathfrak{S}$. In the present section, the representations of the Lie algebra of $\mathfrak{S}$ will be determined, and the corresponding representations of $\mathfrak{S}$ will be constructed in the succeeding §§6–9.

5a. *Introductory remarks.* Let $\mathfrak{H}$ be a complex Hilbert space with elements $f, g, \cdots$. The inner product of two vectors f, g in $\mathfrak{H}$ will be denoted by $(f, g) = \overline{(g, f)}$, the norm of a vector f by $\|f\| = (f, f)^{\frac{1}{2}}$. We assume the inner product to be linear in the *second* factor, so that for any complex number λ

$$(f, \lambda g) = \lambda(f, g) \qquad (\lambda f, g) = \bar{\lambda}(f, g). \tag{5.1}$$

We only use *strong convergence*, i.e., a sequence f_n is said to converge to $f (f_n \to f)$ if $\| f_n - f \| \to 0$.

If A is any linear oprator with dense domain, its adjoint is denoted by A^*. A linear operator B is an extension of A $(B \supset A)$ if B's domain of existence includes that of A and if there $Bf = Af$.

An *irreducible unitary representation* of $\mathfrak{S}$ on $\mathfrak{H}$ will be defined as follows

(5.2)
- (1) *To every $a \in \mathfrak{S}$ corresponds a unitary operator $U(a)$ on $\mathfrak{H}$.*
- (2) *For any two elements $a, b \in \mathfrak{S}$ $U(a)U(b) = U(ab)$. (Representation property)*
- (3) *If the sequence $a_n \in \mathfrak{S}$ converges to a, then $U(a_n)f \to U(a)f$ for every $f \in \mathfrak{H}$. (Continuity)*
- (4) *No proper closed linear subspace of $\mathfrak{H}$ is invariant with respect to all $U(a)$, i.e., if for a closed linear subspace $\mathfrak{K}$ the relation $U(a)\mathfrak{K} \subset \mathfrak{K}$ holds for all $a \in \mathfrak{S}$, then $\mathfrak{K}$ is either $\{0\}$ or $\mathfrak{H}$. (Irreducibility).*[13]

Ad(2). Since $ae = a$, and $aa^{-1} = e$, we have $U(a)U(e) = U(a)$, and $U(a)U(a^{-1} = U(e)$, consequently

$$U(e) = 1, \; U(a^{-1}) = U(a)^{-1} \tag{5.3}$$

Ad (3). As is well known, it is sufficient—on account of the representation property—to require (3) for sequences converging to e.

5b. *Criterion for irreducibility.* We shall make use of the following well known criterion:

LEMMA 1. *Let $\mathfrak{U}$ be a collection of unitary operators on $\mathfrak{H}$ which contains with any U its inverse U^{-1}. The collection $\mathfrak{U}$ is irreducible if and only if any bounded linear operator A which commutes with all U is of the form $A = \alpha \cdot 1$ (where α is a complex number).*

For the sake of completeness we give the proof of this lemma. Let $\mathfrak{K}$ be a closed linear manifold invariant under the transformations $U(\epsilon \mathfrak{U})$ so that $Uf \in \mathfrak{K}$ whenever $f \in \mathfrak{K}$. Denote $\mathfrak{K}$'s projection operator by E, i.e., $E^* = E$, $E^2 = E$, every vector of the form $Ef (f \in \mathfrak{H})$ is in $\mathfrak{K}$, and for every $f \in \mathfrak{K}$ $Ef = f$. We then have $E(UEf) = UEf$ for every f, i.e., $EUE = UE$. The same equation holds for $U^{-1} = U^*$, so that $EU^*E = U^*E$. Taking adjoints on both sides, we find $EUE = EU$, and hence $EU = UE$. Conversely, if all U commute with a projection operator E, then $E(UE) = E(EU) = EU = UE$, and hence the closed linear manifold $\mathfrak{K}$ defined by E is invariant under the transformations U.

(1) Let first $\mathfrak{U}$ be reducible. Then there exists an invariant manifold $\mathfrak{K}$ different from $\mathfrak{H}$ and $\{0\}$, whose projection operator E is therefore different from 1 and 0. Hence all U commute with the bounded operator E not of the form $\alpha \cdot 1$.

(2) If $\mathfrak{U}$ is irreducible, then the only projection operators which commute

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[13] $\{0\}$ is the linear manifold which contains only the element $f = 0$.

with all U are $E = 1$ and $E = 0$. Let B be a *bounded self-adjoint* operator which commutes with all U, and let $B = \int_{c_1}^{c_2} \lambda \, dE(\lambda)$ be its spectral representation (c_1 and c_2 finite). Then every $E(\lambda)$ commutes with all U, and hence the $E(\lambda)$ are either 0 or 1. It follows that $B = \beta \cdot 1$. Let, finally, A be any bounded linear operator commuting with all U. Then $AU = UA$, and also $AU^* = U^*A$ (since $U^* = U^{-1}$ is contained in $\mathfrak{U}$) and consequently $A^*U = UA^*$. It follows that the two self-adjoint bounded operators $B = A^* + A$ and $C = i(A^* - A)$ likewise commute with all U. Hence from the foregoing, $B = \beta \cdot 1$, $C = \gamma \cdot 1$ (with suitable β, γ), and $A = \frac{1}{2}(B + iC) = \alpha \cdot 1$, where $\alpha = \frac{1}{2}(\beta + i\gamma)$ q.e.d.

5c. *One parameter subgroups.* Our discussion is based on Stone's theorem (in its weaker form), which may be stated as follows (cf. [Stone 1, 2], [von Neumann, 2]).

STONE'S THEOREM: (1) *Let $U_t(-\infty < t < \infty)$ be a one parameter group of unitary operators on $\mathfrak{H}$ such that $U_sU_t = U_{s+t}$ and that for every $f \in \mathfrak{H}$ $U_tf \rightarrow U_{t_0}f$ whenever $t \rightarrow t_0$. Then U_t may be expressed in the form $U_t = \exp(-itH)$ with a self-adjoint operator H on $\mathfrak{H}$. If $H = \int_{-\infty}^{+\infty} \lambda \, dE(\lambda)$, $U_t = \int_{-\infty}^{+\infty} e^{-it\lambda} \, dE(\lambda)$.* (2) *Let t_n ($t_n \neq 0$; $n = 1, 2, \cdots$) converge to zero. The sequence $t_n^{-1}(U_{t_n} - 1)f$ ($f \in \mathfrak{H}$) converges if and only if Hf exists. If Hf exists, $t^{-1}(U_t - 1)f \rightarrow -iHf$ for $t \rightarrow 0$. ($t \neq 0$.)*

We return now to the study of a representation of $\mathfrak{S}$ characterized by the conditions (5.2). Let $a(t) = \exp(t\chi)$ be a one parameter subgroup on $\mathfrak{S}$, and let $U_t = U(a(t))$. The operators U_t satisfy the conditions of Stone's theorem and may be expressed in the form $U_t = \exp(-itH_\chi)$ with a self-adjoint operator H_χ depending on χ. This relation corresponds to what we found for finite-dimensional representations (cf. §1e, where L_χ stands for $(-iH_\chi)$). The H_χ will be called the *infinitesimal operators* of our representation. If $\chi = \chi_r$, we denote the corresponding infinitesimal operator by H_r ($r = 0, 1, 2$).

Conjugate subgroups. In the same way as in §1e one can show that $\exp(-itH_{\chi'}) = U(b) \exp(-itH_\chi)U(b)^{-1}$ if $\chi' = b\chi b^{-1}$. Consequently

$$H_{\chi'} = U(b)H_\chi U(b)^{-1}, \qquad \chi' = b\chi b^{-1}. \tag{5.4}$$

If $\mathfrak{D}_\chi$ is the domain of the operator H_χ, i.e., the linear manifold on which H_χ is defined, then by (5.4)

$$\mathfrak{D}_{\chi'} = U(b)\mathfrak{D}_\chi. \tag{5.5}$$

We should like to postulate for the operators H_χ some analogue of the relations (1.36) in §1e which characterize a representation of the Lie algebra $\mathfrak{s}$ of the group $\mathfrak{S}$. However, the operators H_χ must be expected to be unbounded (it will turn out that this is the case), and hence linear combinations and commutators of the H_χ cannot be properly defined unless suitable assumptions about their domains are introduced. This will be done in what follows.

5d. *Assumptions about the infinitesimal representations.* We start with some observations on the domains $\mathfrak{D}_\chi$ of the operators H_χ.

DEFINITION 2. *The intersections of all domains $\mathfrak{D}_\chi$ is denoted by $\mathfrak{A}$, i.e. $\mathfrak{A}$ is the linear manifold of the vectors f to which all H_χ can be applied. The linear manifold of the vectors f to which all products $H_{\chi'} \cdot H_\chi$ can be applied (χ', χ any two elements of $\mathfrak{s}$) is denoted by $\mathfrak{B}$. $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are the closures of $\mathfrak{A}$ and $\mathfrak{B}$ respectively.*

It is evident that $\mathfrak{A} \supset \mathfrak{B}$, $\bar{\mathfrak{A}} \supset \bar{\mathfrak{B}}$.

LEMMA 2. *For an irreducible representation either $\bar{\mathfrak{A}} = \mathfrak{A} = \{0\}$ or $\bar{\mathfrak{A}} = \mathfrak{H}$, and either $\bar{\mathfrak{B}} = \mathfrak{B} = \{0\}$ or $\bar{\mathfrak{B}} = \mathfrak{H}$.*[14]

It is sufficient to prove this for $\bar{\mathfrak{A}}$, since the same arguments apply to $\bar{\mathfrak{B}}$. From (5.4) and (5.5) we infer that $U(b)f \in \mathfrak{A}$ whenever $f \in \mathfrak{A}$ (for every $b \in \mathfrak{S}$), i.e. $U(b)\mathfrak{A} \subset \mathfrak{A}$. Because of the unitary character of $U(b)$ we have also $U(b)\bar{\mathfrak{A}} \subset \bar{\mathfrak{A}}$. Since $\bar{\mathfrak{A}}$ is closed, the assertion follows. (Cf. [Wigner, p. 155].)

Since we want to base our discussion on a representation of the Lie algebra $\mathfrak{s}$ we exclude the possibility $\bar{\mathfrak{B}} = \{0\}$ and hence assume that $\bar{\mathfrak{B}} = \mathfrak{H}$, and, *a fortiori*, $\bar{\mathfrak{A}} = \mathfrak{H}$. In addition we introduce the following assumption. If f is a vector in $\mathfrak{D}_\chi$ there exists a sequence f_n ($f_n \in \mathfrak{B}$) such that $f_n \to f$, $H_\chi f_n \to H_\chi f$. (This already implies that $\bar{\mathfrak{B}} = \mathfrak{H}$ because the linear manifold $\mathfrak{D}_\chi$ is dense in $\mathfrak{H}$.) We shall use this assumption only for the operator H_0 (cf. (5.17)).

Concerning the representation of the Lie algebra it is sufficient to require that $H_{(\alpha\chi+\alpha'\chi')} = \alpha H_\chi + \alpha' H_{\chi'}$ on $\mathfrak{A}$ (for any two constants α, α' and any two elements χ, χ' of $\mathfrak{s}$), or rather that $H_{\chi+\chi'} = H_\chi + H_{\chi'}$, since the relation $H_{(\alpha\chi)} = \alpha H_\chi$ is evidently satisfied. The commutation rules may then be derived. We summarize our conditions as follows.

(5.6) (1) *If f is a vector in $\mathfrak{D}_\chi$, the domain of the operator $\mathfrak{H}_\chi$, there exists a sequence f_n in $\mathfrak{B}$ (see Definition 2 above) such that $f_n \to f$, $H_\chi f_n \to H_\chi f$.*
(2) *For every vector f in $\mathfrak{A}$ $H_{(\chi+\chi')}f = H_\chi f + H_{\chi'} f$.*

It follows from (1) that

$$\bar{\mathfrak{A}} = \bar{\mathfrak{B}} = \mathfrak{H}. \tag{5.7}$$

By (2.17) $b\chi_k b^{-1} = w^l{}_k(b)\chi_l$. Applying (5.4) and the condition (2) above, we obtain for every $f \in \mathfrak{A}$

$$U(b)H_k U(b)^{-1} f = w^l{}_k(b) H_l f \qquad (k = 0, 1, 2). \tag{5.8}$$

It also follows that on $\mathfrak{A}$ every H_χ is a linear combination of H_0, H_1, and H_2. If we denote the domain of H_r by $\mathfrak{D}_r$, we may characterize $\mathfrak{A}$ as the intersection of $\mathfrak{D}_0$, $\mathfrak{D}_1$, and $\mathfrak{D}_2$.

5e. *The operator Q.* We define the operator Q as follows. Its domain is $\mathfrak{B}$, and for every $f \in \mathfrak{B}$

$$Qf = (H_1)^2 f + (H_2)^2 f - (H_0)^2 f = -g^{rs} H_r H_s f \tag{5.9}$$

(This corresponds to the definition (2.23)). Since the H_r are self-adjoint, $(Qf, g) = (f, Qg)$ for any two vectors $f, g \in \mathfrak{B}$, i.e. $Q \subset Q^*$.

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[14] If $\mathfrak{A}(\mathfrak{B})$ contains at least one element $f \neq 0$ it is dense in $\mathfrak{H}$.

From (5.8) we may compute $U(b)QU(b)^{-1}f$, $(f \in \mathfrak{B})$. In fact we have $g^{rs}U(b)H_rH_sU(b)^{-1}f = g^{rs}U(b)H_rU(b^{-1})U(b)H_sU(b)^{-1}f = g^{rs}w^k{}_r(b)w^l{}_s(b)H_kH_lf = g^{kl}H_kH_lf$, using the orthogonality relations (2.3). Hence for every $b \in \mathfrak{S}$

$$U(b)QU(b)^{-1}f = Qf, \qquad (f \in \mathfrak{B}) \tag{5.10}$$

or

$$U(b)Qf = QU(b)f, \qquad (f \in \mathfrak{B}). \tag{5.11}$$

The operator Q may be extended to its closure $\bar{Q}$, whose domain will be denoted by $\mathfrak{E}$. Since $\mathfrak{E}$ contains $\mathfrak{B}$ it is dense in $\mathfrak{H}$. (Whenever for a converging sequence $f_n \to f$ $(f_n \in \mathfrak{B})$ the sequence Qf_n also converges, $f \in \mathfrak{E}$, and $\bar{Q}f = \lim Qf_n$. On $\mathfrak{B}$, we have $\bar{Q}f = Qf$). $\bar{Q}$ is a closed Hermitian operator, and it follows from (5.11) that

$$U(b)\mathfrak{E} \subset \mathfrak{E}, \qquad U(b)\bar{Q}f = \bar{Q}U(b)f, \qquad (f \in \mathfrak{E}). \tag{5.12}$$

We now construct the Cayley transform of $\bar{Q}$, $V = (\bar{Q} - i)(\bar{Q} + i)^{-1}$ (cf. [von Neumann 1, p. 80]). $\mathfrak{D}_V$, the domain of V, consists of all vectors h of the form $h = (\bar{Q} + i)f$ $(f \in \mathfrak{E})$, and on $\mathfrak{D}_V$ $Vh = (\bar{Q} - i)f$. Moreover $\| Vh \| = \| h \|$, and $Vh \neq h$ unless $h = 0$ (i.e., $f = 0$). By (5.12)

$$U(b)\mathfrak{D}_V \subset \mathfrak{D}_V, \qquad U(b)Vh = VU(b)h, \qquad (h \in \mathfrak{D}_V). \tag{5.13}$$

Since $\mathfrak{D}_V$ is a closed linear manifold and different from $\{0\}$, it follows from the first equation (5.13) and the irreducibility of the $U(b)$ that $\mathfrak{D}_V = \mathfrak{H}$. Therefore V is defined everywhere on $\mathfrak{H}$, it is bounded $(\| Vh \| = \| h \|)$ and by the second equation (5.13) it commutes with all $U(b)$. Hence it is of the form $V = \alpha \cdot 1$ (cf. Lemma 1 above), where α has the absolute value 1, but is different from 1. For every $f \in \mathfrak{E}$ we then have $(\bar{Q} - i)f = \alpha(\bar{Q} + i)f$, i.e., $\bar{Q}f = q \cdot f$ where q is the real number $i(1 + \alpha)/(1 - \alpha)$. (It follows that $\mathfrak{E} = \mathfrak{H}$.) This holds in particular for $f \in \mathfrak{B}$, where $\bar{Q}f = Qf$. Consequently

$$Qf = q \cdot f, \qquad (f \in \mathfrak{B}). \tag{5.14}$$

From this equation and from the first condition (5.6) we may draw a conclusion about $\mathfrak{D}_0$, the domain of the operator H_0. For every vector $f \in \mathfrak{B}$ we have $(f, Qf) = q \cdot (f, f)$ and hence from the definition of Q

$$\| H_1 f \|^2 + \| H_2 f \|^2 - \| H_0 f \|^2 = q \cdot \| f \|^2. \tag{5.15}$$

Let g be a vector in $\mathfrak{D}_0$. There exists a sequence $g_n (g_n \in \mathfrak{B})$ such that $g_n \to g$, $H_0 g_n \to H_0 g$. Applying (5.15) to $f = g_m - g_n$ we have

$$\| H_1 g_m - H_1 g_n \|^2 + \| H^2 g_m - H_2 g_n \|_2 = \| H_0 g_m - H_0 g_n \|^2 + q \| g_m - g_n \|^2. \tag{5.16}$$

The right hand side of (5.16) tends to zero as $m, n \to \infty$, and it follows that both sequences $H_1 g_n$ and $H_2 g_n$ are convergent. Since the operators H_1 and H_2 are closed, g is also contained in the two domains $\mathfrak{D}_1$, $\mathfrak{D}_2$ and hence in the intersection of all $\mathfrak{D}_r$, i.e. in $\mathfrak{A}$. This shows that

$$\mathfrak{D}_0 = \mathfrak{A}. \tag{5.17}$$

(Notice that this equality only holds for $\mathfrak{D}_0$ due to the unsymmetric way in which the three operators H_r enter the expression for Q.)

5 f. *Construction of the infinitesimal representations of* $\mathfrak{S}$. The one parameter subgroup $a(t) = \exp(t\chi_0)$ is a *compact* Abelian group. By comparison of (4.11) and (4.12) it is seen that $\exp(4\pi\chi_0) = e$. It follows that $U_t = \exp(-itH_0)$ has a *pure point spectrum*, i.e., there exists a complete orthonormal system of vectors g_n in $\mathfrak{H}$ such that[15]

$$(5.18) \qquad U_t g_n = e^{-i\lambda_n t} g_n ; \qquad H_0 g_n = \lambda_n g_n .$$

Since $a(4\pi) = e$, the proper values λ_n may be *integral* or *half integral*. We may derive (5.18) directly from Stone's Theorem, or we may use the fact that a unitary representation of any compact group may be decomposed into finite-dimensional irreducible parts (which, for Abelian groups, are one-dimensional and of the form (5.18)). (Cf. [Wigner, p. 194]).

Choose one of the proper vectors of H_0. Denote it by g ($\| g \| = 1$) and let λ be the corresponding proper value, so that $H_0 g = \lambda g$. By (5.17) $g \in \mathfrak{A}$, and hence the equations (5.8) may be applied. We first choose $b = \exp(t\chi_0)$, $k = 1, 2$, and set $U_t = \exp(-itH_0)$. The only non-vanishing coefficients $w^l{}_k$ are these: $w^1{}_1 = w^2{}_2 = \cos t$, $w^2{}_1 = -w^1{}_2 = \sin t$ (cf. (4.13a)). Using (5.18) we find

$$(5.19) \quad e^{i\lambda t} U_t H_1 g = (\cos t\, H_1 + \sin t\, H_2) g, \; e^{i\lambda t} U_t H_2 g = (-\sin t\, H_1 + \cos t\, H_2) g.$$

It will be convenient to introduce on $\mathfrak{A}$ the two operators

$$(5.20) \qquad F = H_1 + iH_2 , \qquad G = H_1 - iH_2 .$$

From the equations (5.19) we obtain

$$U_t F g = e^{-i(\lambda+1)t} F g, \qquad U_t G g = e^{-i(\lambda-1)t} G g$$

and also

$$t^{-1}(U_t - 1) F g = t^{-1}(e^{-i(\lambda+1)t} - 1) F g,$$
$$t^{-1}(U_t - 1) G g = t^{-1}(e^{-i(\lambda-1)t} - 1) G g \qquad (t \neq 0).$$

If we let t tend to zero, the right hand sides of the last two equations tend to $-i(\lambda + 1) F g$ and $-i(\lambda - 1) G g$ respectively. The left hand sides must converge to the same limits, and it follows from Stone's Theorem that both Fg and Gg are in $\mathfrak{D}_0$, and hence in $\mathfrak{A}$, and furthermore that

$$(5.21) \qquad H_0(Fg) = (\lambda + 1) F g, \qquad H_0(Gg) = (\lambda - 1) G g.$$

Since the preceding argument was based on the equation $H_0 g = \lambda g$ (we did not

[15] In the following the summation convention will only be applied to quantities introduced in §1 and §2, in particular to χ^i_k, κ^r, g_{rs}, w^i_j. In all other cases a superscript is an exponent unless otherwise indicated and no summation is carried out with respect to repeated indices.

use the fact that $g \neq 0$) we may apply it to Fg and Gg instead, replacing λ by $\lambda + 1$ and $\lambda - 1$ respectively. By induction we find for any non-negative integral power of F and G

$$H_0(F^s g) = (\lambda + s)F^s g, \qquad H_0(G^s g) = (\lambda - s)G^s g \tag{5.22}$$

where $F^s g$ and $G^s g$ are in $\mathfrak{A}$. (For $s = 0$ (5.22) reduces to $H_0 g = \lambda g$.) All non-vanishing vectors $F^s g$ and $G^s g$ are proper vectors of H_0 with the corresponding proper values $\lambda + s$ and $\lambda - s$.

Since Fg and Gg are in $\mathfrak{A}$, the same holds for $H_1 g$ and $H_2 g$, and also for $H_0 g = \lambda g$. This shows that g is even contained in $\mathfrak{B}$. More generally, it may be shown by induction that every finite product of operators H_r can be applied to g.

Commutation rules. If we subtract from the two equations (5.21) the equations $F(H_0 g) = \lambda\, Fg$ and $G(H_0 g) = \lambda\, Gg$ we obtain two equations which may be written in the equivalent forms

$$(H_0 F - F H_0)g = Fg\,, \qquad (H_0 G - G H_0)g = -Gg, \tag{5.22a}$$

$$(H_0 H_1 - H_1 H_0)g = iH_2 g, \qquad (H_2 H_0 - H_0 H_2)g = iH_1 g. \tag{5.22b}$$

It is seen that the equations (5.22b) correspond to the first and third equation (2.22) respectively, since H_r corresponds to iL_r.

To obtain the analogue of the second equation (2.22) we again apply (5.8), but this time in the form

$$H_k U(b)^{-1} g = U(b)^{-1} w^r{}_k(b) H_r g$$

and with $b = \exp(t\chi_2)$, $k = 1$. The coefficients $w^r{}_1$ are given by $w^r{}_2 = \sinh t$, $w^1{}_1 = \cosh t$, $w^2{}_1 = 0$ (cf. (4.13b)). Hence, with $U_t = \exp(itH_2)$,

$$H_1 U_t g = U_t(\sinh t\, H_0 + \cosh t\, H_1)g$$

and

$$H_1\{t^{-1}(U_t - 1)g\} = t^{-1} \sinh t\, U_t H_0 g + t^{-1}(\cosh t\, U_t - 1)H_1 g, \qquad (t \neq 0).$$

We use here the fact that for every vector $f \in \mathfrak{D}_2$ and every differentiable function $\phi(t)$, $\lim_{t\to 0} t^{-1}(\phi(t)U_t - \phi(0))g = i\phi(0)H_2 g + \phi'(0)g$, where $\phi'(t)$ is the derivative of $\phi(t)$. Hence the right hand side of the last equation convergences for $t \to 0$ to $(H_0 + iH_2 H_1)g$, which must coincide with the limit of the left hand side, $iH_1 H_2 g$. Equating these two expressions we obtain the third commutation rule in the two equivalent forms

$$(H_1 H_2 - H_2 H_1)g = -iH_0 g, \qquad (GF - FG)g = 2H_0 g. \tag{5.23}$$

The relations (5.22) and (5.23) hold evidently for all vectors $F^s g$ and $G^s g$. Notice that, on $\mathfrak{B}$, we have

$$2Q = GF + FG - 2(H_0)^2$$

and hence

$$(GF + FG)g = 2(Q + (H_0)^2)g = 2(q + \lambda^2)g.$$

Adding and subtracting the second equation (5.23) we find

(5.24a) $$G(Fg) = \rho_1 \cdot g, \qquad \rho_1 = q + \lambda(\lambda + 1)$$

(5.24b) $$F(Gg) = \sigma_1 \cdot g, \qquad \sigma_1 = q + \lambda(\lambda - 1).$$

Replacing in (5.24a) g by $F^{s-1}g(s \geqq 1)$, λ by $\lambda + s - 1$, and in (5.24b) g by $G^{s-1}g$, λ by $\lambda - s + 1$, we obtain

(5.25a) $$G(F^s g) = \rho_s F^{s-1} g, \qquad \rho_s = q + (\lambda + s - 1)(\lambda + s)$$

(5.25b) $$F(G^s g) = \sigma_s G^{s-1} g, \qquad \sigma_s = q + (\lambda - s + 1)(\lambda - s)$$

We now use these equations to find recursion formulas for the norms of the vectors $F^s g$ and $G^s g$. Since for any two vectors f_1 and f_2 of $\mathfrak{A}$ $(Ff_1, f_2) = (f_1, Gf_2)$ we have

$$\| F^{s+1} g \|^2 = (F^{s+1}g, F^{s+1}g) = (F^s g, GF^{s+1}g) = \rho_{s+1}(F^s g, F^s g)$$

and hence, with the corresponding computation for $G^s g$,

(5.26) $$\| F^{s+1} g \|^2 = \rho_{s+1} \| F^s g \|^2, \quad \| G^{s+1} g \|^2 = \sigma_{s+1} \| G^s g \|^2, \qquad (s \geqq 0).$$

We proceed now with the discussion of the series $F^s g$, $G^s g$ on the basis of the equations (5.22), (5.25) and (5.26). It will be shown in §5h that the classification of these series amounts to the classification of all infinitesimal representations of $\mathfrak{S}$.

5g. *Classification of the possible representations. I. The continuous class.* Assume first that all vectors of the series $F^s g$, $G^s g$ are different from zero. Then all numbers $\lambda \pm s$ are proper values of H_0, and are either all integral or half integral. Moreover, all numbers ρ_s and σ_s are *positive*. (Cf. (5.26)) It follows from the equations (5.25a) and (5.25b) that, apart from a non-vanishing factor, every vector of the series may be obtained from any fixed $F^{s_0}g$ or $G^{s_0}g$ by applying to it either a suitable power of F or a suitable power of G, i.e., any $F^{s_0}g$ or any $G^{s_0}g$ may be substituted for g.

(1) If the sequence $\lambda \pm s$ is *integral*, the value 0 occurs in it; for convenience we choose our basic g as the corresponding proper vector, and assume it to be normalized ($\| g \| = 1$). We then have

$$H_0(F^s g) = s \cdot F^s g, \qquad H_0(G^s g) = -s \cdot G^s g \qquad s \geqq 0.$$

Moreover, we have from (5.26)

(5.27) $$\begin{cases} \| F^s g \| = r_s, & r_s = \prod_{n=1}^{s} \rho_n^{\frac{1}{2}} \qquad s \geqq 1, \\ \| G^s g \| = r'_s, & r'_s = \prod_{n=1}^{s} \sigma_n^{\frac{1}{2}} \qquad s \geqq 1. \end{cases}$$

The requirement that all ρ_s and σ_s be *positive* restricts the admissible values of q. From (5.25) and $\lambda = 0$ we at once obtain the necessary and sufficient condition, $q > 0$. It is also seen that $\rho_s = \sigma_s = q + s(s - 1)$, and hence $r_s = r'_s$.

We finally introduce the following vectors f_m ($m = 0, \pm 1, \pm 2, \cdots$)

$$f_0 = \eta_0 g, \quad f_m = (\eta_m/r_m)F^m g, \quad f_{-m} = (\eta_{-m}/r'_m)G^m g \qquad (m \geqq 1) \tag{5.28}$$

where the η may be any arbitrarily chosen, complex numbers of absolute value 1. The f_m form a set of *orthogonal unit vectors*. In fact, by (5.27) $\|f_m\| = 1$, and since $H_0 f_m = m \cdot f_m$, two different vectors of the set are orthogonal to each other. Using (5.25) (with $\lambda = 0$) we finally obtain the following equations, for $m = 0, \pm 1, \pm 2, \cdots$,

$$\begin{cases} H_0 f_m = m f_m, & F f_m = \omega_{m+1}(q + m(m+1))^{\frac{1}{2}} f_{m+1} \\ G f_m = (1/\omega_m)(q + m(m-1))^{\frac{1}{2}} f_{m-1}, & \omega_m = \eta_{m-1}/\eta_m, \quad |\omega_m| = 1. \end{cases} \tag{5.29}$$

(2) A similar analysis applies to the *half integral* case. The value $\frac{1}{2}$ must occur in the series $\lambda \pm s$, and the corresponding proper vector is chosen as the basic g ($\|g\| = 1$), so that

$$H_0(F^s g) = (\tfrac{1}{2} + s)F^s g, \qquad H_0(G^s g) = (\tfrac{1}{2} - s)G^s g$$

(5.27) remaining unchanged. Here we find from (5.25) (for $\lambda = \frac{1}{2}$ and $s \geqq 1$) $\rho_s = (q - \frac{1}{4}) + s^2$, $\sigma_s = (q - \frac{1}{4}) + (s-1)^2$. Consequently all ρ_s and σ_s are positive if and only if $q > \frac{1}{4}$.

In analogy to (5.28), we introduce a set of normalized orthogonal vectors f_m, where $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots$, by the following definitions:

$$f_{\frac{1}{2}} = \eta_{\frac{1}{2}} \cdot g, \qquad f_{\frac{1}{2}+s} = (\eta_{\frac{1}{2}+s}/r_s)F^s g, \quad f_{\frac{1}{2}-s} = (\eta_{\frac{1}{2}-s}/r'_s)G^s g \qquad (s \geqq 1) \tag{5.30}$$

η_m being complex numbers of absolute value 1. The relations (5.29) remain unchanged, but the numbers m have a different range.

The class considered here is termed "*continuous class*" because the admissible values of q fill an (infinite) interval in the integral as well as in the half integral case.

II. The discrete class.

(1) Assume now that for some positive s $F^s g = 0$. This will also hold for all succeeding s. If $h + 1$ ($h \geqq 0$) is the *smallest* integer for which this occurs then the vectors $g, \cdots, F^h g$ are all different from zero, and it again follows that all vectors in the series may be obtained from $F^h g$ by applying a power of G to it, and hence we may substitute $F^h g$ for the basic vector g of the series. Then $H_0 g = \lambda g$, $Fg = 0$, and $\rho_1 = 0$. Consequently $q = -\lambda(\lambda + 1)$ (cf. (5.25a)). All vectors of the series are of the form $G^s g$, and $\sigma_s = s(s - 1 - 2\lambda)$ (cf. (5.25b)). Since $\sigma_1 \geqq 0$ it follows that $\lambda \leqq 0$.

If $\lambda = 0$, we have $\sigma_1 = 0$, and hence $Gg = 0$. Then the series reduces to the one vector g, and we have the trivial case $H_0 g = H_1 g = H_2 g = 0$. It is seen that $\exp(-itH_r)g = g$ for all r, and since by (4.12) every group element is a product of three such exponentials, $U(b)g = g$ for all $b \in \mathfrak{S}$. The manifold generated by the vector g is invariant under the $U(b)$, and is hence the whole representation space $\mathfrak{H}$.

If $\lambda < 0$, we set $\lambda = -k$, where k is integral or half integral. Then

$\sigma_s = s(s - 1) + 2ks$ is *positive* for all $s \geqq 1$, and hence all $G^s g$ are *different from zero*. The set f_m will be defined for the values $m = -k, -(k + 1), \cdots$ by the following equations:

$$f_{-k} = \eta_{-k} \cdot g, \qquad f_{-(k+s)} = (\eta_{-(k+s)}/r_s')G_s g \qquad (s \geqq 1). \tag{5.31}$$

The equations (5.29) still hold, but the following must be added: For $m = -k$, there appears in the equation for Ff_{-k} the vector $f_{-(k-1)}$ which is not defined. The corresponding coefficient, however, vanishes, and the equation is to be interpreted as $Ff_{-k} = 0$.

We notice the relations

$$q = k(1 - k), \qquad q + m(m + 1) = (m + k)(m - k + 1). \tag{5.32}$$

(2) The case in which some vector $G^s g (s \geqq 1)$ vanishes, is treated in exactly the same way. Disregarding the trivial one-dimensional representation, we may state the result as follows: Let g be the proper vector associated with the *lowest* proper value k $(k > 0)$. Then $Gg = 0$, and we define the vectors f_m $(m = k, k + 1, \cdots)$ by the equations

$$f_k = \eta_k g, \qquad f_{k+s} = (\eta_{k+s}/r_s)F^s g \qquad (s \geqq 1). \tag{5.33}$$

We have again $q = k(1 - k)$, and the equations (5.29) and (5.32) apply.

The term "*discrete class*" refers to the fact that q can only assume the values $k(1 - k)$, $k = \frac{1}{2}, 1, \frac{3}{2}, \cdots$. Observe that except for $k = \frac{1}{2} (q = \frac{1}{4})$ and $k = 1$ $(q = 0)$ all these values are *negative*.

5h. *Discussion.* (1) *In each case the vectors f_m span the Hilbert space $\mathfrak{H}$.* It follows from the equations (5.29) that the operators H_0, H_1, H_2 may be applied to any *finite* linear combination of the f_m and that they again lead to *finite* linear combinations of the f_m. The linear manifold defined by these finite linear combinations is therefore *invariant* with respect to the operators H_r. From this it may be shown that the *closure* of this manifold is invariant with respect to all $U(b)$, and therefore it must be the whole representation space. It follows that the classification of the series considered above amounts to a classification of all possible irreducible infinitesimal representations.

(2) *Characterization of the operators H_r.* Since $H_1 = \frac{1}{2}(F + G)$ and $H_2 = (i/2)(G - F)$, we obtain from (5.29) a set of equations which may be written in the form

$$H_r f_m = \sum_n h_{mn}^{(r)} f_m \qquad (r = 0, 1, 2) \tag{5.34}$$

where $h_{nm}^{(r)} = \overline{h_{mn}^{(r)}}$, and where $h_{nm}^{(r)} = 0$ if $|m - n| > 1$. (It is not necessary to state here the explicit expressions for the $h_{mn}^{(r)}$.) The range of the m has been determined in each case. These equations define the operators H_r on all finite linear combinations of the vectors f_m. It is desirable to characterize the operators H_r more explicitly in terms of these equations. This may be done with the help of the following lemma, which we state without proof.

LEMMA 3. *Let three operators K_r ($r = 0, 1, 2$) on $\mathfrak{H}$ be defined as follows. The domain of K_r consists of all finite linear combinations of the vectors f_m which span $\mathfrak{H}$, and on this domain K_r is defined by the equations $K_r f_m = \sum_n h_{mn}^{(r)} f_m$. The closure $\bar{K}_r$ of K_r is a self-adjoint operator whose domain $\mathfrak{D}_r$ consists of all vectors g for which $\sum_m | \sum_n h_{mn}^{(r)}(f_n, g) |^2$ is finite, and on $\mathfrak{D}_r$ $\bar{K}_r g = \sum_m \{ \sum_n h_{mn}^{(r)}(f_n, g) \} f_m$.*

We see that the operators H_r must coincide with the operators $\bar{K}_r$ of Lemma 3, and we may conclude that the condition (1) of (5.6) is satisfied. The sequence mentioned there may even be chosen to consist of finite linear combinations of the basic vectors f_m which are certainly contained in $\mathfrak{B}$. (It should be mentioned that Lemma 3 holds also when K_r is replaced by any combination of K with real coefficients.)

Concerning the manifolds $\mathfrak{A}$ and $\mathfrak{B}$, the following may be shown from (5.34).

LEMMA 4. *The manifold $\mathfrak{A}$ consists of all vectors g for which $\sum_m m^2 |(f_m, g)|^2$ is finite; the manifold $\mathfrak{B}$ consists of all vectors for which $\sum_m m^4 |(f_m, g)|^2$ is finite. In other words, $\mathfrak{A}$ is the domain of the operator H_0, and $\mathfrak{B}$ is the domain of the operator H_0^2.*

Finally we mention the following result. Define H_r as the closure of the operators K_r of Lemma 3. On $\mathfrak{B}$ we then have the relations

$$(5.35) \qquad [H_0 H_1] = iH_2, \qquad [H_1 H_2] = -iH_0, \qquad [H_2 H_0] = iH_1$$

and $(H_1^2 + H_2^2 - H_0^2)g = q \cdot g$, where q is defined by the coefficients of the equations (5.34).

(3) It has not yet been shown that to every infinitesimal representation which we have found there corresponds a representation of the group $\mathfrak{S}$ itself by unitary operators. This will be proved in the succeeding sections by an actual construction. However we may assert the following.

A unitary representation is uniquely determined by the operators H_r. In fact, by (4.12) every element of $\mathfrak{S}$ may be expressed as

$$a = \exp(2\mu\chi_0) \cdot \exp(2\zeta\chi_2) \cdot \exp(2\nu\chi_0),$$

and hence the corresponding operator U is given by

$$U = \exp(-2i\mu H_0) \cdot \exp(-2i\zeta H_2) \cdot \exp(-2i\nu H_0).$$

A unitary representation for which the operators H_r are defined by the equations (5.34) *is irreducible.* To prove this we have to show that every bounded operator A which commutes with all $U(a)$ is necessarily of the form $A = \alpha \cdot 1$. Apply the equation $U(a)A = AU(a)$ to a vector g in $\mathfrak{A}$, and choose $U(a) = \exp(itH_r) = U_t$. Then $U_t A g = AU_t g$, and $t^{-1}(U_t - 1)Ag = A\{t^{-1}(U_t - 1)\}g$. If we let t tend to zero, we obtain $H_r A g = AH_r g$ for every g in $\mathfrak{A}$. Choose $g = f_m$, $H_r = H_0$. It follows by familiar arguments that $Af_m = \alpha_m f_m$. For H_r we therefore obtain the equations $(\alpha_m - \alpha_n)h_{mn}^{(r)} = 0$ for all m and n. From the explicit expressions (5.29) we find that all α_m are equal to each other, i.e., that $A = \alpha \cdot 1$ q.e.d.

5i. *Summary.* The representations which we have found may be characterized by the value of q and by the range of the m, i.e. by the spectrum of the operator H_0. (It is evident that two representations which differ in both cannot be equivalent.) We also introduce symbols to denote the different representations.

I. Continuous Class.

(1) *Integral case* $C_q^0\{q > 0, m = 0, \pm 1, \pm 2, \cdots\}$

(2) *Half integral case* $C_q^{\frac{1}{2}}\{q > \frac{1}{4}, m = \pm \frac{1}{2}, \pm\frac{3}{2}, \cdots\}$

II. Discrete Class

(1) *Maximal m* $D_k^-\{q = k(1 - k), m = -k, -(k + 1), \cdots\}$ $k = \frac{1}{2}, 1, \frac{3}{2}, \cdots$

(2) *Minimal m* $D_k^+\{q = k(1 - k), m = k, k + 1, \cdots\}$ $k = \frac{1}{2}, 1, \frac{3}{2}, \cdots$

(The one-dimensional case is omitted.) In every case the operators H_r are defined by the equations (5.29).

Remark. We have pointed out in §3e that a representation of $\mathfrak{S}$ is a single-valued representation of $\mathfrak{L}$ if and only if $U(-e) = 1$. Since by (4.11) and (4.12) $(-e) = \exp(2\pi\chi_0)$, we have $U(-e) = \exp(-2\pi i H_0)$, hence by (5.29) $U(-e)f_m = e^{-2\pi i m}f_m$. Consequently $U(-e) = 1$ in the *integral* case, and $U(-e) = -1$ in the *half integral* case.

§6. Representations of the continuous class, C_q^0 ($q \geqq \frac{1}{4}$)

It is easy to construct unitary representations of $\mathfrak{L}$ (and hence of $\mathfrak{S}$) as long as they are not required to be irreducible. For example, one may choose in the Euclidean space of the three variables x^0, x^1, x^2, the manifold $\mathfrak{M}$ (invariant under Lorentz transformations) defined by the equation $g_{kl}x^kx^l = d(=\text{const.})$, and consider the transformations $y = ax$ which are induced on $\mathfrak{M}$ by the Lorentz transformations of the x^k. (Depending on the value of d, $\mathfrak{M}$ is a hyperboloid, of one sheet ($d > 0$), or of two sheets ($d < 0$), or it is a cone ($d = 0$), the light-cone of special relativity). A volume element which is invariant with respect to the standard transformations $T^0(a)f = f(a^{-1}x)$ is readily defined on $\mathfrak{M}$. Therefore the $T^0(a)$ are unitary operators on the Hilbert space of all square-integrable functions over $\mathfrak{M}$, and they furnish a representation of $\mathfrak{L}$ since $T^0(a)T^0(b) = T^0(ab)$.

A simple analysis shows that these representations are *reducible.* It turns out that the operators $T^0(a)$ are particularly simple if $\mathfrak{M}$ is the light-cone. In what follows we shall show that a further reduction leads to *irreducible* representations (of the continuous class C_q^0).

6a. *Transformations on the light-cone.* It is sufficient to consider the *upper half* of the light-cone ($x^0 > 0$), corresponding to the future in the physical interpretation. We therefore define the manifold $\mathfrak{M}$ by the equations

$$g_{kl}x^kx^l = 0, \qquad x^0 > 0. \tag{6.1}$$

On $\mathfrak{M}$ we introduce two sets of parameters, viz. the rectangular coordinates ξ^1, ξ^2

$$x^1 = \xi^1, x^2 = \xi^2, \qquad x^0 = r = ((\xi^1)^2 + (\xi^2)^2)^{\frac{1}{2}}, \qquad 0 < r < \infty \tag{6.2}$$

and the polar coordinates r, ϕ

(6.3) $$x^1 + ix^2 = re^{i\phi}, \qquad x^0 = r; \qquad 0 < r < \infty.$$

The angle ϕ is defined mod. 2π. All functions occurring are assumed periodic in ϕ.

The infinitesimal operators of the standard representation on $\mathfrak{M}$, Λ^0_χ, are obtained from (2.8a). Remembering that $\Lambda^0_0 = \Lambda^0_{12}$, $\Lambda^0_1 = \Lambda^0_{02}$, $\Lambda^0_2 = \Lambda^0_{10}$, we find

(6.4) $$\begin{cases} \Lambda^0_0 = -\xi^1 \dfrac{\partial}{\partial \xi^2} + \xi^2 \dfrac{\partial}{\partial \xi^1} = -\dfrac{\partial}{\partial \phi}, \qquad \Lambda^0_1 = r\dfrac{\partial}{\partial \xi^2} = \cos\phi \dfrac{\partial}{\partial \phi} + \sin\phi\, r\dfrac{\partial}{\partial r} \\ \Lambda^0_2 = -r\dfrac{\partial}{\partial \xi^1} = \sin\phi \dfrac{\partial}{\partial \phi} - \cos\phi\, r \dfrac{\partial}{\partial r} \\ \Lambda^0_1 + i\Lambda^0_2 = r\left(\dfrac{\partial}{\partial \xi^2} - i\dfrac{\partial}{\partial \xi^1}\right) = e^{i\phi}\left(\dfrac{\partial}{\partial \phi} - ir\dfrac{\partial}{\partial r}\right). \end{cases}$$

For the operator corresponding to Q we obtain the simple expression

(6.5) $$Q^0 = (\Lambda^0_0)^2 - (\Lambda^0_1)^2 - (\Lambda^0_2)^2 = -D(D+1) \qquad (D = r\partial/\partial r).$$

Consider now, for a given constant q, the differential equation

(6.6a) $$Q^0 f = q \cdot f$$

where f is a function of r and ϕ which has continuous second derivatives. Its solutions are

(6.6b) $$f = r^{-(\frac{1}{2}+\sigma)} g(\phi) \qquad \sigma = \pm(\tfrac{1}{4} - q)^{\frac{1}{2}}$$

with an arbitrary function $g(\phi)$. Observe that σ is *imaginary* if $q > \frac{1}{4}$, and *real* if $q < \frac{1}{4}$. This distinction will prove significant in the following discussion.

6b. *Transformations on the unit circle.* We have seen in §5 that for every irreducible representation of $\mathfrak{S}$ the operator Q has a constant value q. Therefore the equations (6.6) suggest that we confine ourselves to functions of the form (6.6b) with a constant σ. In fact, it may be seen in various ways that under the transformations $T^0(a)$ on $\mathfrak{M}$ every f of the form (6.6b) is carried into a function of the same form with the same value of σ.

Introduce, for $x^0 > 0$, the variables $\eta^k = x^k/x^0$. Any Lorentz transformation induces a *projective transformation* of η^1 and η^2. On the light-cone $\mathfrak{M}$ $\eta^1 + i\eta^2 = e^{i\phi}$, and this is therefore a projective transformation of the unit circle into itself. (Turning to a physical interpretation, we may also speak of the transformation of *light rays*.) If an element a of $\mathfrak{S}$ carries (r, ϕ) into $(r', \phi') = (ar, a\phi)$, we have, by (3.18) and (3.19)

(6.7) $$\begin{cases} r' = ar = r\,|\,w(a,\phi)\,|^2, \quad e^{i\phi'} = e^{i\phi}\overline{w(a,\phi)}/w(a,\phi) \qquad (\phi' = a\phi) \\ w(a,\phi) = \alpha + \beta e^{i\phi}. \end{cases}$$

We also notice that for a fixed a

(6.8) $$\frac{d\phi'}{d\phi} = \frac{1}{|w(a, \phi)|^2}.$$

From the discussion in §1h it follows at once that

(6.9) $$\mu(a, \phi) = |w(a, \phi)|^2$$

is a *multiplier* of the group of transformations $\phi' = a\phi$. The cone is the manifold $\mathfrak{M}$, and the unit circle corresponds to $\mathfrak{M}^*$.

With the help of the standard representation on $\mathfrak{M}$ we define now a *multiplier representation* on the unit circle $\mathfrak{M}^*$ by the equations (cf. (1.57))

(6.10) $$T^0(a)(r^{-(\frac{1}{2}+\sigma)}f(\phi)) = r^{-(\frac{1}{2}+\sigma)}T_\sigma(a)f(\phi).$$

The muliplier is given by $\{\mu(a, \phi)\}^h$, $h = \sigma + \frac{1}{2}$.

It should be noted that *any real or complex power* of $\mu(a, \phi)$ may be defined in an unambiguous way. In fact, we may write $\mu(a, \phi) = |\alpha|^2(1 + \gamma e^{i\phi})(1 + \bar{\gamma}e^{-i\phi})$, $\mu^h = \exp(h \log \mu)$, and the logarithm of μ is even an analytic function of a and of ϕ, since $|\alpha|^2$ is positive, and $\gamma = \beta/\alpha$ has an absolute value less than one. Therefore (6.10) holds for any real or complex value of σ.

The multiplier transformations $T_\sigma(a)$ may also be written

(6.11) $$T_\sigma(a)f(\phi) = \mu(a, \phi)^{\frac{1}{2}+\sigma}f(a^{-1}\phi).$$

The corresponding infinitesimal transformations are defined by

$$\Lambda^0_\chi(r^{-(\frac{1}{2}+\sigma)}f(\phi)) = r^{-(\frac{1}{2}+)\sigma}\Lambda_\chi f(\phi)$$

(cf.(1.58)). Hence we obtain from (6.4)

(6.12) $$\begin{cases} \Lambda_0 = -\dfrac{\partial}{\partial\phi}, & \Lambda_1 + i\Lambda_2 = e^{i\phi}\left\{\dfrac{\partial}{\partial\phi} + i(\frac{1}{2} + \sigma)\right\} \\ \Lambda_1 = \cos\phi\,\dfrac{\partial}{\partial\phi} - (\frac{1}{2} + \sigma)\sin\phi, & \Lambda_2 = \sin\phi\,\dfrac{\partial}{\partial\phi} + (\frac{1}{2} + \sigma)\cos\phi. \end{cases}$$

Consequently, the infinitesimal multiplier is given by

(6.13) $\tau_0 = 0$, $\tau_1 = -(\frac{1}{2} + \sigma)\sin\phi$, $\tau_2 = (\frac{1}{2} + \sigma)\cos\phi$; $\tau_1 + i\tau_2 = i(\frac{1}{2} + \sigma)e^{i\phi}$.

Finally, we introduce the operator

$$Q = (\Lambda_0)^2 - (\Lambda_1)^2 - (\Lambda_2)^2$$

and we find from (1.58a) and (6.6)

(6.14) $$Qf(\phi) = r^{\frac{1}{2}+\sigma}Q^0(r^{-(\frac{1}{2}+\sigma)}f(\phi) = q\cdot f(\phi)) \qquad q = \tfrac{1}{4} - \sigma^2.$$

6c. *Discussion of the transformations $T_\sigma(a)$.* Let $\mathfrak{H}$ be the Hilbert space of all square-integrable functions $f(\phi)$ over the unit circle, and let the inner product in $\mathfrak{H}$ be defined by

(6.15) $$(f, g) = (2\pi)^{-1}\int_{-\pi}^{\pi}\overline{f(\phi)}g(\phi)\,d\phi.$$

It follows from (6.11) that for any choice of σ and τ

$$(T_\sigma(a)f, T_\tau(a)g) = (2\pi)^{-1} \int_{-\pi}^{\pi} \overline{f(\phi')} g(\phi') [\mu(a, \phi')]^{\bar{\sigma}+\tau+1} \frac{d\phi}{d\phi'} d\phi' \qquad (\phi' = a^{-1}\phi)$$

and since by (6.8) $d\phi/d\phi' = [\mu(a, \phi')]^{-1}$,

$$(T_\sigma(a)f, T_\tau(a)g) = (2\pi)^{-1} \int_{-\pi}^{\pi} \overline{f(\phi')} g(\phi') [\mu(a, \phi')]^{\bar{\sigma}+\tau} d\phi'. \tag{6.16}$$

From (6.16) we may draw the following conclusions:

(1) *If $\sigma = is$ (s any real number), the operators $T_\sigma(a)$ are unitary.* In fact, $(T_\sigma(a))^{-1}$ exists, and $(T_\sigma(a)f, T_\sigma(a)g) = (f, g)$.

(2) *In all cases the operators $T_\sigma(a)$ are bounded.* (Set $\sigma = \tau, f = g$.) It may be easily shown that the bound of $T_\sigma(a)$ is equal to $e^{|(\sigma+\bar{\sigma})\zeta|}$ where ζ is the parameter introduced in §4.

(3) The adjoint of $T_\sigma(a)$ is given by $T_\sigma^*(a) = T_{-\bar{\sigma}}(a^{-1})$.

Matrix elements. We choose on $\mathfrak{H}$ the complete orthonormal set of functions

$$f_m = e^{im\phi} \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{6.17}$$

In this coordinate system the matrix elements of the operator $T_\sigma(a)$ are

$$u_{mn}(a) = (f_m, T_\sigma(a)f_n). \tag{6.18}$$

It is evident that the matrix elements $u_{mn}(a)$ are *analytic functions* on the group manifold. (They are also analytic in the exponent σ.) Clearly $u_{mn}(e) = \delta_{mn}$.

6d. *Representations of the class C_q^0 ($q \geqq \frac{1}{4}$).* We are going to show that the unitary representation $T_\sigma(a)$ ($\sigma = is$) is a representation of the class C_q^0, where $q = \frac{1}{4} - \sigma^2 = \frac{1}{4} + s^2$. According to the discussion in §5h we must prove:

(1) The operators $T_\sigma(a)$ satisfy the continuity condition (3) of (5.2).

(2) The infinitesimal operators satisfy the equations (5.29) where $q = \frac{1}{4} - \sigma^2$.

Ad (1). Consider a fixed vector g of $\mathfrak{H}$, and set for every integer N $g_N = \sum_{m=-N}^{N} (f_m, g)f_m$, $h_N = g - g_N$. We have, with $U(a) = T_\sigma(a)$,

$$\begin{aligned} \| (U(a) - 1)g \| &\leqq \| (U(a) - 1)g_N \| + \| (U(a) - 1)h_N \| \\ &\leqq \| (U(a) - 1)g_N \| + 2 \| h_N \| \end{aligned} \tag{6.19}$$

and, since $U(a)$ is a unitary operator,

$$\begin{cases} \| (U(a) - 1)g_N \|^2 = ((U(a) - 1)g_N, g_N) + (g_N, (U(a) - 1)g_N) \\ \qquad = \text{real part } \{2 \sum_{m,n=-N}^{N} (g, f_m) (u_{mn}(a) - \delta_{mn}) (f_n, g)\}. \end{cases} \tag{6.20}$$

For a prescribed positive ϵ we may first choose a number N such that $2 \| h_N \| < \epsilon$, and secondly, a neighborhood of e on $\mathfrak{S}$ such that for every element a in this neighborhood the finite sum in (6.20) is smaller than ϵ^2 (since the matrix elements are continuous in a and $u_{mn}(e) = \delta_{mn}$). By (6.19) we have $\| (U(a) - 1)g \| < 2\epsilon$ for these elements a, q.e.d.

Ad (2). Once the continuity of the operators $U(a) = T_\sigma(a)$ has been proved, the existence of infinitesimal operators H_χ follows from Stone's theorem. Let $U_t = \exp(-itH_\chi) = T_\sigma(\exp t\chi)$, and let $g(\phi)$ be an *analytic* function of ϕ. Then for every ϕ $t^{-1}(U_t - 1)g(\phi)$ converges to $\Lambda_\chi g(\phi)$ as $t \to 0$ (cf. (1.30). Since the convergence is *uniform* over the unit circle, this implies convergence in the mean, i.e., convergence in the Hilbert space $\mathfrak{H}$. Consequently, for every analytic function g (in particular for $f_m = e^{im\phi}$)

$$H_\chi g = i\Lambda_\chi g. \tag{6.21}$$

Computing $\Lambda_r f_m$ ($r = 0, 1, 2$) from the definitions (6.12) we find

$$\begin{cases} i\Lambda_0 f_m = mf_m, \qquad i(\Lambda_1 + i\Lambda_2)f_m = -(m + \frac{1}{2} + \sigma)f_{m+1} \\ i(\Lambda_1 - i\Lambda_2)f_m = -(m - \frac{1}{2} - \sigma)f_{m-1} \end{cases} \tag{6.22}$$

$$\begin{cases} \Lambda_1 f_m = \dfrac{i}{2}(m + \frac{1}{2} + \sigma)f_{m+1} + \dfrac{i}{2}(m - \frac{1}{2} - \sigma)f_{m-1} \\ \Lambda_2 f_m = \frac{1}{2}(m + \frac{1}{2} + \sigma)f_{m+1} - \frac{1}{2}(m - \frac{1}{2} - \sigma)f_{m-1}. \end{cases} \tag{6.22a}$$

The equations (6.22) coincide with (5.29) if we choose the constants $\eta_m(\sigma)$ as follows ($q = \frac{1}{4} - \sigma^2 = \frac{1}{4} + s^2$):

$$\begin{cases} \eta_0(\sigma) = 1, \\ \eta_m(\sigma) = (-1)^m \displaystyle\prod_{h=1}^{m} \frac{h - \frac{1}{2} - \sigma}{((h - \frac{1}{2})^2 - \sigma^2)^{\frac{1}{2}}} = (-1)^m \prod_{h=1}^{m} \left(\frac{h - \frac{1}{2} - \sigma}{h - \frac{1}{2} + \sigma}\right)^{\frac{1}{2}} \\ \eta_{-m}(\sigma) = (-1)^m \eta_m(\sigma) \qquad\qquad (m \geqq 1) \end{cases} \tag{6.23}$$

so that for all values of m

$$\omega_m(\sigma) = \eta_{m-1}(\sigma)/\eta_m(\sigma) = -\frac{m - \frac{1}{2} - \sigma}{((m - \frac{1}{2})^2 - \sigma^2)^{\frac{1}{2}}} = -\left(\frac{m - \frac{1}{2} + \sigma}{m - \frac{1}{2} - \sigma}\right)^{\frac{1}{2}}. \tag{6.24}$$

For imaginary σ all $\eta_m(\sigma)$ have the absolute value 1.

So far it has been shown that for analytic functions the operator H_χ coincides with $i\Lambda_\chi$. From the discussion in §5h (cf. in particular Lemma 3) it is seen that H_χ is the *closure* of $i\Lambda_\chi$. For analytic functions the condition $H_{(\chi+\chi')} = H_\chi + H\chi'$ likewise holds, because it is satisfied by the operators Λ_χ, and it can be extended to the linear manifold $\mathfrak{A}$ defined in §5. The same is true for the commutation rules (5.35) which may also be deduced from (1.48). This shows that the representation $T_\sigma(a)$ satisfies all conditions stated in §5. We may state our result as follows.

THEOREM 1. *The multiplier representation $T_\sigma(a)$ ($\sigma = is$) is an irreducible unitary representation of $\mathfrak{S}$ on the Hilbert space $\mathfrak{H}$ defined by* (6.15). *It belongs to the class C_q^0 where $q = \frac{1}{4} + s^2$.*

The remaining representations C_q^0 where q is in the "exceptional interval" $0 < q < \frac{1}{4}$ are treated in §8.

6e. *A new coordinate system in* $\mathfrak{H}$. The functions

$$g_m = (1/\eta_m(\sigma))f_m \tag{6.25}$$

evidently form a complete orthonormal system of vectors in $\mathfrak{H}$, if $\sigma = is$. With the definitions (6.23) it follows at once that

$$\begin{cases} i\Lambda_0 g_m = m g_m, \quad i(\Lambda_1 + i\Lambda_2)g_m = (q + m(m+1))^{\frac{1}{2}}\, g_{m+1} \\ i(\Lambda_1 - i\Lambda_2)g_m = (q + m(m-1))^{\frac{1}{2}}\, g_{m-1} \end{cases} \tag{6.26}$$

$$\begin{cases} \Lambda_1 g_m = \frac{i}{2}(q + m(m+1))^{\frac{1}{2}}\, g_{m+1} + \frac{i}{2}(q + m(m-1))^{\frac{1}{2}}\, g_{m-1} \\ \Lambda_2 g_m = \frac{1}{2}(q + m(m+1))^{\frac{1}{2}}\, g_{m+1} - \frac{1}{2}(q + m(m-1))^{\frac{1}{2}}\, g_{m-1}. \end{cases} \tag{6.27}$$

The matrix elements of $T_\sigma(a)$ referred to the system g_m will be denoted by

$$v_{mn}(a) = (g_m, T_\sigma(a)g_n). \tag{6.28}$$

By (6.18) we have

$$v_{mn}(a) = (\eta_m(\sigma)/\eta_n(\sigma))\, u_{mn}(a). \tag{6.29}$$

While the vectors f_m are the same for all σ, the vectors g_m depend explicitly on the representation $T_\sigma(a)$.

REMARK. For later use (§8d) we notice the following. If σ is real, and $\sigma^2 < \frac{1}{4}$, we may still define the g_m by (6.25). They will not be normalized in $\mathfrak{H}$, but the equations (6.26) and (6.27) will hold.

§7. Representations of the continuous class $C_q^{\frac{1}{2}}$

7a. *The representations* $T'_\sigma(a)$. It has been shown at the end of §5i that in the half integral case $U(-e) = -1$, where the group element $(-e)$ has the parameters $\alpha = -1, \beta = 0$. If we want to obtain a multiplier representation

$$T'_\sigma(a)\, f(\phi) = \mu'_\sigma(a, a^{-1}\phi)\, f(a^{-1}\phi) \tag{7.1}$$

(on the unit circle) of class $C_q^{\frac{1}{2}}$ we must have

$$\mu'_\sigma(-e, \phi) = -1 \tag{7.2}$$

since $a^{-1}\phi = \phi$ for $a = -e$. We set

$$\mu'_\sigma(a, \phi) = \mu(a, \phi)^{\frac{1}{2}+\sigma}\, \nu(a, \phi) \tag{7.3}$$

where $\mu(a, \phi)$ is defined by (6.8) and $\nu(a, \phi)$ is given by

$$\nu(a, \phi) = w(a, \phi)/\,|\,w(a, \phi)\,| \qquad w(a, \phi) = \alpha + \beta e^{i\phi} \tag{7.4}$$

(cf. (6.7)). The equation (7.1) may be written in the form

$$T'_\sigma(a)\, f(\phi) = \nu(a, a^{-1}\phi)\cdot T_\sigma(a)\, f(\phi) \tag{7.5}$$

where $T_\sigma(a)$ is the operator defined in the preceding section.

To prove that ν is a multiplier it is sufficient to show that w is a multiplier (since $\nu(a, \phi) = w(a, \phi)\cdot\mu(a, \phi)^{-\frac{1}{2}}$), and this follows from the fact that $w(a, \phi)$ coincides with the multiplier $\alpha + \beta z$ (cf. (4.25)) if $z = e^{i\phi}$. (It is clear that the discussion in §4e applies to the points $|z| = 1$.) Moreover, $\mu(-e, \phi) = 1$, and $\nu(-e, \phi) = -1$, so that (7.2) holds. Consequently (7.5) defines a multiplier representation, and $T'_\sigma(-e) = -1$. Notice that $\nu(a, \phi)$ depends *analytically* on a and ϕ.

By the second equation (6.7) the square of $\nu(a, \phi)$ equals $e^{-i\phi'}/e^{-i\phi}(\phi' = a\phi)$. If we restrict ϕ to a sufficiently small neighborhood of some ϕ_0, and a to a sufficiently small neighborhood of the unit element on $\mathfrak{S}$ we may write

$$(7.6) \qquad \nu(a, \phi) = \rho(a\phi)/\rho(\phi) \qquad \rho(\phi) = e^{-i\phi/2}.$$

Hence $\nu(a, \phi)$ is *locally* a multiplier of the special form (1.43).

The relation (7.6) will now be used to derive the infinitesimal operator Λ'_χ of the representation $T'_\sigma(a)$. By (7.3), (7.5) and (7.6) we have *locally*

$$T'_\sigma(a)\, f(\phi) = \rho(\phi)\; T_\sigma(a)(f(\phi)/\rho(\phi))$$

and hence

$$\chi T'_\sigma(a)f = \Lambda'_\chi(T'_\sigma(a)f) = \rho\cdot\chi(T_\sigma(a)(f/\rho)) = \rho\Lambda_\chi(T_\sigma(a)(f/\rho))$$

where $T_\sigma(a)$ and Λ_χ are defined by (6.11) and (6.12). In particular, for $a = e$,

$$(7.7) \qquad \Lambda'_\chi\, f(\phi) = \rho(\phi)\Lambda_\chi\,(f(\phi)/\rho(\phi)) \qquad \rho(\phi) = e^{-i\phi/2}$$

For the operator $Q' = (\Lambda'_0)^2 - (\Lambda'_1)^2 - (\Lambda'_2)^2$ we find from (7.7) and (6.14)

$$(7.8) \qquad Q' f(\phi) = \rho(\phi)\; Q\,(f(\phi)/\rho(\phi)) = q\cdot f(\phi) \qquad q = \tfrac{1}{4} - \sigma^2.$$

7b. *Representations of the class* $C_q^{\frac{1}{2}}$. For $\sigma = is$, $T'_\sigma(a)$ is a *unitary* operator on $\mathfrak{H}$ because $T_\sigma(a)$ is then unitary, and $\nu(a, \phi)$ has the absolute value 1. We use on $\mathfrak{H}$ the same orthonormal set of vectors as before, but it will be convenient to denote it differently, viz.,

$$(7.9) \qquad f'_m = e^{i(m-\frac{1}{2})\phi} \qquad (m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \cdots).$$

If we compute the expressions $\Lambda'_r f'_m$ ($r = 0, 1, 2$) from (7.7) we obtain again the equations (6.22) and (6.22a), with the difference, however, that the numbers m are *half integral*, and that the constants $\eta_m(\sigma)$ of absolute value 1 will be chosen as follows:

$$(7.10) \qquad \begin{cases} \eta_{\frac{1}{2}} = 1, \\ \eta_{m+\frac{1}{2}} = (-1)^m \prod_{h=1}^{m} \dfrac{h-\sigma}{(h^2-\sigma^2)^{\frac{1}{2}}} = (-1)^m \prod_{h=1}^{m}\left(\dfrac{h-\sigma}{h+\sigma}\right)^{\frac{1}{2}} & (m \geqq 1) \\ \eta_{-\frac{1}{2}} = \dfrac{-\sigma}{|\sigma|}, \qquad \eta_{-(m+\frac{1}{2})} = (-1)^{m-1}\dfrac{\sigma}{|\sigma|}\cdot\eta_{m+\frac{1}{2}} & (m \geqq 1). \end{cases}$$

The expressions for $\omega_m(\sigma)$ are the same as in (6.24). We assume here that $\sigma \neq 0$. For $\sigma = 0$ we obtain a unitary representation, but it is *reducible*, since the manifold spanned by the vectors f'_m with positive m and its orthogonal complement are invariant.

THEOREM 2. *The multiplier representation $T'_\sigma(a)$ ($\sigma = is$, $s \neq 0$) is an irreducible unitary representation of $\mathfrak{S}$ on the Hilbert space $\mathfrak{H}$ defined by* (6.15). *It belongs the class $C_q^{\frac{1}{2}}$ where $q = \frac{1}{4} + s^2$.*

The proof is literally the same as the proof of Theorem 1 given in §6d.

7c. *A new coordinate system in $\mathfrak{H}$.* With the help of the $\eta_m(\sigma)$ defined by (7.10) we may introduce the vectors

$$g'_m = (1/\eta_m(\sigma))\, f'_m \tag{7.11}$$

which form a complete orthonormal system, and which satisfy the equations (6.26) and (6.27).

The matrix elements of $T'_\sigma(a)$ referred to the systems f'_m and g'_m will be denoted by

$$u_{mn}(a) = (f'_m\,,\, T'_\sigma(a) f'_n) \qquad v_{mn}(a) = (g'_m\,,\, T'_\sigma(a) g'_n) \tag{7.12}$$

$$v_{mn}(a) = (\eta_m(\sigma)/\eta_n(\sigma))\, u_{mn}(a). \tag{7.13}$$

7d. *Remarks on multi-valued representations.* It may be mentioned here that by a similar procedure one obtains irreducible unitary representations of the covering group $\mathfrak{C}$ which are no longer single-valued representations of $\mathfrak{S}$. We choose them in the form (7.5), with $\sigma = is$, where the multiplier ν, however, must be defined in terms of the parameters (γ, ω) introduced in §4, viz.,

$$\nu(a,\phi) = \left(\frac{w(a, \phi)}{\bar{w}(a, \phi)}\right)^h = e^{2ih\omega}\left(\frac{1 + \gamma e^{i\phi}}{1 + \bar{\gamma} e^{-i\phi}}\right)^h. \tag{7.14}$$

The exponent h may be any real number; h and $h + 1$ give rise to *equivalent* representations. (The multiplier (7.4) used above is obtained for $h = \frac{1}{2}$.) We have again $Qf = q \cdot f$, $q = \frac{1}{4} + s^2$.

Let a and b be two elements of $\mathfrak{C}$ which correspond to the same element of $\mathfrak{S}$, so that their respective parameters are given by (γ, ω) and $(\gamma, 2l\pi + \omega)$, $l = 0, \pm 1, \pm 2, \cdots$. Then the unitary operators $U(a)$ and $U(b)$ of the representation considered satisfy the relation

$$U(b) = e^{4il h\pi}\, U(a) \tag{7.15}$$

which characterizes the multiplicity of the representation. The spectrum of H_0 consists of the numbers $m = h \pm l$, and for $f'_m = e^{i(m-h)\phi}$ one obtains the equations (6.22).

§8. Representations of the continuous class C_q^0. The exceptional interval $0 < \frac{1}{4} < q$.

8a. *Determination of the kernel $L(\phi, \psi)$.* As we have seen in §6 the operators $T_\sigma(a)$ are not unitary on $\mathfrak{H}$ if the exponent σ is *real* and different from zero. It will now be shown that by a different definition of the inner product, i.e., by a different definition of the Hilbert space, they may be made unitary if $0 < \sigma < \frac{1}{2}$. The inner product will be defined by a *positive definite integral form*

$$(f, g)_\sigma \equiv (2\pi)^{-2} \int \int L_\sigma(\phi, \psi)\overline{f(\phi)}g(\psi)\, d\phi\, d\psi \qquad -\pi \leqq \phi, \psi \leqq \pi \tag{8.1}$$

with a suitably chosen kernel $L_\sigma(\phi, \psi)$ depending on σ. We must require this inner product to be invariant under the transformations $T_\sigma(a)$, so that

$$(T_\sigma(a)f, T_\sigma(a)g)_\sigma = (f, g)_\sigma . \tag{8.2}$$

This leads to the problem of "*invariant densities*" treated in §1i. The manifold $\mathfrak{M}$ of §1i is the product of two unit circles; hence x corresponds to the pair (ϕ, ψ), and the multiplier considered there is to be replaced by the product $\mu(a, \phi)^{\sigma+\frac{1}{2}} \cdot \mu(a, \psi)^{\sigma+\frac{1}{2}}$. Therefore the operator $\mathbf{M}_r$ is the sum of two such operators, the first acting on the variable ϕ, the second acting on the variable ψ. Consequently if we assume that $L_\sigma(\phi, \psi)$ is differentiable we find from (1.61) and (6.12) the following differential equations:

$$\begin{cases} \mathbf{M}_0 L_\sigma = \dfrac{\partial L_\sigma}{\partial \phi} + \dfrac{\partial L_\sigma}{\partial \psi} = 0 \\ (\mathbf{M}_1 + i\mathbf{M}_2)L_\sigma = -\dfrac{\partial}{\partial \phi}(e^{i\phi} L_\sigma) - \dfrac{\partial}{\partial \psi}(e^{i\psi} L_\sigma) + i(\tfrac{1}{2} + \sigma)(e^{i\phi} + e^{i\psi})L_\sigma = 0. \end{cases} \tag{8.3}$$

It follows from the first equation that L_σ is a function of the difference $(\phi - \psi)$, and from the second that it has the form

$$L_\sigma(\phi, \psi) = \mathfrak{L}_\sigma(\phi - \psi) = c \cdot (1 - \cos(\phi - \psi))^{\sigma - \frac{1}{2}} = c \cdot [2d(\phi, \psi)]^{\sigma - \frac{1}{2}} \tag{8.4}$$

with a constant c to be determined later. ($d(\phi, \psi)$ is the distance between the two points ϕ and ψ).

Before discussing the properties of the function L_σ we mention another derivation which is more easily generalized to the case of the group $\mathfrak{L}_4$ to be treated in Part II of this paper. From (1.60) we obtain the condition

$$[\mu(a, \phi)\mu(a, \psi)]^{\sigma+\frac{1}{2}} \frac{d\phi'}{d\phi} \frac{d\psi'}{d\psi} L_\sigma(\phi', \psi') = L_\sigma(\phi, \psi), \qquad (\phi' = a\phi, \psi' = a\psi).$$

Inserting the expression (6.8) for the derivatives we find

$$[\mu(a, \phi)\, \mu(a, \psi)]^{\sigma-\frac{1}{2}} L_\sigma(a\phi, a\psi) = L_\sigma(\phi, \psi). \tag{8.5}$$

Let x and y be two points on the light-cone with polar coordinates (r, ϕ) and (R, ψ) respectively. The transformation a carries them into (r', ϕ') and (R', ψ') where by (6.7) and (6.9) $r' = \mu(a, \phi)r$, $R' = \mu(a, \psi)R$. Since their scalar product $g_{kl}x^k x^l = rR(1 - \cos(\phi - \psi))$ remains invariant, this leads to the equation

$$\mu(a, \phi)\, \mu(a, \psi)\, (1 - \cos(\phi' - \psi')) = 1 - \cos(\phi - \psi) \quad (\phi' = a\phi, \psi = a\psi) \tag{8.6}$$

which shows that the expression (8.4) is a solution of (8.5). It is sufficient to know *one* solution of (8.5). In fact, if $\rho(\phi, \psi)$ is the ratio of two solutions, then $\rho(a\phi, a\psi) = \rho(\phi, \psi)$, and outside the lower dimensional region $\phi - \psi = 0$ any pair (ϕ, ψ) may be transformed into any other pair (by a suitably chosen a). Hence ρ is a constant for all pairs for which $\phi \neq \psi$.

The positive definite character of the kernel L_σ is discussed below. (cf. (8.14)).

8b. *Properties of* $L_\sigma(\phi, \psi)$. The function (8.4) is *finite* as long as $\sigma \geqq \frac{1}{2}$. If $\sigma < \frac{1}{2}$ it is infinite for $\phi = \psi$, however, its integral over ϕ and ψ remains *finite* for $\sigma > 0$. Therefore the expression (8.1) is certainly defined for *bounded* functions f, g. Moreover, we may choose the constant in (8.4) in such a way that $(f, g)_\sigma = 1$ if $f(\phi) = g(\phi) = 1$, which leads to the definition

$$(8.7) \qquad \mathfrak{L}_\sigma(\phi) = 2^{\frac{1}{2}-\sigma}\, \pi(\mathrm{B}(\sigma, \tfrac{1}{2}))^{-1}\, (1 - \cos\phi)^{\sigma-\frac{1}{2}} \qquad \sigma > 0.$$

($\mathrm{B}(x, y)$ is Euler's function $\Gamma(x)\,\Gamma(y)/\Gamma(x + y)$.)

We may show now that the integral (8.1) is defined for any *square-integrable* functions f and g. By (f, g) (without the subscript σ) we denote the inner product introduced by (6.15) for the Hilbert space $\mathfrak{H}$, and correspondingly by $\| f \|$ the norm $(f, f)^{\frac{1}{2}}$.

From (8.1) and (8.7) we obtain

$$| (f, g)_\sigma | \leqq (2\pi)^{-2} \int\int \mathfrak{L}_\sigma(\phi - \psi)\, | f(\phi) | \cdot | g(\psi) |\, d\phi\, d\psi$$

$$\leqq (2\pi)^{-1} \int_{-\pi}^{\pi} \mathfrak{L}_\sigma(\zeta) \left\{ (2\pi)^{-1} \int_{-\pi}^{\pi} | f(\zeta + \psi) | \cdot | g(\psi) |\, d\psi \right\} d\zeta$$

where the variable $\zeta = \phi - \psi$ has been introduced. By Schwarz' inequality the inner integral is at most equal to $\| f \| \cdot \| g \|$, and since the integral of $\mathfrak{L}_\sigma$ has been normalized to one we find

$$(8.8) \qquad | (f, g)_\sigma | \leqq \| f \| \cdot \| g \| .$$

In particular, it follows from the inequality (8.8) by familiar arguments that $(f, g)_\sigma$ is continuous in both Hilbert vectors f, g, i.e., if

$$\| f_n - f \| \to 0, \quad \| g_n - g \| \to 0,$$

then $(f_n, g_n)_\sigma \to (f, g)_\sigma$.

Let $f_m = e^{im\phi}$ for all integers m. Then

$$(8.9) \quad (f_m, f_n)_\sigma = 0 \quad \text{if} \quad m \neq n, \qquad (f_m, f_m)_\sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathfrak{L}_\sigma(\phi) e^{-im\phi}\, d\phi = \lambda_m(\sigma).$$

To compute the Fourier coefficients we notice that $\lambda_m = \lambda_{-m}$ and that in the integral (8.9) the exponential function may be replaced by $\cos m\phi$. Expressing $\cos m\phi$ ($m \geqq 0$) by the Tchebysheff polynomial $T_m(\cos\phi)$,

$$T_m(z) = (-1)^m \prod_{l=1}^{m} (2l - 1)^{-1} \cdot (1 - z^2)^{\frac{1}{2}} \cdot \frac{d^m}{dz^m} [(1 - z^2)^{m-\frac{1}{2}}]$$

we find by partial integration

$$\lambda_0(\sigma) = 1, \qquad \lambda_m(\sigma) = \prod_{l=1}^{m} \left(\frac{l - \frac{1}{2} - \sigma}{l - \frac{1}{2} + \sigma}\right) < 1 \qquad (m \geqq 1), \tag{8.10}$$

$$\lambda_{-m}(\sigma) = \lambda_m(\sigma).$$

With the help of Γ-functions we also may write for all m

$$\lambda_m(\sigma) = \frac{\Gamma(\frac{1}{2} + \sigma)}{\Gamma(\frac{1}{2} - \sigma)} \cdot \frac{\Gamma(|m| + \frac{1}{2} - \sigma)}{\Gamma(|m| + \frac{1}{2} + \sigma)}. \tag{8.11}$$

From (8.10) and (8.11) we may conclude

(1) If $0 < \sigma < \frac{1}{2}$, all $\lambda_m(\sigma)$ are *positive*. If $\sigma \geqq \frac{1}{2}$, at least one coefficient is smaller than or equal to zero.

(2) For large $|m|$ we have by Stirling's formula the asymptotic expression

$$\lambda_m(\sigma) \sim (\Gamma(\tfrac{1}{2} + \sigma)/\Gamma(\tfrac{1}{2} - \sigma)) \, |m|^{-2\sigma}. \tag{8.12}$$

Let $g = \sum_{m=-\infty}^{\infty}(f_m, g)f_m$ and $h = \sum_{m=-\infty}^{\infty}(f_m, h)f_m$ be two elements of the Hilbert space $\mathfrak{H}$. It follows from the continuity of $(g, h)_\sigma$, from (8.9) and from (8.10), that

$$(g, h)_\sigma = \sum_{m=-\infty}^{\infty} \lambda_m(\sigma)(g, f_m)(f_m, h)$$

$$(g, g)_\sigma = \sum_{m=-\infty}^{\infty} \lambda_m(\sigma) \, |(f_m, g)|^2 \leqq (g, g) \tag{8.13}$$

(8.14) If $0 < \sigma < \frac{1}{2}$, and $g \in \mathfrak{H}$, then $(g, g)_\sigma > 0$, if $g \neq 0$.

(8.14) follows from the fact that all $\lambda_m(\sigma) > 0$. (It is this property that excludes larger values of σ.)

8c. *The Hilbert space* $\mathfrak{H}_\sigma$. In what follows we assume that $0 < \sigma < \frac{1}{2}$.

DEFINITION 3. *The linear manifold* $\mathfrak{H}_\sigma^0$ *consists of all vectors of* $\mathfrak{H}$. *The inner product of two vectors* f, g *of* $\mathfrak{H}_\sigma^0$ *is defined by* (8.1), *and the norm of a vector* f *is defined by* $\| f \|_\sigma = (f, f)_\sigma^{\frac{1}{2}}$. *The closure of* $\mathfrak{H}_\sigma^0$, *which is a Hilbert space, is denoted by* $\mathfrak{H}_\sigma$.

It has been shown (cf. (8.14)) that $(f, g)_\sigma$ has all the required properties of an inner product. The linear manifold $\mathfrak{H}_\sigma^0$ is not closed with respect to the metric $\| f - g \|_\sigma$. Since the $\lambda_m(\sigma)$ tend to zero (cf. (8.12)) there are sequences h_m in $\mathfrak{H}_\sigma^0$ for which $\| h_m - h_n \|_\sigma \to 0$ as $m, n, \to \infty$ which have, however, no limit in $\mathfrak{H}_\sigma^0$. The closure $\mathfrak{H}_\sigma$ is obtained in the familiar way of adjoining these sequences (as "ideal" elements). Most of the following discussion may be carried out for $\mathfrak{H}_\sigma^0$, because $\mathfrak{H}_\sigma^0$ is dense in $\mathfrak{H}_\sigma$.

An orthonormal basis for $\mathfrak{H}_\sigma$. By (8.9) the vectors f_m, though orthogonal, are not normalized. A comparison of (8.10) with (6.23) shows that $\lambda_m(\sigma) = (\eta_m(\sigma))^2$ for all m. We obtain therefore an orthonormal set of vectors by the definition

$$g_m = (1/\eta_m(\sigma))f_m, \qquad (\eta_m(\sigma))^2 = \lambda_m(\sigma) \tag{8.15}$$

For any $h \in \mathfrak{H}_\sigma^0$ we find by (8.13) $(g_m, h)_\sigma = \eta_m(\sigma)(f_m, h)$, and hence for any two $h', h'' \in \mathfrak{H}_\sigma^0$

$$(8.16) \qquad (h', h'')_\sigma = \sum_{m=-\infty}^{\infty} (h', g_m)_\sigma (g_m, h'')_\sigma .$$

This equation may be extended to any two vectors in $\mathfrak{H}_\sigma$, and shows the *completeness* of the orthonormal set g_m.

8d. *Representations of the class* C_q^0 $(q = \frac{1}{4} - \sigma^2)$. On $\mathfrak{H}_\sigma^0$ the transformation $T_\sigma(a)$ is *unitary*, because it leaves the inner products $(f, g)_\sigma$ invariant and has an inverse. By continuity it is extended to a unitary transformation $T_\sigma(a)$ on $\mathfrak{H}_\sigma$. Likewise the transformation property $T_\sigma(a)T_\sigma(b) = T_\sigma(ab)$ extends from $\mathfrak{H}_\sigma^0$ to its closure $\mathfrak{H}_\sigma$.

The proof given in §6d can again be applied to show that the multiplier representation $T_\sigma(a)$ is a representation of class C_q^0, because the analytic functions with which we have to operate are in $\mathfrak{H}_\sigma^0$ so that the explicit definition of the inner product by (8.1) may be used. We have only to remember that $Qf = (\frac{1}{4} - \sigma^2)f$ (cf. (6.14)) and that the expressions $\Lambda_r g_m$ are given by (6.26) as has been remarked at the end of §8d. The matrix elements

$$(8.17) \qquad v_{mn}(a) = (g_m, T_\sigma(a) g_n)_\sigma$$

are *analytic* functions on $\mathfrak{S}$.

Theorem 3. *The multiplier representation $T_\sigma(a)$ $(0 < \sigma < \frac{1}{2})$ is an irreducible unitary representation of $\mathfrak{S}$ on the Hilbert space $\mathfrak{H}_\sigma$. It belongs to the class C_q^0 where $q = \frac{1}{4} - \sigma^2$.*

§9. Representations of the discrete classes D_k^+ and D_k^- [16]

9a. *The multiplier representations T_l.* We consider here as the manifold $\mathfrak{M}$ the open unit circle in the complex plane ($z\bar{z} < 1$), i.e., the manifold $\mathfrak{M}^*$ of §4e. On $\mathfrak{M}$ the group $\mathfrak{S}$ is realized by the conformal transformations of the unit circle onto itself. We have

$$(9.1) \qquad z' = az = \frac{\bar{\alpha} z + \bar{\beta}}{\beta z + \alpha} \quad (\alpha\bar{\alpha} - \beta\bar{\beta} = 1) \qquad (a \in \mathfrak{S}, z \in \mathfrak{M})$$

(cf. (4.23)). A multiplier of (1) is given by

$$(9.2) \qquad \mu(a, z) = \alpha + \beta z$$

(cf. (4.25)). We observe that

$$(9.3) \qquad \frac{dz'}{dz} = \mu(a, z)^{-2}\, 1 - z'\bar{z}' = |\mu(a, z)|^{-2}(1 - z\bar{z}) \qquad (z' = az).$$

[16] The construction described in this section is closely related to Dirac's construction of the expansor representation [Dirac 2]. Cf. also the Appendix to Part II of this paper.

The functions $f(z)$ introduced here will be analytic functions of z which are *regular* on $\mathfrak{M}$. For an integral exponent l we define the multiplier representation

$$(9.4) \qquad T_l(a)f(z) = \mu(a, a^{-1}z)^l \cdot f(a^{-1}z).$$

The invariant density ω_l. We shall now determine a continuous positive real function $\omega_l(z)$ (which is not analytic in z) such that the inner product

$$(9.5) \qquad (f, g)_l = \int_{\mathfrak{M}} \omega_l(z)\overline{f(z)}g(z)\, dS$$

remains *invariant* under the transformations $T_l(a)$. Here $dS = dx\, dy$ where x and y are the real and imaginary parts of z. From (9.4) we obtain the condition

$$\omega_l(az) \cdot | \mu(a, z) |^{2l} \cdot J_a(z) = \omega_l(z)$$

where $J_a(z)$ is the Jacobian of the transformation (9.1), i.e., $J_a(z) = | dz'/dz |^2 = | \mu(a, z) |^{-4}$ (by 9.3). Hence we may write

$$(9.6) \qquad \omega_l(az) \, | \mu(a, z) |^{2l-4} = \omega_l(z).$$

The second equation (9.3) shows that

$$(9.7) \qquad \omega_l(z) = \text{const. } (1 - z\bar{z})^{l-2}$$

is a solution of (9.6). Since the transformation group (9.1) is *transitive* on $\mathfrak{M}$ it follows that no other solutions exist (cf. the remarks at the end of §8a).

9b. *The Hilbert space* $\mathfrak{H}_l$. For the present we consider the expression (9.5) for *arbitrary positive numbers* $l > 1$. (If $l \leqq 1$, the singularity of $\omega_l(z)$ at the unit circle $| z | = 1$ is too high.) The constant in (9.7) can be so chosen that $(f, g)_l \equiv 1$ if both $f(z) \equiv 1$ and $g(z) \equiv 1$. We then have

$$(9.8) \qquad \omega_l(z) = ((l - 1)/\pi)(1 - z\bar{z})^{l-2}$$

and we define

$$(9.9) \qquad (f, g)_l = \frac{l-1}{\pi} \int_{\mathfrak{M}} (1 - z\bar{z})^{l-2}\overline{f(z)}g(z)\, dS \qquad (l > 1).$$

This definition may now be extended to the case $l = 1$:

$$(9.10) \qquad (f, g)_1 = \lim_{l \to 1} \left\{ \frac{l-1}{\pi} \int_{\mathfrak{M}} (1 - z\bar{z})^{l-2}\overline{f(z)}g(z)\, dS \right\}.$$

The inner products defined above are easily computed for the powers of z by introducing polar coordinates. Set $f = z^m$, $g = z^n$, then for $l \geqq 1$

$$(9.11) \quad (z^m, z^n)_l = 0 \text{ if } m \neq n \qquad (z^m, z^m)_l = \binom{l-1+m}{m}^{-1} \quad (m = 0, 1, \cdots).$$

If $f(z)$ has the power series $\sum_{m=0}^{\infty} c_m z^m$, then

$$(9.12) \qquad (f, f)_l = \sum_{m=0}^{\infty} \binom{l-1+m}{m}^{-1} | c_m |^2.$$

This equation is to be interpreted as follows: Either both sides have the same finite value, or both sides are infinite. (9.12) is readily obtained from (9.11) by first considering the integral $(f, f)_l$ over a circle $|z| \leqq \rho < 1$, and then taking the limit for $\rho \to 1$. Moreover, every power series with coefficients c_m for which the right hand side of (9.12) converges has evidently a radius of convergence at least equal to one and hence defines an analytic function regular over $\mathfrak{M}$. It is now evident that (9.9) and (9.10) determine a Hilbert space.

DEFINITION 4. *The Hilbert space* $\mathfrak{H}_l$ $(l \geqq 1)$ *consists of those analytic functions regular on the open unit circle for which* $(f, f)_l$ *(defined by* (9.9) *and* (9.10)) *is finite. The inner product is defined by* (9.9) *(or* (9.10)) *and the norm of an element of* $\mathfrak{H}_l$ *is given by* $\| f \|_l = (f, f)_l^{\frac{1}{2}}$.

For two elements $f = \sum_{m=0}^{\infty} c_m z^m$ and $g = \sum_{m=0}^{\infty} d_m z^m$ in $\mathfrak{H}_l$ we have

$$(f, g)_l = \sum_{m=0}^{\infty} \binom{l - 1 + m}{m}^{-1} \bar{c}_m d_m . \tag{9.13}$$

The functions

$$h_m(z) = \binom{l - 1 + m}{m}^{\frac{1}{2}} z^m \qquad m = 0, 1, \cdots \tag{9.14}$$

form a *complete orthonormal* set in $\mathfrak{H}_l$, and

$$(f, g)_l = \sum_{m=0}^{\infty} (f, h_m)_l \, (h_m , g)_l$$

for any two elements f, g of $\mathfrak{H}_l$.

An inequaliy. Let $f(z) = \sum_{m=0}^{\infty} c_m z^m$ be an element of $\mathfrak{H}_l$. Then, by Schwarz' inequality,

$$|f(z)| \leqq \sum_{m=0}^{\infty} |c_m| \, |z|^m$$
$$\leqq \left(\sum_{m=0}^{\infty} \binom{l - 1 + m}{m}^{-1} |c_m|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{m=0}^{\infty} \binom{l - 1 + m}{m} |z|^{2m} \right)^{\frac{1}{2}}.$$

Consequently, by (9.12),

$$|f(z)| \leqq (1 - z\bar{z})^{-(l/2)} \, \| f \|_l \tag{9.15}$$

for every $z \in \mathfrak{M}$. By applying this inequality to the difference of two functions we see that convergence in $\mathfrak{H}_l$ implies *pointwise* convergence of the corresponding functions over $\mathfrak{M}$.

9c. *Representations of the class* D_k^+. We now restrict l to positive *integral* values, so that the transformations $T_l(a)$ are properly defined. It follows from §9a that they are *unitary*.

Infinitesimal operators. From our results in §4e it follows that

$$\Lambda_\chi f(z) = -\lambda_\chi(z) \frac{df(z)}{dz} + l\tau_\chi(z) f(z)$$

where λ_χ and τ_χ are defined by (4.24) and (4.26) respectively. In particular

$$i\Lambda_0 = z \frac{d}{dz} + \tfrac{1}{2}l, \quad i(\Lambda_1 + i\Lambda_2) = -z^2 \frac{d}{dz} - lz, \quad i(\Lambda_1 - i\Lambda_2) = -\frac{d}{dz}. \tag{9.16}$$

We now obtain

$$(9.17) \qquad Qf(z) = ((\Lambda_0)^2 - (\Lambda_1)^2 - (\Lambda_2)^2)\, f(z) = q \cdot f(z) \qquad q = \frac{l}{2}\left(1 - \frac{l}{2}\right).$$

The operators Λ_r can be applied to any analytic function $f(z)$, but the resultant function will not necessarily be an element of $\mathfrak{H}_l$, even though $f \in \mathfrak{H}_l$. It is clear from the definitions (9.9) and (9.10) that $f \in \mathfrak{H}_l$ if it has a radius of convergence *greater than one*. The same will then hold for all $\Lambda_r f$ and for all $T_l(a)f$. These functions may therefore be used in the proof that $T_l(a)$ is a representation of the class D_k^+ $(k = \frac{1}{2}l)$ instead of the analytic functions which were used in §6d. To show that the equations (5.29) hold we set

$$(9.18) \qquad g_m = (-1)^{m-k}\, h_{m-k}\ (m = k, k+1, \cdots)\ k = \tfrac{1}{2}l.$$

The g_m form an orthonormal basis for $\mathfrak{H}_l$, and on applying the operators Λ_r (cf. (9.16)) we obtain equations of the form (6.26) for $m = k, k + 1, \cdots$, and with $q = k\,(1 - k)$. Thus we have

THEOREM 4. *The multiplier representation $T_l(a)$ $(l = 1, 2, \cdots)$ is an irreducible unitary representation of $\mathfrak{S}$ on the Hilbert svace $\mathfrak{H}_l$. It belongs to the class D_k^+ $(k = \frac{1}{2}l)$.*

The matrix elements

$$(9.19) \qquad v_{mn}(a) = (g_m\,,\ T_l(a)\, g_n)_l$$

are *analytic* functions on $\mathfrak{S}$.

9d. *Representations of the class D_k^-.* The representations D_k^- may be treated quite briefly. The definition of the Hilbert space $\mathfrak{H}_l$ is the same, but the transformations $z' = az$ and the multiplier $\mu(a, z)$ are defined differently, viz.,

$$(9.20) \qquad z' = az = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \qquad \mu(a, z) = \bar{\alpha} + \bar{\beta} z.$$

The infinitesimal operators (9.16) are replaced by

$$(9.21) \quad i\Lambda_0 = -\left(z\frac{d}{dz} + \tfrac{1}{2}l\right), \qquad i(\Lambda_1 + i\Lambda_2) = \frac{d}{dz}, \qquad i(\Lambda_1 - i\Lambda_2) = z^2\frac{d}{dz} + lz$$

and we choose

$$(9.22) \qquad g_m = h_{k-m} \quad (m = -k, -(k+1), \cdots) \quad k = \tfrac{1}{2}l.$$

The equations (9.17) are unchanged.

THEOREM 5. *The multiplier representation $T_l(a)$ $(l = 1, 2, \cdots)$ defined by* (9.20) *is an irreducible unitary representation of $\mathfrak{S}$ on the Hilbert space $\mathfrak{H}_l$. It belongs to the class D_k^- $(k = \frac{1}{2}l)$.*

REMARK. Non-integral values of l $(l > 1)$ lead to representations of the covering group $\mathfrak{C}$ which are *multi-valued* representations of $\mathfrak{S}$. The multiplier must be defined in terms of the parameters (γ, ω) as pointed out in §7d. The corresponding value of q is $k(1 - k)$ where $k = \frac{1}{2}l$. The spectrum of H_0 consists of the numbers $k, k + 1, \cdots$, or $-k, -(k + 1), \cdots$, depending on whether (9.1) or (9.20) are used.

§10. Explicit expressions for the matrix elements

We shall derive now explicit expressions for the matrix elements $v_{mn}(a)$ of the representations which have been constructed in the preceding sections. In particular we shall be interested in their dependence on the invariant q, and in their asymptotic behavior (for large values of the parameters ζ) on the group manifold. The results of this section will be used in §12 to obtain orthogonality relations.

10a. *Differential relations.* In each case the representation was defined by unitary operators $T(a)$ on some Hilbert space, so that

$$(f, g) = (T(a)f, T(a)g). \tag{10.1}$$

The notation (10.1) is meant to cover the different inner products introduced in §§6–9, and hence any subscripts are omitted.

In the present discussion, we only consider functions f which are *analytic* in the case of $\mathfrak{H}$ and $\mathfrak{H}_\sigma$ (§§6, 7, 8) or which have a radius of convergence *greater than one* in the case of $\mathfrak{H}_l$ (§9). $\Lambda_\chi f$ and $T(a)f$ are then functions of the same type, and in the inner products we may interchange differentiation with respect to the group parameters with the integration over the manifold on which these functions are defined. We always have

$$Qf = ((\Lambda_0)^2 - (\Lambda_1)^2 - (\Lambda_2)^2)f = q \cdot f \tag{10.2}$$

where the number q is characteristic of the representation in question. The matrix elements are given by

$$v_{mn}(a) = (g_m, T(a)\, g_n). \tag{10.3}$$

The set of orthonormal functions g_m is defined for each particular case. Since $\chi(T(a)f) = \Lambda_\chi(T(a)f)$ we find

$$\chi v_{mn}(a) = (g_m, \Lambda_\chi(T(a)\, g_n)). \tag{10.4}$$

By a repeated application of (10.4) we obtain from (10.2)

$$\Omega v_{mn}(a) = (g_m, Q(T(a)\, g_n)) = q(g_m, T(a)\, g_n) = q \cdot v_{mn}(a). \tag{10.5}$$

Ω is the operator $(\chi_0)^2 - (\chi_1)^2 - (\chi_2)^2$ defined in §4 (cf. (4.19)). This shows that all matrix elements of a given representation satisfy the same second order partial differential equation. (Incidentally, this equation will be the main tool for determining the functions $v_{mn}(a)$).[17]

On applying the operator χ to (10.1) and setting $a = e$ we find

$$0 = (\Lambda_\chi f, g) + (f, \Lambda_\chi g).$$

[17] This corresponds to Casimir's method in the compact case. Cf. [Casimir].

We use this relation to transform (10.4) and obtain

$$\chi v_{mn}(a) = -(\Lambda_\chi g_m, T(a)g_n) = -\textstyle\sum_r(\Lambda_\chi g_m, g_r)(g_r, T(a)g_n)$$

(10.6) $$\chi v_{mn}(a) = \textstyle\sum_r \lambda_{mr}v_{rn}(a); \qquad \lambda_{mr} = (g_m, \Lambda_\chi g_r) = -\overline{\lambda_{rm}}.$$

In our case every equation of the infinite set (10.6) contains at most *three* terms, since $\lambda_{mr} = 0$ if $| m - r | > 1$ (cf. (6.22)).

If $a(t)$ is a one-parameter subgroup of $\mathfrak{S}$ and if we set $v_{mn}(a(t)) = v_{mn}(t)$, then $\chi v_{mn} = dv_{mn}/dt$, and hence

(10.7) $$\frac{dv_{mn}}{dt} = \sum_r \lambda_{mr} v_{rn}(t) \qquad \lambda_{mr} = (g_m, \Lambda_\chi g_r)$$

((10.6) and (10.7) correspond to the differential equations for *finite-dimensional* representations discussed in §1e). Since $v_{mn}(0) = \delta_{mn}$ we obtain from (10.7) the power series for $v_{mn}(t)$ by successive derivations. Using the fact that $\lambda_{mr} = 0$ if $| m - r | > 1$ we find in particular

(10.8) $$\begin{cases} v_{nn}(t) = 1 + \cdots \\ v_{n+h,n}(t) = \left(\prod_{j=1}^h \lambda_{n+j,n+j-1}\right)\dfrac{t^h}{h!} + \cdots & (h = 1, 2, \cdots) \\ v_{n-h,n}(t) = \left(\prod_{j=1}^h \lambda_{n-j,n-j+1}\right)\dfrac{t^h}{h!} + \cdots & (h = 1, 2, \cdots) \end{cases}$$

10b. *Form of the matrix elements.* To the decomposition

$$a = \exp(2\mu\chi_0)\ \exp(2\zeta\chi_2)\ \exp(2\nu\chi_0)$$

of every element $a \in \mathfrak{S}$ (cf. (4.12)) corresponds the following decomposition of $v_{mn}(a)$. Since in every case $H_0 g_m = mg_m$, we have $\exp(-2i\,\mu H_0)g_m = e^{-2im\mu}g_m$. If we now set

(10.9) $$V_{mn}(\zeta) = v_{mn}\ (\exp 2\zeta\chi_2)$$

we obtain

(10.10) $$v_{mn}(a) = e^{-2im\mu}\, e^{-2in\nu}\, V_{mn}(\zeta).$$

The equations (10.8) may be applied to $V_{mn}(\zeta)$, with $\chi = \chi_2$, and $t = 2\zeta$. By (6.27), $\lambda_{m,m-1} = -\lambda_{m-1,m} = \frac{1}{2}(q + m(m-1))^{\frac{1}{2}}$. Consequently,

(10.11) $$\begin{cases} V_{nn}(\zeta) = \Theta_{nn}(q) + \cdots & \Theta_{nn}(q) = 1 \\ V_{n+h,n}(\zeta) = \Theta_{n+h,n}(q)\zeta^h + \cdots & \Theta_{n+h,n}(q) = \dfrac{1}{h!}\displaystyle\prod_{j=1}^h (q + (n+j)(n+j-1))^{\frac{1}{2}} \\ V_{n-h,n}(\zeta) = \Theta_{n-h,n}(q)\zeta^h + \cdots & \Theta_{n-h,n}(q) = (-1)^h\Theta_{n,n-h}(q). \end{cases}$$

Furthermore, it follows from (10.7) that the power series for V contains only *odd* or only *even* powers of ζ depending on the parity of h (because only the matrix elements $\lambda_{m,m-1}$ and $\lambda_{m-1,m}$ are different from zero). Hence

$$V_{mn}(-\zeta) = (-1)^{m-n} V_{mn}(\zeta). \tag{10.12}$$

REMARK. Since the λ_{mn} are real, *all functions* $V_{mn}(\zeta)$ *are real.*

10c. *Discussion of the functions* $V_{mn}(\zeta)$. It is sufficient to consider only *positive* values of ζ, and it will prove advantageous to introduce the new variable

$$y = (\sinh \zeta)^2 = \beta\bar{\beta} \qquad 0 \leqq y < \infty \tag{10.13}$$

on $\mathfrak{S}$. Notice that

$$\frac{dy}{d\zeta} = \sinh 2\zeta = 2\{y(1 + y)\}^{\frac{1}{2}}$$

and that the volume element on $\mathfrak{S}$ may be written

$$da = (2\pi)^{-2} \, dy \, d\mu \, d\nu \tag{10.14}$$

(cf. (4.20)). If we set

$$V_{mn}(\zeta) = W_{mn}(y) \tag{10.15}$$

and apply the differential equation $\Omega v_{mn}(a) = q v_{mn}(a)$ to the matrix element (10.9) we obtain from (4.19)

$$\begin{gathered} (\Omega_{mn} + q) W_{mn}(y) = 0; \\ \Omega_{mn} = \frac{d}{dy}\left(y(1 + y)\frac{d}{dy}\right) - (y(1 + y))^{-1}\left\{\left(\frac{m - n}{2}\right)^2 - mny\right\} \end{gathered} \tag{10.16}$$

This equation is transformed into a *hypergeometric equation* for the independent variable $(-y)$, if we introduce the function $Y_{mn}(y)$ by the definition

$$W_{mn}(y) = y^{\frac{1}{2}|m-n|}(1 + y)^{-\frac{1}{2}|m+n|} \, Y_{mn}(y). \tag{10.17}$$

In fact, we find

$$y(1 + y)\frac{d^2 Y_{mn}}{dy^2} + ((c_1 + c_2 + 1)y + c_3)\frac{dY_{mn}}{dy} + c_1 c_2 Y_{mn} = 0 \tag{10.18}$$

$$\left.\begin{matrix} c_1 \\ c_2 \end{matrix}\right\} = \tfrac{1}{2}(1 + |m - n| - |m + n|) \pm \sigma, \tag{10.19}$$

$$c_3 = 1 + |m - n|, \qquad \sigma = (\tfrac{1}{4} - q)^{\frac{1}{2}}.$$

For small ζ, $y \sim \zeta^2$, hence we have split off a factor $\zeta^{|m-n|}$ in (10.17), and it follows by comparison with (10.11) that $Y_{mn}(y)$ is a power series of the form

$\Theta_{mn} + \cdots$. We must therefore choose that solution of (10.18) which is regular for $y = 0$, i.e., we have

$$Y_{mn}(y) = \Theta_{mn}(q) \cdot F(c_1, c_2, c_3, -y). \tag{10.20}$$

From the well known identity

$$F(a, b, c, z) = (1 - z)^{c-a-b} F(c - b, c - a, c, z)$$

we obtain a second expression

$$Y_{mn}(y) = \Theta_{mn}(q)(1 + y)^{|m+n|} F(c_1', c_2', c_3, -y) \tag{10.21}$$

$$\left.\begin{matrix} c_1' \\ c_2' \end{matrix}\right\} = \tfrac{1}{2}(1 + | m - n | + | m + n |) \pm \sigma. \tag{10.22}$$

10d. *Remarks on hypergeometric functions.* Since the hypergeometric func tion has no singularities on the negative real axis the functions in (10.20) and (10.21) are given by their familiar power series for $y < 1$ and are obtained by analytic continuation for larger values of y. For $y > 1$ we may write [18]

$$\begin{aligned} F(a, b, c, -y) = {} & y^{-a} \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} F(a, 1 + a - c, 1 + a - b, -1/y) \\ & + y^{-b} \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} F(b, 1 + b - c, 1 + b - a, -1/y) \end{aligned} \tag{10.23}$$

where the functions on the right hand side may be expressed by power series in $(1/y)$. This equation holds whenever $b - a$ is not an integer (positive, negative, or zero). It is particularly suitable for the discussion of F for *large* values of y.

In the case of a hypergeometric *polynomial*, if a is equal to a negative integer, say $a = -l$, the second term in (10.23) vanishes because $1/\Gamma(-l) = 0$, and we have

$$F(-l, b, c, -y) = \frac{(b, l)}{(c, l)} y^l F(-l, 1 - l - c, 1 - l - b, -1/y). \tag{10.24}$$

We use Appell's symbol (a, s) defined for non-negative integers s by

$$(a, s) \equiv \frac{\Gamma(a + s)}{\Gamma(a)} \equiv \left\{\begin{matrix} 1 & (s = 0) \\ a(a + 1) \cdots (a + s - 1) & (s > 0) \end{matrix}\right\} \tag{10.25}$$

$$(-a, s) = (-1)^s(a + 1 - s, s).$$

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[18] [Whittaker-Watson, p. 289]. It should be noted that the formula given there is not correct. The differences $a - b$, $a - c$, $b - a$, $b - c$, which appear as arguments of the Γ-function must be replaced by $b - a$, $c - a$, $a - b$, $c - b$ respectively.

(10.24) is, of course, elementary and may be readily verified. It holds as long as $(b, l) \neq 0$. We also notice that a hypergeometric polynomial may be expressed as[19]

$$(10.26) \qquad F(-l, b, c, -y) = \frac{y^{1-c}(1+y)^{l+c-b}}{(c, l)} \frac{d^l}{dy^l} \{y^{c+l-1}(1+y)^{b-c}\}.$$

10e. *Representations of the continuous class.* We may now combine (10.10) and (10.17) with (10.20) and (10.21) to obtain the matrix elements $v_{mn}(a)$ in terms of the parameters α, β, where the relation $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$ is used. The results are as follows:

$$(10.27a) \quad \begin{cases} v_{mn}(a) = \Theta_{mn}(q)\alpha^{m+n}\beta^{m-n} \\ \qquad F(\tfrac{1}{2}+m+\sigma, \tfrac{1}{2}+m-\sigma, 1+m-n, -\beta\bar{\beta}) \\ \quad = \Theta_{mn}(q)\bar{\alpha}^{-(m+n)}\beta^{m-n} \\ \qquad F(\tfrac{1}{2}-n+\sigma, \tfrac{1}{2}-n-\sigma, 1+m-n, -\beta\bar{\beta}) \end{cases} \qquad (m \geqq n)$$

$$(10.27b) \quad \begin{cases} v_{mn}(a) = \Theta_{mn}(q)\alpha^{m+n}\bar{\beta}^{n-m} \\ \qquad F(\tfrac{1}{2}+n+\sigma, \tfrac{1}{2}+n-\sigma, 1+n-m, -\beta\bar{\beta}) \\ \quad = \Theta_{mn}(q)\,\bar{\alpha}^{-(m+n)}\bar{\beta}^{n-m} \\ \qquad F(\tfrac{1}{2}-m+\sigma, \tfrac{1}{2}-m-\sigma, 1+n-m, -\beta\bar{\beta}) \end{cases} \qquad (m \leqq n).$$

These expressions depend *analytically* on a as well as on q. As for the dependence on q we see first that the hypergeometric functions are symmetric in σ and $-\sigma$ and are therefore analytic in $q = \frac{1}{4} - \sigma^2$. (This also follows from the differential equation (10.18) because its coefficients contain only q and not σ itself.) Moreover, the constants $\Theta_{mn}(q)$ are analytic in q (cf. (10.11)) as long as q varies in the open interval $(0, \infty)$ in the integral case, (i.e., including the exceptional interval $(0, \frac{1}{4})$) and in the open interval $(\frac{1}{4}, \infty)$ in the half integral case, since m may assume all integral or all half integral values respectively. Notice that by (10.11) all $\Theta_{mn}(q)$ are different from zero. It is noteworthy that the matrix elements $v_{mn}(a)$ do not show any peculiarity for values of q in the exceptional interval $(0, \frac{1}{4})$. Later we shall see, however, that they differ in their *asymptotic* behavior from the matrix elements of other representations. Observe that the above statements refer to a particular choice of the basic functions g_m.

10f. *Representations of the discrete class.*

I. D_k^+ Here $q = k(1-k)$, $\sigma = k - \frac{1}{2}$, and $m = k, k+1, \cdots$. We may write

$$(10.28a) \quad \begin{cases} v_{mn}(a) = \Theta_{mn}(q)\bar{\alpha}^{-(m+n)}\beta^{m-n} \\ \qquad F(k-n, 1-n-k, 1+m-n, -\beta\bar{\beta}) \qquad (m \geqq n) \\ \quad = \Theta_{mn}(q)\,\bar{\alpha}^{-(m+n)}\bar{\beta}^{n-m} \\ \qquad F(k-m, 1-m-k, 1+n-m, -\beta\bar{\beta}) \qquad (m \leqq n) \end{cases}$$

[19] [Courant-Hilbert, p. 77]

(10.28b) $\Theta_{mn}(q) = \frac{1}{(m-n)!}\left(\frac{\Gamma(m+1-k)\Gamma(m+k)}{(\Gamma n+1-k)\Gamma(n+k)}\right)^{\frac{1}{2}}$ if $m \geqq n$

$$\Theta_{nm}(q) = (-1)^{m-n}\Theta_{mn}(q).$$

II. D_k^- $q = k(1-k), \sigma = k - \frac{1}{2}, m = -k, -(k+1), \cdots$

(10.29a)
$$\begin{cases} v_{mn}(a) = \Theta_{mn}(q)\, \alpha^{m+n}\, \beta^{m-n} \\ \qquad F(k+m, 1+m-k, 1+m-n, -\beta\bar{\beta}) \quad (m \geqq n) \\ \quad = \Theta_{mn}(q)\, \alpha^{m+n}\, \bar{\beta}^{n-m} \\ \qquad F(k+n, 1+n-k, 1+n-m, -\beta\bar{\beta}) \quad (m \leqq n) \end{cases}$$

(10.29a) $\Theta_{mn}(q) = \frac{1}{(m-n!)}\left(\frac{\Gamma(1-k-n)\Gamma(k-n)}{\Gamma(1-k-m)\Gamma(k-m)}\right)^{\frac{1}{2}}$ if $m \geqq n$

$$\Theta_{nm}(q) = (-1)^{m-n}\Theta_{mn}(q).$$

It is seen that in each of the equations (10.28a) and (10.29a) the function F is a hypergeometric *polynomial* whose degree is the minimum of the two numbers $(|m| - k, |n| - k)$.

Denote, more specifically, the matrix elements of D_k^+ and D_k^- by $v_{mn}(a \mid k)$ and $v_{mn}(a \mid -k)$ respectively. Then it follows from (10.28) and (10.29) that

(10.29c) $$v_{mn}(a \mid k) = (-1)^{m-n}\, \overline{v_{-m,-n}(a \mid -k)}.$$

10g. *Remarks on finite-dimensional representations of* $\mathfrak{S}$. Although we are concerned in this paper with the *infinite-dimensional* unitary representations of $\mathfrak{S}$ we include here, for the sake of comparison, a short account of the *irreducible finite-dimensional* representations of $\mathfrak{S}$. They are obtained in the same way as the representations of the rotation group. (Cf. [v.d. Waerden, §16].)

Let the group $\mathfrak{S}$ operate on two complex variables ξ, η in the form

(10.30) $$\xi' = \alpha\xi + \beta\eta, \qquad \eta' = \bar{\beta}\xi + \bar{\alpha}\eta, \qquad \alpha\bar{\alpha} - \beta\bar{\beta} = 1$$

and form, for every integral or half integral j, the $(2j+1)$ monomials

(10.31) $$z_m = \frac{\xi^{j+m}\eta^{j-m}}{((j+m)!(j-m)!)^{\frac{1}{2}}} \qquad m = -j, -j+1, \cdots, j.$$

The transformations (10.30) induce transformations

(10.32) $$z_m' = \textstyle\sum_n v_{mn}(a) z_n$$

which leave the indefinite form $\sum_m (-1)^{j-m}\bar{z}_m z_m$ invariant. This representation is irreducible, the operator Q has the value $q = -j(j+1)$. The matrix elements $v_{mn}(a)$ may be directly computed from (10.30) and (10.31) or from the differential equation (10.5). We find

$$(10.33)\quad \begin{cases} v_{mn}(a) = \Theta_{mn}(q)\, \alpha^{m+n}\, \beta^{m-n}\, F(m-j, 1+m+j, 1+m-n, -\beta\bar{\beta}) \\ \qquad\qquad (m+n \geqq 0, m \geqq n) \\ \quad = \Theta_{mn}(q)\, \alpha^{m+n}\, \bar{\beta}^{n-m}\, F(n-j, 1+n+j, 1+n-m, -\beta\bar{\beta}) \\ \qquad\qquad (m+n \geqq 0, m \leqq n) \\ \quad = \Theta_{mn}(q)\, \bar{\alpha}^{-(m+n)}\, \beta^{m-n}\, F(-n-j, 1-n+j, 1+m-n, -\beta\bar{\beta}) \\ \qquad\qquad (m+n \leqq 0, m \geqq n) \\ \quad = \Theta_{mn}(q)\, \bar{\alpha}^{-(m+n)}\, \bar{\beta}^{n-m}\, F(-m-j, 1+j-m, 1+n-m, -\beta\bar{\beta}) \\ \qquad\qquad (m+n \leqq 0, m \leqq n) \end{cases}$$

$$(10.34)\quad \Theta_{mn}(q) = \frac{1}{(m-n)!}\left(\frac{(j+m)!(j-n)!}{(j+n)!(j-m)!}\right)^{\frac{1}{2}} \quad \text{if } m \geqq n,$$

$$\Theta_{nm}(q) = \Theta_{mn}(q).$$

These are *all* continuous irreducible finite-dimensional representations of $\mathfrak{S}$.

REMARK. If we replace $\bar{\beta}$ by $(-\bar{\beta})$ in (10.30) and (10.33) we obtain the corresponding expressions for the rotation group or rather its spinor group $\mathfrak{S}_R$.

10h. *Functional relations.* Any relation between the matrix elements of a representation gives rise to relations between the hypergeometric functions in terms of which the matrix elements are defined. We mention (1) the equations (10.6), which are equivalent to linear relations between contiguous hypergeometric functions, and (2) the equations

$$(10.35)\qquad v_{mn}(ab) = \sum\nolimits_r v_{mr}(a)\, v_{rn}(b)$$

which express the representation property. They correspond to addition theorems for hypergeometric functions. (In the case of the finite-dimensional single-valued irreducible representations of the rotation group, the equation (10.35), for $m = n = 0$, is equivalent to the well known addition theorem of spherical harmonics.)

§11. The asymptotic behavior of the matrix elements.

According to the decomposition (10.10) of the matrix elements $v_{mn}(a)$ the asymptotic behavior of $v_{mn}(a)$ for large values of ζ or y is determined by the function $V_{mn}(\zeta) = W_{mn}(y)$. Moreover, it follows from the explicit expressions (10.17) and (10.20), and from the relation $\Theta_{nm}(q) = (-1)^{m-n}\Theta_{mn}(q)$ (cf. (10.11)) that

$$(11.1)\qquad W_{nm}(y) = (-1)^{m-n} W_{mn}(y).$$

Therefore we may confine ourselves to the case $m \geqq n$.

11a. *Representations of the discrete class.* If we apply the equation (10.24) to transform the expression (10.28a) we obtain

$$(11.2)\quad W_{mn}(y) = \frac{(-1)^{n-k}}{\Gamma(2k)}\left(\frac{\Gamma(m+k)\Gamma(n+k)}{\Gamma(m+1-k)\Gamma(n+1-k)}\right)^{\frac{1}{2}} y^{-k}\left(\frac{y}{1+y}\right)^{\frac{m+n}{2}} F(k-m,\, k-n,\, 2k,\, -1/y)$$

for the representation D_k^+. The leading term is const. y^{-k} or const. $e^{-2k\zeta}$. Therefore these matrix elements are *square-integrable* over the group manifold $\mathfrak{S}$ (with the volume element da (defined by (10.14)) if and only if $k > \frac{1}{2}$.

By (10.29c) the matrix elements of D_k^- show the same asymptotic behavior.

11b. *Representations of the continuous class.* Transforming the second equation (10.27a) by (10.23) if $\sigma \neq 0$[20] we obtain a result which may be expressed as follows:

$$(11.3)\quad \begin{cases} W_{mn}(y) = y^{-\frac{1}{2}}(y/(1+y))^{m+n/2}\{\beta_{mn}(\sigma, -1/y)y^{\sigma} + \beta_{mn}(-\sigma, -1/y)y^{-\sigma}\} \\ \beta_{mn}(\sigma, x) = \left\{\dfrac{\Gamma(\frac{1}{2}+m-\sigma)}{\Gamma(\frac{1}{2}+n+\sigma)\Gamma(\frac{1}{2}+n-\sigma)\Gamma(\frac{1}{2}+m+\sigma)}\right\}^{\frac{1}{2}} \\ \qquad\qquad \cdot \dfrac{\Gamma(2\sigma)}{\Gamma(\frac{1}{2}-n+\sigma)} F(\tfrac{1}{2}-m-\sigma, \tfrac{1}{2}-n-\sigma, 1-2\sigma, x). \end{cases}$$

If $q > \frac{1}{4}$, and hence $\sigma = is$, we have asymptotically

$$(11.4)\quad W_{mn}(y) \sim 2y^{-\frac{1}{2}}\mathrm{Re}\{\beta_{mn}(is,\, 0)y^{is}\} \sim 4e^{-\zeta}\mathrm{Re}\{4^{-is}\beta_{mn}(is,\, 0)e^{2is\zeta}\}$$

where Re stands for "*real part*". The functions $v_{mn}(a)$ are not square-integrable, but due to their oscillatory character it is possible to obtain square-integrable functions by forming "*wave packets*", i.e., by an integration with respect to the variable s.

In the exceptional interval $0 < q < \frac{1}{4}$, where $0 < \sigma < \frac{1}{2}$, we have the asymptotic expression

$$(11.5)\qquad W_{mn}(y) \sim \beta_{mn}(\sigma, 0)y^{\sigma-\frac{1}{2}} \sim 2^{1-2\sigma}\beta_{mn}(\sigma, 0)e^{(2\sigma-1)\zeta}.$$

These functions are not oscillatory in the parameter ζ, and they decrease more slowly than those outside the exceptional interval. Therefore their square integrals cannot be made to converge.

REMARK. It may be shown that the matrix elements (10.33) of the *finite-dimensional* representations increase as y^j.

11c. *Computation of* $|\beta_{mn}(is, 0)|^2$. In the next section we shall need the absolute value of $\beta_{mn}(is,0)$. By (11.1) $|\beta_{mn}(is, 0)| = |\beta_{nm}(is, 0)$, and it suffices to treat the case $m \geqq n$. We obtain from (11.3)

$$(11.6)\quad |\beta_{mn}(is, 0)|^2 = |\Gamma(2is)|^2/b_n, \quad b_n = |\Gamma(\tfrac{1}{2}-n+is)\Gamma(\tfrac{1}{2}+n+is)|^2$$

It is easily seen that $b_n = b_{n+1}$, therefore we may replace b_n by b_0 (for C_q^0) or by

[20] If $\sigma = 0$ the equation (10.23) does not apply and an additional logarithmic term appears.

$b_{\frac{1}{2}}$ (for $C_q^{\frac{1}{2}}$). Consequently the expression (11.6) is the *same* for all matrix elements of a given representation, and it will be denoted by $c_0(s)$ and $c_{\frac{1}{2}}(s)$ respectively.

Since, for every real number ξ, $| \Gamma(i\xi) |^2 = \pi/\xi \sinh \xi$, $| \Gamma(\frac{1}{2} + i\xi) |^2 = \pi/\cosh \xi$, $| \Gamma(1 + i\xi) |^2 = \xi^2 | \Gamma(i\xi) |^2$, we find

$$(11.7a) \qquad | \beta_{mn}(is, 0) |^2 = c_0(s) = \frac{\coth \pi s}{4\pi s} \quad \text{for} \quad C_q^0$$

$$q = \tfrac{1}{4} + s^2.$$

$$(11.7b) \qquad | \beta_{mn}(is, 0) |^2 = c_{\frac{1}{2}}(s) = \frac{\tanh \pi s}{4\pi s} \quad \text{for} \quad C_q^{\frac{1}{2}}$$

§12. Orthogonality relations

12a. We want to discuss now the analogue of the orthogonality relations which hold for irreducible unitary representations of *finite* and of *compact topological* groups. (These representations are necessarily finite-dimensional.) In the case of a compact topological group $\mathfrak{G}$ they may be stated as follows

(1) Consider a continuous irreducible representation of $\mathfrak{G}$ by unitary transformations $U(a)$ ($a \in \mathfrak{G}$) of an r-dimensional linear space E^r into itself. Then, for any four vectors f, g, f', g' of E^r,

$$(12.1) \qquad \int_{\mathfrak{G}} \overline{(f, U(a)g)}(f', U(a)g')\, da = (1/r)\overline{(f, f')}(g, g')$$

where by da we denote invariant integration on $\mathfrak{G}$, and where the total volume of the group has been normalized to one.

(2) Let $U'(a)$ define a second continuous irreducible representation of $\mathfrak{G}$ by unitary transformations of an r'-dimensional linear space $E^{r'}$ into itself, and assume this representation to be *inequivalent* to $U(a)$. If f, g, are two vectors in E^r, and f', g' are two vectors in $E^{r'}$, then

$$(12.2) \qquad \int_{\mathfrak{G}} \overline{(f, U(a)g)}(f', U'(a)g')\, da = 0.$$

Let $f_m (1 \leqq m \leqq r)$ and $f'_m (1 \leqq m \leqq r')$ be sets of orthonormal vectors on E^r and $E^{r'}$ respectively. Applying (12.1) to any four vectors of the set f_m, we obtain

$$(12.3) \qquad \int_{\mathfrak{G}} \overline{u_{mn}(a)} u_{m'n'}(a)\, da = (1/r)\delta_{mm'}\delta_{nn'} \qquad (1 \leqq m, n, m', n' \leqq r)$$

where $u_{mn}(a) = (f_m, U(a)f_n)$. In a similar way we obtain from (12.2)

$$(12.4) \qquad \int_{\mathfrak{G}} \overline{u_{mn}(a)} u'_{m'n'}(a)\, da = 0 \qquad (1 \leqq m, n \leqq r, \quad 1 \leqq m', n' \leqq r')$$

with $u'_{m'n'}(a) = (f'_{m'}, U'(a)f'_{n'})$. It is clear that (12.3) is equivalent to (12.1) and that (12.4) is equivalent to (12.2).

With respect to the unitary representations of $\mathfrak{S}$ the following is evident. The same type of orthogonality relations can only hold if all matrix elements of the representation considered are *square-integrable* over the group manifold. (Set $f = f', g = g'$ in (12.1).) Consequently we may expect this for the *discrete class* D_k^+ and D_k^- where $k > \frac{1}{2}$ (cf. §11a). Moreover, the factor $(1/r)$ in (12.1) and (12.3) which is determined by the dimension of the representation space must be replaced by some constant characteristic of the representation. For the representations of the *continuous class* the orthogonality relations must be suitably modified.

12b. *The representations* D_k^+, D_k^- $(k > \frac{1}{2})$. We shall first obtain the orthogonality relations in the form (12.3) and (12.4) and later pass to (12.1) and (12.2). To distinguish the matrix elements of the different representations we denote them more explicitly by $v_{mn}(a \mid k)$ and $v_{mn}(a \mid -k)$. Since, by (10.10),

$$v_{mn}(a \mid \pm k) = e^{-2im\mu} e^{-2in\nu} W_{mn}(y \mid \pm k)$$

and by (10.14)

$$da = (2\pi)^{-2}\, dy\, d\mu\, d\nu \quad 0 \leqq y < \infty, \qquad -\pi \leqq \mu, \nu \leqq \pi$$

it is evident that an integral of the form

$$\int_{\mathfrak{S}} \overline{v_{mn}(a \mid \pm k)} v_{m'n'}(a \mid \pm k')\, da \tag{12.5}$$

must vanish unless

$$m = m', \qquad n = n'. \tag{12.6}$$

In particular, the matrix elements of a representation D_k^+ (where m, n are *positive*) are orthogonal to the matrix elements of a representation $D_{k'}^-$ (where m', n' are *negative*). We may therefore discuss the representations D_k^+ and apply our results to D_k^- with the help of the relations (10.29c). Because of the restrictions (12.6) we are left with the integrals

$$\int_{\mathfrak{S}} \overline{v_{mn}(a \mid k)} v_{mn}(a \mid k')\, da = \int_0^\infty W_{mn}(y \mid k) W_{mn}(y \mid k')\, dy. \tag{12.7}$$

By (11.1) it is sufficient to consider the case $m \geqq n$. The integrals (12.7) may then be evaluated by the following procedure commonly used in the treatment of hypergeometric polynomials. Assume $k \leqq k'$, replace the hypergeometric polynomial in the expression (10.28a) for $W_{mn}(y \mid k)$ by the right hand side of (10.26), and apply to (12.7) $(n - k)$ successive partial integrations. The result is

$$\begin{aligned} &\int_0^\infty \mid W_{mn}(y \mid k) \mid^2 dy = (2k - 1)^{-1}, \\ &\qquad\qquad \int_0^\infty W_{mn}(y \mid k) W_{mn}(y \mid k')\, dy = 0 \qquad (k \neq k'). \end{aligned} \tag{12.8}$$

In combination with the conditions (12.6) the second part of this equation implies that the matrix elements of two *inequivalent* representations D_k^+ and $D_{k'}^+$ $(k \neq k')$ are orthogonal. This could also have been inferred from the fact that, by (10.5), $v_{mn}(a \mid k)$ and $v_{m'n'}(a \mid k')$ are proper functions of the self-adjoint differential operator Ω belonging to two *different* proper values $q = k(1 - k)$ and $q' = k'(1 - k')$. (Because of the restriction $m = m'$, $n = n'$, this is most easily seen from (10.16).) The first equation (12.8) implies equations of the type (12.3), with r replaced by $(2k - 1)$.

To sum up, we have found

$$\text{(12.9)} \qquad \int_{\mathfrak{S}} \overline{v_{mn}(a)} v_{m'n'}(a)\, da = (2k - 1)^{-1} \delta_{mm'} \delta_{nn'} \qquad (v_{mn}(a) = v_{mn}(a \mid \pm k));$$

$$\text{(12.10)} \qquad \int_{\mathfrak{S}} \overline{v_{mn}(a)} v'_{m'n'}(a)\, da = 0$$

if v_{mn} and $v'_{m'n'}$ belong to *inequivalent* representations.

REMARK. For D_k^+ and D_k^-, $(2k - 1) = (1 - 4q)^{\frac{1}{2}}$. It is worth mentioning that the dimension $r = 2j + 1$ which appears in the orthogonality relations for the spinor rotation group $\mathfrak{S}_R$ has the same form, because with our definition (2.19) the operator Q has the value $q = -j(j + 1)$.

12c. *Orthogonality relations for arbitrary vectors.* As in §10a we denote by $T(a)$ the operators on a Hilbert space $\mathfrak{H}$ which define a representation $D_k^\pm$, and by g_m a complete orthonormal set of vectors on $\mathfrak{H}$, so that $v_{mn}(a) = (g_m, T(a)g_n)$. Let f, g be two vectors in $\mathfrak{H}$, with components $\eta_m = (g_m, f)$ and $\zeta_m = (g_m, g)$ respectively. By (12.9) the functions $(2k - 1)^{\frac{1}{2}} v_{mn}(a)$ form a set of orthonormal functions on $\mathfrak{S}$. Since

$$\text{(12.11)} \qquad (f, T(a)g) = \textstyle\sum_{m,n} \rho_{mn}(2k - 1)^{\frac{1}{2}} v_{mn}(a), \qquad \rho_{mn} = (2k - 1)^{-\frac{1}{2}} \overline{\eta_m} \zeta_n$$

and since

$$\textstyle\sum_{m,n} | \rho_{mn} |^2 = (2k - 1)^{-1} \sum_m | \eta_m |^2 \sum_n | \zeta_n |^2 = (2k - 1)^{-1} \| f \|^2 \| g \|^2,$$

it follows from the Riesz-Fischer theorem that $(f, T(a)g)$ is a *square-integrable* function[21] on $\mathfrak{S}$ and that the sum in (12.11) converges to $(f, T(a)g)$ in the mean over $\mathfrak{S}$. Consequently

$$\text{(12.12)} \qquad \int_{\mathfrak{S}} | (f, T(a)g) |^2\, da = (2k - 1)^{-1} \| f \|^2 \| g \|^2.$$

Replacing in (12.12) f by $\gamma f + \gamma' f'$, g by $\delta g + \delta' g'$ where f, f', g, g' are any vectors in $\mathfrak{H}$, and $\gamma, \gamma', \delta, \delta'$ any complex numbers, we obtain the analogue of (12.1)

$$\text{(12.13)} \qquad \int_{\mathfrak{S}} \overline{(f, T(a)g)} (f', T(a)g')\, da = (2k - 1)^{-1} \overline{(f, f')} (g, g')$$

because (12.12) holds for any choice of $\gamma, \gamma', \delta, \delta'$.

[21] Due to the continuity of the operators $T(a)$ on $\mathfrak{S}$ this function is continuous though in general not analytic.

Let the operators $T'(a)$ on a Hilbert space $\mathfrak{H}'$ define a representation $D^{\pm}_{k'}$, inequivalent to the one considered above, and let $v'_{m'n'} = (g'_{m'}, T'(a)g'_{n'})$ be its matrix elements in terms of the orthonormal set $g'_{m'}$. By (12.10) the functions $v'_{m'n'}(a)$ are orthogonal to the functions $v_{mn}(a)$. Choose any vectors $f, g \in \mathfrak{H}$, and $f', g' \in \mathfrak{H}'$ Then $(f', T'(a)g')$ is a linear combination of the matrix elements $v'_{m'n'}(a)$ (cf. (12.11)) and hence orthogonal to the function $(f, T(a)g)$. Therefore

$$\int_{\mathfrak{S}} \overline{(f, T(a)g)}(f', T'(a)g')\, da = 0. \tag{12.14}$$

THEOREM 6. *If the unitary operators $T(a)$ on a Hilbert space $\mathfrak{H}$ define a representation of $\mathfrak{S}$ of the discrete class $D^{\pm}_{k}(k > \frac{1}{2})$, then for any $f, g \in \mathfrak{H}$ the function $(f, T(a)g)$ is square-integrable over $\mathfrak{S}$, the value of the integral being given by* (12.12). *Let the unitary operators $T'(a)$ on $\mathfrak{H}'$ define an inequivalent representation $D^{\pm}_{k}$, $(k' > \frac{1}{2})$, and let f', g' be any vectors of $\mathfrak{H}'$. Then the function $(f', T(a)g')$ is orthogonal to $(f, T(a)g)$ on $\mathfrak{S}$.*

REMARK. This theorem applies in particular to the analytic functions $f(z)$ and the operators $T_l(a)$ discussed in §9.

12d. *Representations of the continuous class* $(q > \frac{1}{4})$. *The operators B.* In the following discussion we shall use the matrix elements $u_{mn}(a)$ defined by (6.18) and (7.12) (we omit the prime) because they refer—for *different* values of q—to the *same* set of orthonormal vectors viz., to $f_m = e^{im\phi}$ in the integral and to $f_m = e^{i(m-\frac{1}{2})\phi}$ in the half integral case (cf. (6.17) and (7.9)). Since we have to distinguish between different values of q and hence of s we write more explicitly $u_{mn}(a \mid s)$, where s is assumed *positive*, and where $q = \frac{1}{4} + s^2$. In each case

$$u_{mn}(a \mid s) = \omega_{mn}(s)v_{mn}(a \mid s) \qquad \omega_{mn}(s) = \eta_n(is)/\eta_m(is) \tag{12.15}$$

(cf. (6.29) and (7.13)) where $\eta_n(is)$ are complex numbers of absolute value 1, defined by (6.23) and (7.10) respectively. According to the decomposition (10.10) we may write

$$u_{mn}(a \mid s) = \omega_{mn}(s)e^{-2im\mu}e^{-2in\nu}V_{mn}(\zeta \mid s) \qquad \mid \omega_{mn}(s) \mid = 1. \tag{12.16}$$

As we have indicated in §11b we must integrate the matrix elements (12.16) with respect to s in order to obtain square-integrable functions on $\mathfrak{S}$. We proceed as follows. Let $\psi(s)$ be a complex-valued *square-integrable* function which vanishes outside the closed interval $I = [s_1, s_2]$ where $0 < s_1 < s_2 < \infty$. Set

$$b_{mn}(a) = \int_I \psi(s)u_{mn}(a \mid s)\, ds. \tag{12.17}$$

These equations are meant to hold either for *all integral* or for *all half integral* values of m, n. From (12.16) we infer that

$$b_{mn}(a) = e^{-2im\mu}e^{-2in\nu}B_{mn}(\zeta); \qquad B_{mn}(\zeta) = \int_I \psi(s)\omega_{mn}(s)V_{mn}(\zeta \mid s)\, ds. \tag{12.18}$$

The $b_{mn}(a)$ are the matrix elements of an operator $B(a)$, viz.,

$$(12.19) \qquad b_{mn}(a) = (f_m, B(a)f_n); \qquad B(a) = \int_I \psi(s)T_{is}(a)\, ds$$

where the operators $T_{is}(s)$ are given by (6.11) for the integral and by (7.5) for the half integral case. (For integrals of the type (12.19) it is sufficient to define for any two $f, g \in \mathfrak{H}$ the inner product $(f, B(a)g) = \int \psi(s)(f, T_{is}(a)g)\, ds$.) From the explicit definitions (6.11) and (7.5) we find

$$(12.20) \quad B(a)f(\phi) = p(a, a^{-1}\phi)f(a^{-1}\phi); \quad p(a, \phi) = \int_I \psi(s)\mu(a, \phi)^{\frac{1}{2}+is}\, ds$$

for the integral case, and

$$(12.21) \qquad B(a)f(\phi) = \nu(a, a^{-1}\phi)p(a, a^{-1}\phi)f(a^{-1}\phi)$$

for the half integral case.

The operators $B(a)$ are uniformly bounded by

$$\int_I |\psi(s)|\, ds$$

(cf. (12.19)) because the unitary operators T have the bound 1. Moreover, it is easily seen that they are *continuous* in a.

12e. *Orthogonality relations for the operators B.* We start with a discussion of the function $B_{mn}(\zeta)$ for a fixed pair of integral or of half integral indices m, n. Introducing in (10.16) the variable ζ instead of y we have the differential equation

$$(12.22) \quad \begin{cases} \dfrac{d}{d\zeta}\left(\sinh 2\zeta \dfrac{dV_{mn}(\zeta \mid s)}{d\zeta}\right) - K_{mn}(\zeta)V_{mn}(\zeta \mid s) = -4s^2 \sinh 2\zeta V_{mn}(\zeta \mid s) \\ K_{mn}(\zeta) = (\sinh 2\zeta)^{-1}[m^2 - 2mn \cosh 2\zeta + n^2] - \sinh 2\zeta, \qquad q = \frac{1}{4} + s^2 \end{cases}$$

Applying (12.22) to two functions $V_{mn}(\zeta \mid s)$ and $V_{mn}(\zeta \mid t)$ we find

$$(12.23) \quad 4(t^2 - s^2) \sinh 2\zeta\, V_{mn}(\zeta \mid s)V_{mn}(\zeta \mid t) = \frac{d}{d\zeta}[\sinh 2\zeta \{V_{mn}(\zeta \mid t), V_{mn}(\zeta \mid s)\}]$$

where the abbreviation

$$\{F, G\} = F\frac{dG}{d\zeta} - G\frac{dF}{d\zeta}$$

is used. If we set $\phi_{mn}(s) = \omega_{mn}(s)\,\psi(s)$ we obtain from (12.18) and (12.23)

$$\int_0^\zeta |B_{mn}(\zeta')|^2 \sinh 2\zeta'\, d\zeta'$$

$$= \int_{s_1}^{s_2}\int_{s_1}^{s_2} \overline{\phi_{mn}(s)}\phi_{mn}(t)\left[\int_0^\zeta V_{mn}(\zeta' \mid s)V_{mn}(\zeta' \mid t)\sinh 2\zeta'\, d\zeta'\right] ds\, dt$$

$$= \int_{s_1}^{s_2}\int_{s_1}^{s_2} \overline{\phi_{mn}(s)}\, \phi_{mn}(t) \sinh 2\zeta\, \{V_{mn}(\zeta \mid t), V_{mn}(\zeta \mid s)\}(4(t^2 - s^2))^{-1}\, ds\, dt.$$

It can be shown by familiar arguments that in the limit $\zeta \to \infty$ the functions $V_{mn}(\zeta) = W_{mn}(y)$ may be replaced by their *asymptotic values* (11.4), more specifically, that we have, with $\gamma_{mn}(s) = \beta_{mn}(is, 0)\phi_{mn}(s)$,

$$\int_0^\infty |B_{mn}(\zeta)|^2 \sinh 2\zeta \, ds = 2 \lim_{\zeta\to\infty} \int_{s_1}^{s_2} \int_{s_1}^{s_2} \frac{\sin 2\zeta(s-t)}{s-t} \overline{\gamma_{mn}(s)}\gamma_{mn}(t) \, ds \, dt$$

$$= 2\pi \int_{s_1}^{s_2} |\gamma_{mn}(s)|^2 \, ds.$$

We have seen in §11c that, independent of m and n, $|\beta_{mn}(is, 0)|^2 = c(s)$, where $c(s)$ stands for $c_0(s)$ in the integral and for $c_{\frac{1}{2}}(s)$ in the half integral case (cf. 11.7)). Since $|\omega_{mn}(s)| = 1$, we have therefore $|\gamma_{mn}(s)|^2 = c(s)|\psi(s)|^2$, and hence

$$\int_0^\infty |B_{mn}(\zeta)|^2 \sinh 2\zeta \, d\zeta = 2\pi \int_I c(s)|\psi(s)|^2 \, ds. \tag{12.24}$$

This equation may be generalized as follows. Let $B^{(\alpha)}(a)$ $(\alpha = 1, 2)$ be two operators defined by the functions $\psi^{(\alpha)}(s)$ which vanish outside $I^{(\alpha)}$, and assume that $B^{(1)}(a)$ and $B^{(2)}(a)$ are both defined either for the integral or for the half integral case. Then

$$\int_0^\infty \overline{B_{mn}^{(1)}(\zeta)} B_{mn}^{(2)}(\zeta) \sinh 2\zeta \, d\zeta = 2\pi \int_J c(s)\overline{\psi^{(1)}(s)}\psi^{(2)}(s) \, ds \tag{12.25}$$

where J is the intersection of the two intervals I_1, I_2.

Orthogonality relations for the matrix elements $b_{mn}(a)$. Since $da = (2\pi)^{-2} \sinh 2\zeta \, d\zeta \, d\mu \, d\nu$, it follows from (12.18) and from (12.24) that the matrix elements $b_{mn}(a)$ are square-integrable over $\mathfrak{S}$. Moreover, it is seen that $b_{mn}^{(1)}(a)$ and $b_{m'n'}^{(2)}(a)$ are orthogonal on $\mathfrak{S}$ unless $m = m'$, $n = n'$. This implies in particular that any $b_{mn}^{(1)}(a)$ with integral m, n, is orthogonal to any $b_{m'n'}^{(2)}(a)$ with half integral m', n'. Consequently we may write

$$\int_{\mathfrak{S}} \overline{b_{mn}^{(1)}(a)} b_{m'n'}^{(2)}(a) \, da = \left(2\pi \int_J c(s)\overline{\psi^{(1)}(s)}\psi^{(2)}(s) \, ds\right)\delta_{mm'}\delta_{nn'} \tag{12.26}$$

if (m, n) and (m', n') are both integral or both half integral while the integral (12.26) vanishes in any other case.

The transition to relations of the more general form (12.13) and (12.14) is now immediate because the discussion in §12c applies equally well to the operators B. In stating the final result we shall denote by $B_0(a)$ and $B_{\frac{1}{2}}(a)$ operators which are defined for the integral and for the half integral case respectively.

THEOREM 7. *Let $B_0(a)$ and $B_{\frac{1}{2}}(a)$ be operators on the Hilbert space $\mathfrak{H}$ of square-integrable functions over the unit circle defined by* (12.20) *and* (12.21) *respectively. For any two elements f, $g \in \mathfrak{H}$ $(f, B_0(a)g)$ and $(f, B_{\frac{1}{2}}(a)g)$ are continuous square-integrable functions on $\mathfrak{S}$. For any admissible choice of $\psi^{(1)}(s)$, $\psi^{(2)}(s)$, and for any vectors $f, g, f', g' \in \mathfrak{H}$ we have*

$$\int_{\mathfrak{S}} \overline{(f, B_0^{(1)}(a)g)}(f', B_0^{(2)}(a)g')\,da = \left(2\pi \int_J c_0(s)\overline{\psi^{(1)}(s)}\psi^{(2)}(s)\,ds\right)\overline{(f, f')}(g, g')$$

$$\int_{\mathfrak{S}} \overline{(f, B_{\frac{1}{2}}^{(1)}(a)g)}(f', B_{\frac{1}{2}}^{(2)}(a)g')\,da = \left(2\pi \int_J c_{\frac{1}{2}}(s)\overline{\psi^{(1)}(s)}\psi^{(2)}(s)\,ds\right)\overline{(f, f')}(g, g')$$

$$\int_{\mathfrak{S}} \overline{(f, B_{\frac{1}{2}}^{(1)}(a)g)}(f', B_0^{(2)}(a)g')\,da = 0$$

$$c_0(s) = \coth \pi s/4\pi s \qquad c_{\frac{1}{2}}(s) = \tanh \pi s/4\pi s.$$

12f. *Orthogonality of* $b_{mn}(a)$ *and* $v_{mn}(a \mid \pm k)$. $(k > \frac{1}{2})$.

THEOREM 8. *Let* $B(a)$ *operate on a Hilbert space* $\mathfrak{H}$, *and let the unitary operators* $T(a)$ *on* $\mathfrak{H}'$ *define a representation of the discrete class* $D_k^{\pm}(k > \frac{1}{2})$. For any $f, g \in \mathfrak{H}, f', g' \in \mathfrak{H}'$ *the functions* $(f, B(a)g)$ *and* $(f', T(a)g')$ *are orthogonal on* $\mathfrak{S}$.

It is evidently sufficient to prove the orthogonality of the corresponding matrix elements. Since the product $u_{mn}(a \mid s)v_{m'n'}(a \mid k)$ decreases at least as $e^{-3\mathfrak{r}}(k \geqq 1!)$ one proves the orthogonality of $u_{mn}(a \mid s)$ and $v_{m'n'}(a \mid k)$ from the fact that these are proper functions of the self-adjoint operator Ω to the different proper values $q = \frac{1}{4} + s^2$ and $q' = k(1 - k) \leqq 0$. Integration with respect to s gives the desired result.

REMARK. The orthogonality of the matrix elements of *inequivalent* representations is partly related to the different range of the values m (i.e., to the different spectra of H_0) and partly to the fact that they are proper functions of Ω to different proper values. The matrix elements of D_k^+ and $D_k^-(k > \frac{1}{2})$ belong to the *discrete* spectrum, those of $C_q^0(q > \frac{1}{4})$ and of $C_q^{\frac{1}{2}}$ to the *continuous* spectrum of Ω. The orthogonality relations for the matrix elements of the *same* representation cannot be obtained by these arguments alone.

§13. Completeness of the matrix elements on S

In the case of a *compact* topological group it follows from the theorem of Peter-Weyl [cf. Weyl, 2] that the matrix elements of its irreducible representations are complete in the Hilbert space of all square-integrable functions over the group manifold. We want to show that a similar result holds in our case. The matrix elements which we have considered are of the form

$$v_{mn}(a) = e^{-2im\mu}e^{-2in\nu}W_{mn}(y) \tag{13.1}$$

where m and n are both integral or both half integral, i.e., $2m$ and $2n$ both *even* or both *odd*. As we have observed in §4c the parameters used here cover the group manifold *twice*, and any function on $\mathfrak{S}$ is unchanged if (μ, ν) are replaced by $(\mu \pm \pi, \nu \pm \pi)$. It is easily seen that the Fourier decomposition of such a function contains only terms in which the two exponents $2m$ and $2n$ have the same parity. Therefore we have to show that for a fixed pair (m, n) linear combinations (or integrals over a continuous parameter) of the functions $W_{mn}(y)$ which we have found are dense in the Hilbert space of square integrable functions over the half line $0 \leqq y < \infty$. The $W_{mn}(y)$ are solutions of the differential equation (10.16). Fortunately, this equation has been discussed by H. Weyl in his investigations on singular differential equations [Weyl, 1, p. 454–455]. His results are stated in terms of three parameters α, γ, λ which in our case are the following (cf. (10.22)):

$$\alpha = c_1' + c_2' = 1 + |m - n| + |m + n|, \qquad \gamma = c_3 = 1 + |m - n|, \qquad \lambda = q - \tfrac{1}{4}. \tag{13.2}$$

Weyl's criterion $\alpha \geqq \gamma \geqq 1$ is evidently satisfied. The equation (10.16) has a *continuous* spectrum consisting of all *positive* λ if

$$\gamma - (\alpha/2) \geqq 0 \tag{13.3}$$

and if $\gamma - (\alpha/2) < 0$ it has in addition to this a *discrete* spectrum[22] which consists of the values

$$\lambda = -((\alpha/2) - \gamma - l)^2 \; l = 0, 1, \cdots \qquad ((\alpha/2) - \gamma - l > 0). \tag{13.4}$$

Moreover, the solution to be chosen is the one given by (10.21). A simple discussion shows that we obtain the matrix elements of the *continuous class* outside the exceptional interval $0 < q < \frac{1}{4}$ for the *continuous* spectrum, and the matrix elements of the *discrete* class with $k > \frac{1}{2}$ for the *discrete* spectrum (in the latter case those which are compatible with the values of m and n). For the group manifold we may state this result as follows.

THEOREM 9. *The functions* $(f, B_0(a)g)$ *and* $(f, B_{\frac{1}{2}}(a)g)$ *defined in* §12, *and the functions* $(f, T(a)g)$ *where* $T(a)$ *belongs to the discrete class* D_k^+ *or* D_k^- $(k > \frac{1}{2})$ *span the Hilbert space of square-integrable functions over* $\mathfrak{S}$.

PROOF. It is sufficient to observe that the functions mentioned are linear combinations and integrals (with respect to the parameter s) of the matrix elements which satisfy Weyl's criteria.

REMARK. It might have been expected that the matrix elements of the unitary representations of $\mathfrak{S}$ *suffice* to span the Hilbert space of Theorem 9, but it is noteworthy that only *part* of the representations occur since the representations C_q^0 with q in the exceptional interval $(0, \frac{1}{4})$ as well as the representations $D_{\frac{1}{2}}^+$ and $D_{\frac{1}{2}}^-$ are excluded.

APPENDIX

The spectra of the infinitesimal operators

We want to add, without proof, a few remarks on the spectra of the infinitesimal operators H_χ for the different representations. As we have seen at the end of §2, the one-parameter subgroups $\exp(t\chi)(\chi \neq 0)$ fall into three different classes which have been called elliptic, hyperbolic, and parabolic. Within each class any two subgroups are conjugate. The same holds for the generating elements χ, with the difference, however, that χ and $\alpha\chi$ $(\alpha \neq 0)$ are not distinguished. For the elliptic class we may take χ_0 as representative. Since the operator H_0

[22] The criterion stated in Weyl's paper is not correct.

has been sufficiently discussed, we disregard it here. For the remaining classes the following may be shown:

I. *Hyperbolic class*. For the representations C_q^0 and $C_q^{\frac{1}{2}}$ H_χ has a continuous spectrum of *multiplicity two*, extending from $-\infty$ to $+\infty$. For the representations $D_k^{\pm}$ H_χ has a *simple* continuous spectrum extending from $-\infty$ to $+\infty$.

II. *Parabolic class*. For the representations C_q^0 and $C_q^{\frac{1}{2}}$ H_χ has a *simple* continuous spectrum extending from $-\infty$ to $+\infty$. For the representations $D_k^{\pm}$ H_χ has a *simple* continuous spectrum which extends either from 0 to $+\infty$ or from $-\infty$ to 0, depending on the element χ chosen.

Added in proof. In the meantime the interesting note by L. Gelfand and M. Neumark (*Journal of Physics* (USSR), Vol. X, pp. 93–94, 1946) on the irreducible unitary representations of the Lorentz group ($\mathfrak{S}_4$ in our notation) has arrived in this country. The results on the classification of the representations which the authors announce are stronger than ours (cf. the introduction to the present paper) since no assumptions about the *infinitesimal* representations are introduced. There is no discussion, however, of the matrix elements as functions on the group manifold.

The representations obtained by the authors are the same as those mentioned in our introduction, and even the realization of the representing linear operators by functional operators is the same—if the space of light rays (or the unit sphere) is used, as described at the end of their note.

The *second class* of Gelfand and Neumark consists of the representations C_q^0 where q is in the exceptional interval $0 < q < 1$, their parameter ρ equals $2(1 - q)^{\frac{1}{2}}$. The *first class* consists of all the remaining representations, viz., $C_q^0(q \geqq 1)$, where $n = 0$, $\rho = \pm 2(q-1)^{\frac{1}{2}}$, and $C_{k,r}$, where $n = \pm 2k$, $\rho = \pm 2r$.

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