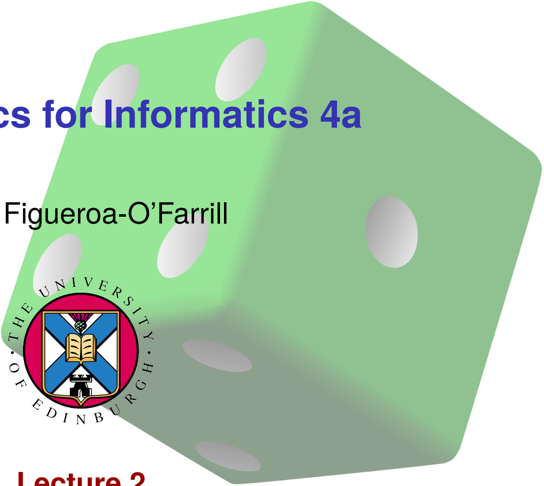
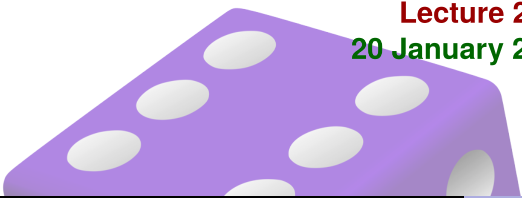


Mathematics for Informatics 4a

José Figueroa-O'Farrill

Lecture 2

20 January 2012



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- the event A is a subset of the set of outcomes of some experiment, and
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The set of outcomes is denoted Ω and the possible events define a σ-field $\mathcal{F}$ of subsets of Ω; that is, a family of subsets which contains $\varnothing$ and Ω and is closed under complementation, countable union and countable intersection.

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the number $\mathbb{P}(A)$ obeys $0 \leqslant \mathbb{P}(A) \leqslant 1$ and is called the **probability** of A occuring.

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Since $N(\Omega) = N$, we have $\mathbb{P}(\Omega) = 1$.
If $A \cap B = \varnothing$, $N(A \cup B) = N(A) + N(B)$, whence $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$. By induction, if A_i is a finite family of pairwise disjoint events,

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) .$$

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As usual, it is (theoretically) convenient to extend this to countable families.

Probability measures

Definition

A **probability measure** on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying

- $\mathbb{P}(\Omega) = 1$
- if $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$ are such that $A_i \cap A_j = \varnothing$ for all $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) .$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

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Remark

Since $\Omega = A \cup A^c$ is a disjoint union, $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$. In particular, $\mathbb{P}(\varnothing) = 0$.

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Let us call the two outcomes generically "success" (S) and "failure" (F), so that $\Omega = \{S, F\}$. Then we have

$$\mathbb{P}(\{S\}) = p \qquad \text{and} \qquad \mathbb{P}(\{F\}) = q \ .$$

Since $\Omega = \{S\} \cup \{F\}$ is a disjoint union, it follows that $q = 1 - p$.

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Notation

We will often drop the $\{\quad\}$ when talking about events consisting of a single outcome and will write $\mathbb{P}(S) = p$ and $\mathbb{P}(F) = 1 - p$.

Fair coins and fair dice

Tossing a coin has $\Omega = \{H, T\}$. Let $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = 1 - p$. The coin is **fair** if $p = \frac{1}{2}$, so that both H and T are equally probable.

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Similarly, a **fair** die is one where every outcome has the same probability. Since there are six outcomes, each one has probability $\frac{1}{6}$:

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Fair coins and fair dice are examples of “uniform probability spaces”: those whose outcomes are all equally likely.

Uniform probability measures

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$$1 = \mathbb{P}\left(\bigcup_{\omega\in\Omega} \omega\right) = \sum_{\omega\in\Omega} \mathbb{P}(\omega) = \sum_{\omega\in\Omega} p = p|\Omega| ,$$

whence $p = 1/|\Omega|$.

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Now let $A \subseteq \Omega$ be an event:

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{\omega\in A} \omega\right) = \sum_{\omega\in A} \mathbb{P}(\omega) = \sum_{\omega\in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

Example

You draw a number at random from $\{1, 2, \ldots, 30\}$. *What is the probability of the following events:*

1. $A =$ *the number drawn is even*
2. $B =$ *the number drawn is divisible by 3*
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3. $C = \{1, 2, 3, \ldots, 11\}$, so $|C| = 11$ and hence $\mathbb{P}(C) = \frac{11}{30}$.

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Therefore $\mathbb{P}(10) > \mathbb{P}(9)$.

Example (Continued)

We could have answered this without enumeration. The average score of rolling three dice is $10\frac{1}{2}$. Since 10 is closer than 9 to the average, $\mathbb{P}(10) \geqslant \mathbb{P}(9)$ as a consequence of the "central limit theorem".

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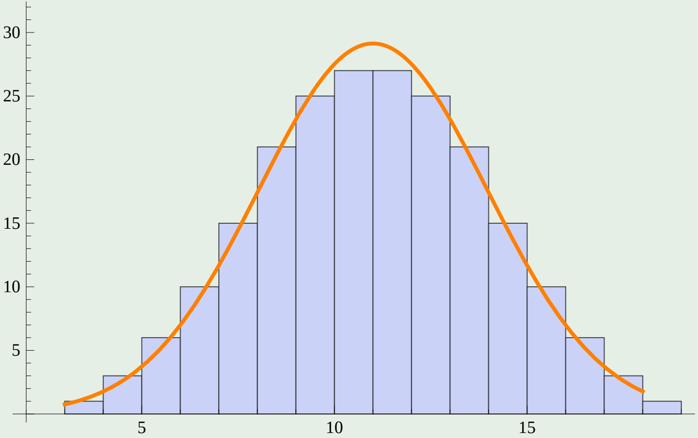

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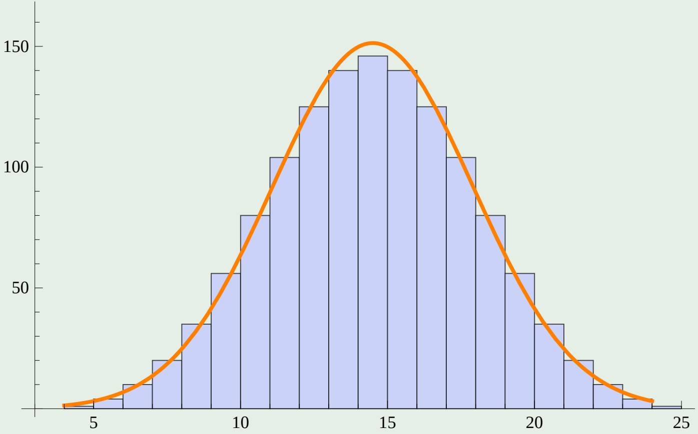

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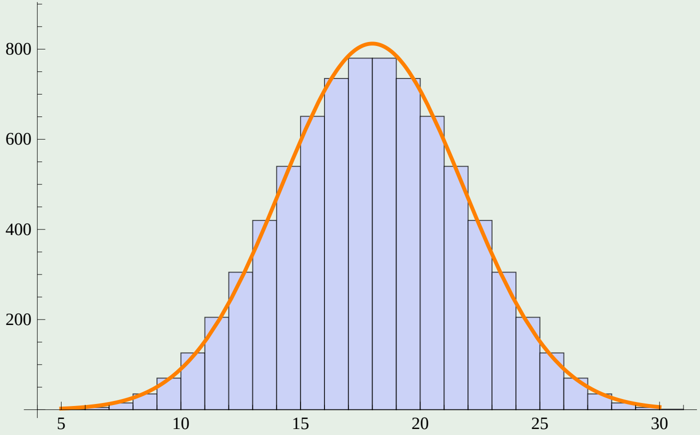

Example (Alice and Bob's game)

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Similarly, since $B = \bigcup_{n=1}^{\infty}\{\omega_{2n}\}$ is also a disjoint union,

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Finally, since $\mathbb{P}(A) + \mathbb{P}(B) = 1$, we see that $\mathbb{P}(C) = 0$.

Warning

Although $\mathbb{P}(C) = 0$, the event C is **not** impossible.

Basic properties of probability measures

Theorem

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
2. *if* $B \supseteq A$ *then* $\mathbb{P}(B) \geqslant \mathbb{P}(A)$

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Proof.

1. $\Omega = A \cup A^c$ is a disjoint union, whence

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Proof.

1. $\Omega = A \cup A^c$ is a disjoint union, whence

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2. Write B as the disjoint union $B = (B \setminus A) \cup A$, whence

$$\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A) \geqslant \mathbb{P}(A) .$$

$\square$

Example (The Birthday problem)

What is the probability that among n people chosen at random, there are at least 2 people sharing the same birthday?

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There are 365^n possible outcomes to the birthdays of n people and $365 \times 364 \times \cdots \times (365 - n + 1)$ possible outcomes consisting of n different birthdays, hence

$$\mathbb{P}(A_n^c) = \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n} = \prod_{i=1}^{n-1} \left(1 - \tfrac{i}{365}\right)$$

and

$$\mathbb{P}(A_n) = 1 - \prod_{i=1}^{n-1} \left(1 - \tfrac{i}{365}\right) .$$

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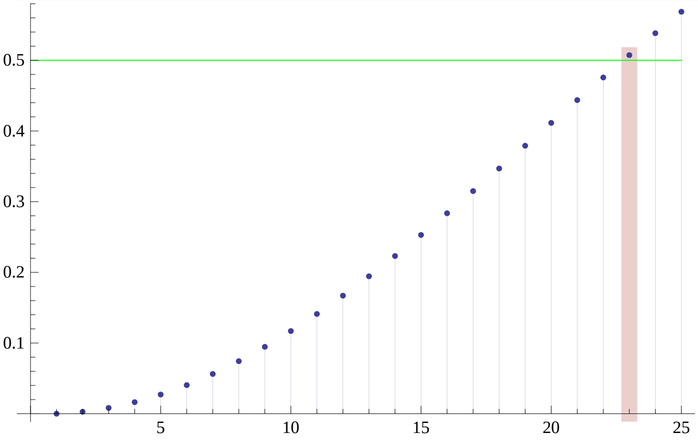

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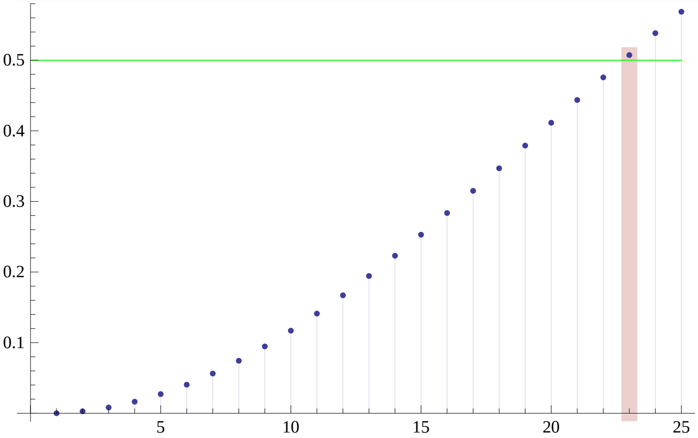

Answer: **23** (!)

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Proof.

- $A = (A \setminus B) \cup (A \cap B)$ whence $\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$

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$$
\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(A \bigtriangleup B) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)
\end{aligned}
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Example

Historical meteorological records for a certain seaside location show that on New Year's day there is a 30% chance of rain, 40% chance of being windy and 20% chance of both rain and wind. What is the chance of it being dry? dry and windy? wet or windy?

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Boole's inequality

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$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leqslant \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n)$

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Proof.

$A_1 \cup A_2 \cup \cdots \cup A_n = (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n$, and by the inclusion-exclusion rule,

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leqslant \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) + \mathbb{P}(A_n)$$

But now $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = (A_1 \cup A_2 \cup \cdots \cup A_{n-2}) \cup A_{n-1}$, so that

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leqslant \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_{n-2}) + \mathbb{P}(A_{n-1}) + \mathbb{P}(A_n)$$

et cetera. $\square$

General inclusion-exclusion rule

Theorem

$$
\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1\leqslant i<j\geqslant n} \mathbb{P}(A_i \cap A_j)
+ \sum_{1\leqslant i<j<k\leqslant n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^n \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)
$$

Proof.

Use induction from the simple inclusion-exclusion rule. □

Continuity

Theorem

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2. Take complements of the previous proof.

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Probability spaces with Ω finite and $\mathbb{P}$ uniformly distributed ("all outcomes equally likely") are particularly amenable to counting techniques from combinatorial analysis.