Mathematics for Informatics 4a

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Lecture 3
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The thought of the day

Probability is a measure of our ignorance and hence, when our knowledge about a system changes, the probability of events should also change to reflect the new knowledge.

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There is now only one positive outcome: namely, (⚄, ⚅).

But the sample space has changed as well.

Once we know that the first die is ⚄, the sample space consists of six outcomes:

{(⚄, ⚀), (⚄, ⚁), (⚄, ⚂), (⚄, ⚃), (⚄, ⚄), (⚄, ⚅)}

and hence $\mathbb{P}(A) = \frac{1}{6}$.

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Now suppose that we know that one of the children is a girl. What is the probability now?

The sample space is now $\{(\female, \female), (\female, \male), (\male, \female)\}$, whereas there is still only one positive outcome, whence $\mathbb{P}(A) = \frac{1}{3}$.

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What about if we know that the oldest child is a girl?

The sample space is now $\{(♀, ♀), (♀, ♂)\}$, whence $\mathbb{P}(A) = \frac{1}{2}$.

Conditional probability

Definition

Let A, B be events. The **conditional probability** $\mathbb{P}(A|B)$ of the event of "A occurs given that B occured" is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

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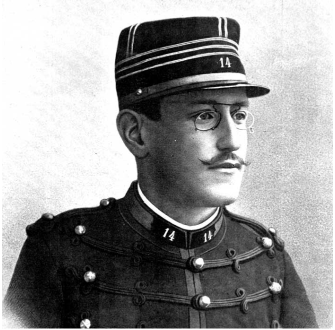
Warning!

$\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$, unless $\mathbb{P}(A) = \mathbb{P}(B)$. The incorrect equality formed the basis of the prosecution's case in the infamous *affaire Dreyfus* in France at the turn of the 20th century. To find out more, visit my colleague Andrew Ranicki's page on this subject: `http://www.maths.ed.ac.uk/~aar/dreyfus.htm`.



This Dreyfus...

... not *this* Dreyfus!

Conditional probability from uniform probability

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a uniform probability space (e.g., Ω finite and all outcomes equally probable) and A, B are two events, we have

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} \qquad \text{and} \qquad \mathbb{P}(B) = \frac{|B|}{|\Omega|} \; .$$

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If B occurs, the possible outcomes are those in B and they remain equally likely with probability $1/|B|$.
The event A occurs if and only if $A \cap B$ occurs, whence the probability that A occurs given that B occurs is

$$\frac{|A \cap B|}{|B|} = \frac{|A \cap B|}{|\Omega|} \Big/ \frac{|B|}{|\Omega|} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B) .$$

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$$\frac{N(A \cap B)}{N(B)} = \frac{N(A \cap B)}{N} \bigg/ \frac{N(B)}{N} \to \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B) \,.$$

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We will now revisit the previous examples in the language of conditional probability.

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Let A be the event "the second die shows a greater score than the first" and let B be the event "the first die shows a $\epsdice{5}$".
Then

$$
\begin{aligned}
B &= \{(\epsdice{5},\epsdice{1}),(\epsdice{5},\epsdice{2}),(\epsdice{5},\epsdice{3}),(\epsdice{5},\epsdice{4}),(\epsdice{5},\epsdice{5}),(\epsdice{5},\epsdice{6})\}\\
A \cap B &= \{(\epsdice{5},\epsdice{6})\}
\end{aligned}
$$

whence

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{|A \cap B|}{|B|} = \frac{1}{6}.$$

Example (Alice and Bob have children)

Let

$$A = \{(\female, \female)\}$$ "both are girls"

$$B = \{(\female, \female), (\female, \male), (\male, \female)\}$$ "at least one is a girl"

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There are 3 coins and hence 6 possible outcomes, of which 3 are heads. Therefore $\mathbb{P}(H) = \frac{3}{6} = \frac{1}{2}$.

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Therefore

$$\mathbb{P}(D|H) = \frac{\mathbb{P}(D \cap H)}{\mathbb{P}(H)} = \frac{1}{3}\Big/\frac{1}{2} = \frac{2}{3} .$$

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In fact, we can define a new probability space $(B, \mathcal{F}', \mathbb{P}')$, where

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Recall that $\mathcal{F}'$ is again a σ-field (cf. Tutorial Sheet 1), only this time of subsets of B.

Multiplication rule

From the ▸ definition of conditional probability there follows the

Multiplication rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

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Let $P =$ "the pair is black" and $B =$ "the first sock is black". Then clearly $P \subset B$, so $P = P \cap B$.

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Let $P =$ "the pair is black" and $B =$ "the first sock is black". Then clearly $P \subset B$, so $P = P \cap B$. Also $\mathbb{P}(B) = \frac{3}{8}$ and $\mathbb{P}(P|B) = \frac{2}{7}$

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Let P = "the pair is black" and B = "the first sock is black". Then clearly $P \subset B$, so $P = P \cap B$. Also $\mathbb{P}(B) = \frac{3}{8}$ and $\mathbb{P}(P|B) = \frac{2}{7}$, whence

$$\mathbb{P}(P) = \mathbb{P}(P \cap B) = \mathbb{P}(P|B)\mathbb{P}(B) = \frac{2}{7} \times \frac{3}{8} = \frac{3}{28}.$$

Extended multiplication rule

Theorem

Let $A_1, A_2, \ldots, A_n$ *be events with* $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) > 0$. *Then*

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_n | A_{n-1} \cap \cdots \cap A_1) \times \\ \times \mathbb{P}(A_{n-1} | A_{n-2} \cap \cdots \cap A_1) \cdots \mathbb{P}(A_2 | A_1) \mathbb{P}(A_1)$$

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Proof.

Just use that for $k = 2, \ldots, n$,

$$\mathbb{P}(A_k | A_{k-1} \cap \cdots \cap A_1) = \frac{\mathbb{P}(A_k \cap \cdots \cap A_1)}{\mathbb{P}(A_{k-1} \cap \cdots \cap A_1)}$$

and multiply the terms in the RHS to obtain the LHS. $\square$

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By the extended multiplication rule,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(C|A \cap B)\mathbb{P}(B|A)\mathbb{P}(A)$$

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where $\mathbb{P}(A) = \frac{6}{15} = \frac{2}{5}$, $\mathbb{P}(B|A) = \frac{4}{14} = \frac{2}{7}$ and $\mathbb{P}(C|A \cap B) = \frac{5}{13}$

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By the extended multiplication rule,

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where $\mathbb{P}(A) = \frac{6}{15} = \frac{2}{5}$, $\mathbb{P}(B|A) = \frac{4}{14} = \frac{2}{7}$ and $\mathbb{P}(C|A \cap B) = \frac{5}{13}$, whence

$$\mathbb{P}(A \cap B \cap C) = \frac{2}{5} \times \frac{2}{7} \times \frac{5}{13} = \frac{4}{91} \simeq 4.4\%$$

Example

A bag contains 26 cards, each one with a letter in $\{a, b, c, \ldots, z\}$*. You take 7 cards at random without replacement. What is the probability that you can spell "dreyfus" with them?*

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&= \binom{26}{7}^{-1} = \tfrac{1}{657800}.
\end{aligned}$$

Independence

Heuristically, two events A, B are said to be independent if the chance of one occurring is not altered by the other's occurrence; that is, $\mathbb{P}(A|B) = \mathbb{P}(A)$.

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Example

"Not winning the lottery last week" and "winning the lottery this week" are independent events.

Suppose that we have three events A, B, C. We say that they are **independent** if any two are independent:

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The reason we impose all conditions is that independence means also that no two events occurring can influence the chance of a third event occurring.

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$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{6}{15} \times \frac{4}{15} \times \frac{5}{15} = \frac{8}{225} \simeq 3.6\%$$

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$$\begin{aligned}\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2) \\ &= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}.\end{aligned}$$

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A fair coin is tossed 10 *times. What is the probability of getting all heads? of getting* HTHTHTHTHT*? of getting* TTTTTTTTTH*?*

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What about if the coin is not fair?

Let $\mathbb{P}(\mathrm{H}) = p$ and $\mathbb{P}(\mathrm{T}) = q = 1 - p$. Then by independence,

$$\begin{aligned}
\mathbb{P}(\mathrm{HHHHHHHHHH}) &= \mathbb{P}(\mathrm{H})^{10} = p^{10} \\
\mathbb{P}(\mathrm{HTHTHTHTHT}) &= \mathbb{P}(\mathrm{H})\mathbb{P}(\mathrm{T})\mathbb{P}(\mathrm{H})\ldots\mathbb{P}(\mathrm{T}) = p^5(1-p)^5 \\
\mathbb{P}(\mathrm{TTTTTTTTTH}) &= \mathbb{P}(\mathrm{T})^9\mathbb{P}(\mathrm{H}) = (1-p)^9 p
\end{aligned}$$

Independence vs. pairwise independence

Definition

A family $A_1, A_2, \ldots, A_n$ of events is **pairwise independent** if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \qquad \forall \quad 1 \leqslant i < j \leqslant n$$

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and it is **independent** if, for any $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$,

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Whereas independence implies pairwise independence, the converse **is not true.**

Example

Two fair coins are tossed. Consider the events

- $A =$ “the first toss is a head”
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Then $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$ and

$$\begin{aligned} \mathbb{P}(A \cap B) &= \tfrac{1}{4} = \mathbb{P}(A)\mathbb{P}(B) \\ \mathbb{P}(A \cap C) &= \tfrac{1}{4} = \mathbb{P}(A)\mathbb{P}(C) \\ \mathbb{P}(B \cap C) &= \tfrac{1}{4} = \mathbb{P}(B)\mathbb{P}(C) , \end{aligned}$$

whence the events are pairwise independent.

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\end{aligned}$$

whence the events are pairwise independent. However

$$\mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) ,$$

and the events are **not** independent.

Summary

- The conditional probability of A occuring given that B occurs is
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- Events A, B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
- More generally, a countable family $\{A_i\}$, $i \in I$, of events are independent if
$$\mathbb{P}\Big(\bigcap_{j \in J} A_j\Big) = \prod_{j \in J} \mathbb{P}(A_j)$$
for every finite subset $J \subseteq I$