Problem Sheet 2

Problem 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. We say that two smooth vector fields $X \in \mathscr{X}(M)$ and $Y \in \mathscr{X}(N)$ are f-*related* if for all $p \in M$, $(f_*)_p(X_p) = Y_{f(p)}$.

1. Show that X and Y are f-related if and only if for every smooth function $g : N \to \mathbb{R}$,
$$(Yg) \circ f = X(g \circ f) \ .$$
2. Suppose that $X_i \in \mathscr{X}(M)$ and $Y_i \in \mathscr{X}(N)$ are f-related for $i = 1, 2$. Then show that $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.
3. Deduce that if X, Y are left-invariant vector fields on a Lie group, then so is $[X, Y]$, and similarly for right-invariant vector fields.
4. Let G be a Lie group and H a Lie subgroup and let $i : H \to G$ be the inclusion. Since $i_* : H_e \to G_e$ has zero kernel, we can think of H_e as a vector subspace of G_e. Show that it is closed under the bracket and is hence a Lie subalgebra of the Lie algebra of G. (There is a converse to this result, but uses slightly more technology than we have developed so far.)
5. Let $\phi : G \to H$ be a smooth homomorphism between Lie groups. Show that $(\phi_*)_e$ is a Lie algebra homomorphism.
6. Let G be a Lie group acting on the left on a smooth manifold M. Let $\mathfrak{g}$ be the Lie algebra of G and for every $X \in \mathfrak{g}$, let $\widetilde{X} \in \mathscr{X}(M)$ denote the corresponding vector field on M. Show that $[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}$.

Problem 2.2. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra.

1. Show that the vector fields generating the action of G on itself by right (resp. left) translations are the left- (resp. right-) invariant vector fields.
2. Let $X \in \mathfrak{g}$ and let X_L and X_R be, respectively, the left- and right-invariant vector fields which agree with X at the identity. How are $X_L(g)$ and $X_R(g)$ related for $g \in G$?
3. Let ω be the $\mathfrak{g}$-valued 1-form on G defined by
$$\omega(g)(X_L(g)) = X \in \mathfrak{g} \ .$$
If X_i is a basis for $\mathfrak{g}$, then $\omega = \sum_i \omega^i X_i$, where the ω^i are 1-forms. Show that ω^i are left-invariant. (Recall that this means that for all $g \in G$, $L_g^* \omega^i = \omega^i$.) The form ω is called the *left-invariant Maurer–Cartan form*.

4. Show that for any two left-invariant vector fields $U, V \in \mathscr{X}(G)$,

$$d\omega(U,V) = [\omega(U), \omega(V)] \ . \tag{1}$$

Show that in terms of the 1-forms ω^i, this can be written as

$$d\omega^i = \tfrac{1}{2} f_{jk}{}^i \omega^j \wedge \omega^k \ . \tag{2}$$

5. Now consider $\widetilde{\omega}$, also a $\mathfrak{g}$-valued 1-form, defined analogously to ω but using right-invariant vector fields:

$$\widetilde{\omega}(g)(X_R(g)) = X \in \mathfrak{g} \ .$$

What are the analogues of equations (1) and (2) for $\widetilde{\omega}$?

Problem 2.3. Let $SO(3)$ act on the unit 2-sphere in $\mathbb{R}^3$ by restricting the linear action on $\mathbb{R}^3$. Let (x^1, x^2, x^3) be the standard coordinates on $\mathbb{R}^3$. Let L_{ij} for $1 \le i < j \le 3$ denote the following basis for $\mathfrak{so}(3)$:

$$L_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad L_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ .$$

1. Show that the corresponding vector fields on $\mathbb{R}^3$ are given by

$$\widetilde{L_{12}} = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$$
$$\widetilde{L_{13}} = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}$$
$$\widetilde{L_{23}} = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2}$$

and check that

$$[\widetilde{L_{ij}}, \widetilde{L_{k\ell}}] = \widetilde{[L_{ij}, L_{k\ell}]} \ .$$

2. Introduce local coordinates θ, ϕ for the sphere via

$$x^1 = \sin\theta\cos\phi$$
$$x^2 = \sin\theta\sin\phi$$
$$x^3 = \cos\theta \ ,$$

and work out the expression of the vector fields $\widetilde{L_{ij}}$ in terms of these local coordinates. Show that at all points x in the sphere covered by these local coordinates (which points are *not* covered?), the values of these vector fields at x span the tangent space to the sphere at x.

Problem 2.4. Let M denote Minkowski spacetime, P the Poincaré group and $\mathfrak{P}$ its Lie algebra.

1. Show that P acts transitively on M and determine the subgroup P_x of P which leaves invariant a point $x \in M$. The answer should depend explicitly on the coordinates of the point being left invariant. Show that for any two points x and y, their stabilizer subgroups P_x and P_y are isomorphic — in fact, conjugate in P. Find an element of P conjugating P_x into P_y.

2. Let $g : M \to P$ be defined by $g(x) = \exp(x^\mu P_\mu)$. Show that this is a good coset representative. Work out the expression for $\theta = g^*\omega$, the pullback by g of the left-invariant Maurer–Cartan form on P. Prove that $d\theta = 0$. (This is equivalent to flatness of Minkowski spacetime.)

Problem 2.5. Let M denote four-dimensional Minkowski spacetime with coordinates x^μ and let $H \subset M$ denote the hypersurface defined by the equation

$$\eta_{\mu\nu} x^\mu x^\nu = -R^2$$

for R some nonzero real number. The geometry induced on H by the ambient Minkowski spacetime turns it into (three-dimensional) hyperbolic space.

1. Show that $SL(2, \mathbb{C})$ acts transitively on H. (Refer to Problem 1.1, part 5, for the action of $SL(2, \mathbb{C})$ on H.)

2. Find a point $x \in H$ whose stabilizer subgroup is $SU(2) < SL(2, \mathbb{C})$.

3. Exhibiting H as the coset space $SL(2, \mathbb{C})/SU(2)$, find a coset representative $\sigma : H \to SL(2, \mathbb{C})$ defined over most of H. (Coset representatives will generally fail to be defined everywhere.)