Mathematics for Informatics 4a

José Figueroa-O'Farrill

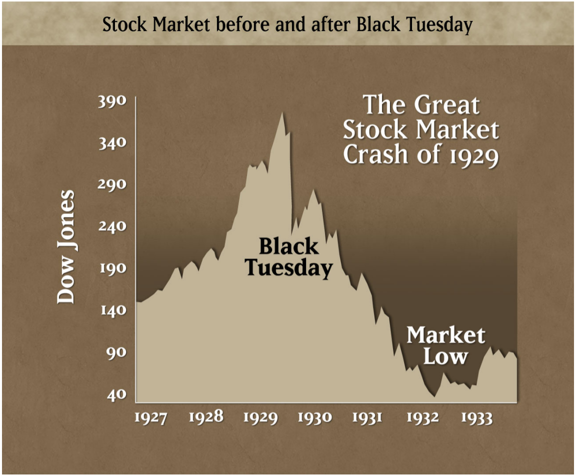
Lecture 1

18 January 2012

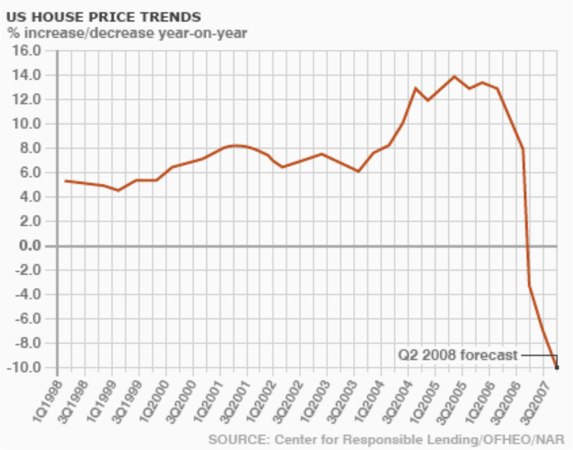
Introduction

- **Topic**: Probability and random processes
- **Lectures**: AT3 on Wednesday and Friday at 12:10pm
- **Office hours**: by appointment (email) at JCMB 6321
- **Email**: `j.m.figueroa@ed.ac.uk`

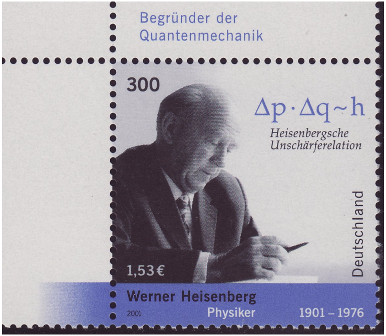
The future was uncertain



The future is **still** uncertain



The **universe** is fundamentally uncertain!



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Some notable names

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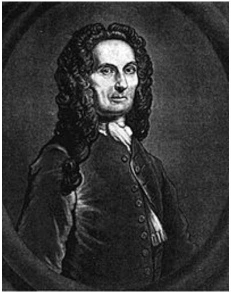

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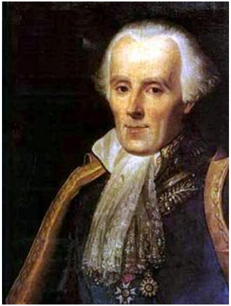

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- Andrei Markov (1866-1922)
- Andrei Kolmogorov (1903-1987)
- Claude Shannon (1916-2001)

What it is all about

Mathematical probability aims to formalise everyday sentences of the type:

"The chance of A is p"

where A is some "event" and p is some "measure" of the likelihood of occurrence of that event.

Example

"There is a 20% chance of snow."
"There is 5% chance that the West Antarctic Ice Sheet will collapse in the next 200 years."
"There is a low probability of Northern Rock having a liquidity problem."

Trials and outcomes

This requires introducing some language.

Definition/Notation

By a **trial** (or an **experiment**) we mean any process which has a well-defined set Ω of **outcomes**. Ω is called the **sample space**.

Example

Tossing a coin: $\Omega = \{\mathrm{H}, \mathrm{T}\}$.
Tossing two coins: $\Omega = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{H}), (\mathrm{T}, \mathrm{T})\}$.

Example (Rolling a (6-sided) die)

$\Omega = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$.

"Events are what we assign a probability to"

Definition

An **event** A is a subset of Ω. We say that an event A has (not) occurred if the outcome of the trial is (not) contained in A.

Example

Tossing a coin and getting a head: $A = \{H\}$.
Tossing two coins and getting at least one head:
$A = \{(H, H), (H, T), (T, H)\}$.
Rolling a die and getting an even number: $A = \{$⚁, ⚃, ⚅$\}$
Rolling two dice and getting a total of 5:
$A = \{($⚀, ⚃$), ($⚁, ⚂$), ($⚂, ⚁$), ($⚃, ⚀$)\}$

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Warning (for infinite Ω)

Not all subsets of Ω need be events!

The language of sets

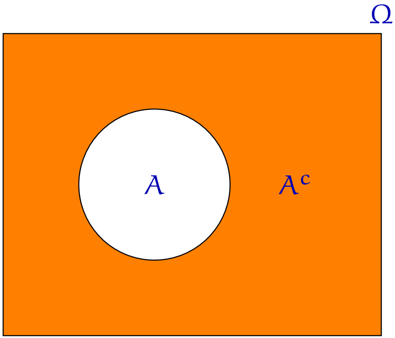
Let us consider subsets of a set Ω.

Definition

The **complement** of $A \subset \Omega$ is denoted $A^c \subset \Omega$:

$$\omega \in A^c \iff \omega \notin A$$

Clearly $(A^c)^c = A$.



Example (The empty set)

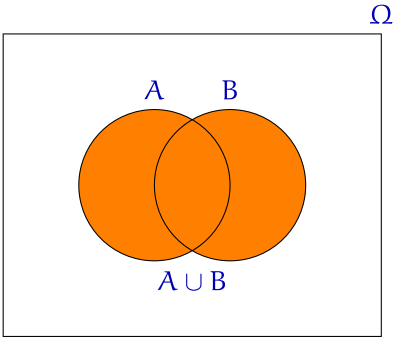
The complement of Ω is the **empty set** $\emptyset$:

$$\omega \notin \emptyset \qquad \forall \omega \in \Omega$$

Definition

The **union** of A and B is denoted $A \cup B$:

$$\omega \in A \cup B \iff \omega \in A \quad \text{or} \quad \omega \in B \quad \text{or both}$$



Remark

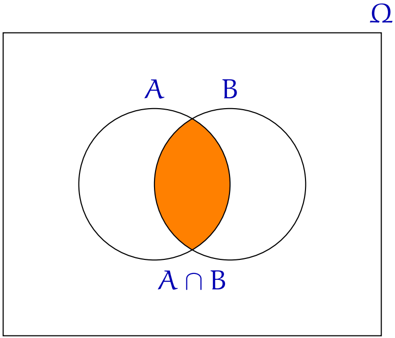
For all subsets A of Ω, $A \cup \varnothing = A$ and $A \cup \Omega = \Omega$.

Definition

The **intersection** of A and B is denoted $A \cap B$:

$$\omega \in A \cap B \iff \omega \in A \quad \text{and} \quad \omega \in B$$

If $A \cap B = \varnothing$ we say A and B are **disjoint**.



Remark

For all subsets A of Ω, $A \cap \varnothing = \varnothing$ and $A \cap \Omega = A$.

Distributivity identities

Union and intersection obey distributive properties.

Theorem

Let $(A_i)_{i\in I}$ *be a family of subsets of* Ω *indexed by some index set* I *and let* $B \subset \Omega$*. Then*

$$\bigcup_{i\in I}(B \cap A_i) = B \cap \bigcup_{i\in I} A_i$$

and

$$\bigcap_{i\in I}(B \cup A_i) = B \cup \bigcap_{i\in I} A_i$$

Proof.

$$
\begin{aligned}
\omega \in \bigcap_{i\in I}(B \cup A_i) &\iff \forall i \in I, \quad \omega \in B \cup A_i \\
&\iff \forall i \in I, \quad \omega \in B \quad \text{or} \quad \omega \in A_i \\
&\iff \omega \in B \quad \text{or} \quad \omega \in A_i \quad \forall i \in I \\
&\iff \omega \in B \quad \text{or} \quad \omega \in \bigcap_{i\in I} A_i \\
&\iff \omega \in B \cup \bigcap_{i\in I} A_i \,.
\end{aligned}
$$

The other equality is proved similarly. □

De Morgan's Theorem

Union and intersection are "dual" under complementation.

Theorem (De Morgan's)

Let $(A_i)_{i \in I}$ *be a family of subsets of* Ω *indexed by some index set* I*. Then*

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \qquad \textit{and} \qquad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$$

Remark

This shows that complementation together with either union or interesection is enough, since, e.g.,

$$A \cup B = (A^c \cap B^c)^c$$

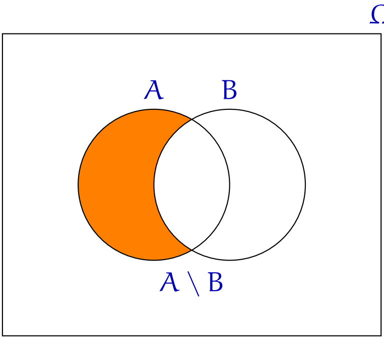
Proof.

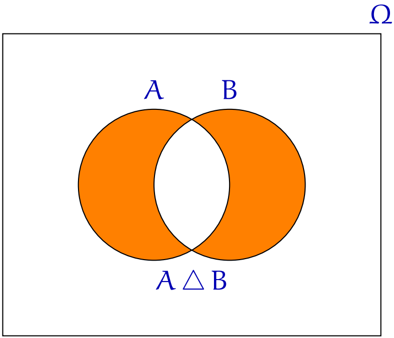
$$\begin{aligned}
\omega \in \left(\bigcup_{i\in I} A_i\right)^c &\iff \omega \notin \bigcup_{i\in I} A_i \\
&\iff \omega \notin A_i \quad \forall i \in I \\
&\iff \omega \in A_i^c \quad \forall i \in I \\
&\iff \omega \in \bigcap_{i\in I} A_i^c \,.
\end{aligned}$$

The other equality if proved similarly. □

Definition

The **difference** $A \setminus B = A \cap B^c$ and the **symmetric difference** $A \triangle B = (A \setminus B) \cup (B \setminus A)$.





Remark

Notice that $A \setminus B = A \setminus (A \cap B)$.

Probability/Set theory dictionary

Notation	Set-theoretic language	Probabilistic language
Ω	Universe	Sample space
$\omega \in \Omega$	member of Ω	outcome
$A \subset \Omega$	subset of Ω	some outcome in A occurs
A^c	complement of A	no outcome in A occurs
$A \cap B$	intersection	Both A and B
$A \cup B$	union	Either A or B (or both)
$A \setminus B$	difference	A, but not B
$A \triangle B$	symmetric difference	Either A or B, but not both
$\varnothing$	empty set	**impossible** event
Ω	whole universe	**certain** event

Which subsets can be events?

- For finite Ω, any subset can be an event.
- For infinite Ω, it is not always sensible to allow all subsets to be events. (Trust me!)
- If A is an event, it seems reasonable that A^c is also an event.
- Similarly, if A and B are events, it seems reasonable that $A \cup B$ and $A \cap B$ should also be events.

In summary, the collection of events must be closed under complementation and pairwise union and intersection. By induction, it must also be closed under finite union and intersection: if $A_1, \dots, A_N$ are events, so should be $A_1 \cap A_2 \cap \cdots \cap A_N$ and $A_1 \cup A_2 \cup \cdots \cup A_N$.

The following example, shows that this is not enough.

Example

Alice and Bob play a game in which they toss a coin in turn. The winner is the first person to obtain H. Intuition says that the the person who plays first has an advantage. We would like to quantify this intuition. Suppose Alice goes first. She wins if and only if the first H turns out after an odd number of tosses. Let ω_i be the outcome $\underbrace{TT\cdots T}_{i-1}H$. Then the event that Alice wins is $A = \{\omega_1, \omega_3, \omega_5, \dots\}$, which is a disjoint union of a countably infinite number of events. In order to compute the likelihood of Alice winning, it had better be the case that A is an event, so one demands that the family of events be closed under **countably infinite** unions; that is, if A_i, for $i = 1, 2, \dots$, are events, then so is $\bigcup_{i=1}^{\infty} A_i$.

σ-fields

Definition

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Remark

It follows from De Morgan's theorem that for a σ-field $\mathcal{F}$, if $A_1, A_2, \cdots \in \mathcal{F}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$. Also $\Omega \in \mathcal{F}$, since $\Omega = \emptyset^c$. Finally, a σ-field is closed under (symmetric) difference.

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Example

The smallest σ-field is $\mathcal{F} = \{\varnothing, \Omega\}$. The largest is the **power set** of Ω (i.e., the collection of all subsets of Ω).

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- There is a dictionary between the languages of set theory and of probability. In particular, we will use the set-theoretic language freely.