Mathematics for Informatics 4a

José Figueroa-O'Farrill

THE UNIVERSITY OF EDINBURGH

Lecture 4
27 January 2012

Public Service Announcement

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3. Plot the polynomial and estimate the root(s) from the plot!

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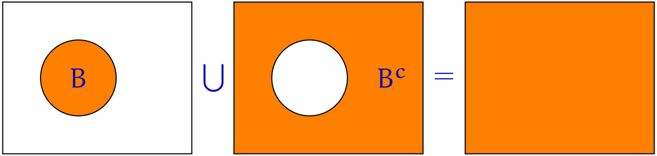
- The method of hurdles:

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_2 \cap A_1)\cdots$$

Partition theorems

For any event B, we have that $B \cup B^c = \Omega$:



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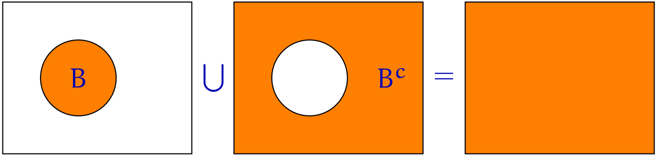
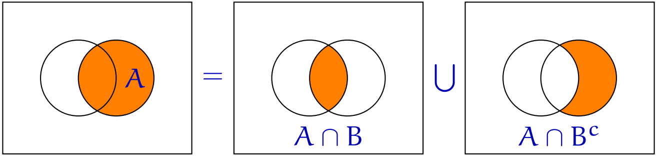


So if A is any other event, $A = (A \cap B) \cup (A \cap B^c)$:



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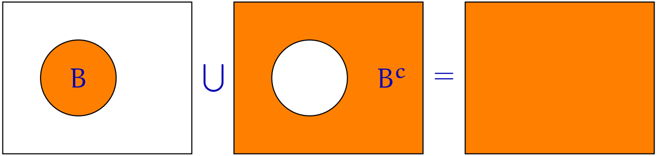

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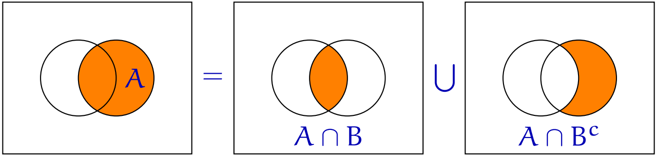

Formally,

$$A = A \cap \Omega = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$$

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$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) .$$

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Theorem (The partition rule)

For any two events A, B, we have

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$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c) \; .$$

Remark

This is also called the *rule of total probability* or the *rule of alternatives*.

A geometric analogy

Consider $\mathbb{R}^2$ with the standard dot product:

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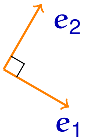
Let $(\mathbf{e}_1, \mathbf{e}_2)$ be an orthonormal basis for $\mathbb{R}^2$:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



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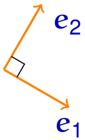

Any vector $\mathbf{x} \in \mathbb{R}^2$ can be decomposed in an unique way as

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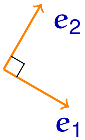

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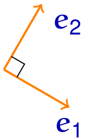

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The analogue of orthogonality is now $\mathbb{P}(B|B^c) = 0$.

A general partition rule

Definition

By a (finite) **partition** of Ω we mean events $\{B_1, B_2, \ldots, B_n\}$ such that $B_i \cap B_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^n B_i = \Omega$.

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Theorem (General partition rule)

Let $\{B_1, \ldots, B_n\}$ *be a partition of* Ω. *Then for any event* A,

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A|B_i)\mathbb{P}(B_i) .$$

Proof.

This is proved in exactly the same way as in the case of the partition $\{B, B^c\}$. □

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Then by the (general) partition rule

$$\begin{aligned}\mathbb{P}(H) &= \mathbb{P}(H|A)\mathbb{P}(A) + \mathbb{P}(H|B)\mathbb{P}(B) + \mathbb{P}(H|C)\mathbb{P}(C) \\ &= (1 \times \tfrac{3}{10}) + (0 \times \tfrac{2}{10}) + (\tfrac{1}{2} \times \tfrac{5}{10}) \\ &= \tfrac{3}{10} + \tfrac{1}{4} = \tfrac{11}{20}\end{aligned}$$

Example (Medical tests)

A virus infects a proportion p *of individuals in a given population. A test is devised to indicate whether a given individual is infected. The probability that the test is positive for an infected individual is 95%, but there is a 10% probability of a false positive. Testing an individual at random, what is the chance of a positive result?*

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(Not a very good test: if p is very small, most positive results are false positives.)

Example (Noisy channels)

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$$\therefore \quad \mathbb{P}(S_0|R_0) = \frac{1}{2}\Big/\left(\frac{1}{2} + \frac{1}{14}\right) = \frac{1}{2}\Big/\frac{4}{7} = \frac{7}{8}$$

Conditional partition rule

Theorem

Let $\{B_1, \ldots, B_n\}$ *be a partition of* Ω *and let* C *be an event with* $\mathbb{P}(C) > 0$*. Then for any event* A*,*

$$\mathbb{P}(A|C) = \sum_{i=1}^{n} \mathbb{P}(A|B_i \cap C)\mathbb{P}(B_i|C) \ .$$

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Proof

The partition rule holds in *any* probability space, so in particular it holds for the conditional probability $\mathbb{P}'(A \cap C) = \mathbb{P}(A|C)$. Since $\{B_1 \cap C, \ldots, B_n \cap C\}$ is a partition of C,

$$\mathbb{P}'(A \cap C) = \sum_{i=1}^{n} \mathbb{P}'(A \cap C|B_i \cap C)\mathbb{P}'(B_i \cap C)$$

Proof (continued)

We rewrite $\mathbb{P}'(A \cap C) = \sum_{i=1}^{n} \mathbb{P}'(A \cap C|B_i \cap C)\mathbb{P}'(B_i \cap C)$ as

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$$\mathbb{P}'(A \cap C|B_i \cap C) = \frac{\mathbb{P}'(A \cap B_i \cap C)}{\mathbb{P}'(B_i \cap C)} = \frac{\mathbb{P}(A \cap B_i|C)}{\mathbb{P}(B_i|C)}$$

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$$\begin{aligned}\mathbb{P}'(A \cap C|B_i \cap C) &= \frac{\mathbb{P}'(A \cap B_i \cap C)}{\mathbb{P}'(B_i \cap C)} = \frac{\mathbb{P}(A \cap B_i|C)}{\mathbb{P}(B_i|C)} \\ &= \frac{\mathbb{P}(A \cap B_i \cap C)}{\mathbb{P}(C)} \Bigg/ \frac{\mathbb{P}(B_i \cap C)}{\mathbb{P}(C)} \\ &= \frac{\mathbb{P}(A \cap B_i \cap C)}{\mathbb{P}(B_i \cap C)}\end{aligned}$$

Proof (continued)

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&= \mathbb{P}(A|B_i \cap C) .
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□

Example

There are number of different drugs to treat a disease and each drug may give rise to side effects. A certain drug C *has a 99% success rate in the absence of side effects and side effects only arise in 5% of cases. If they do arise, however, then* C *has only a 30% success rate. If* C *is used, what is the probability of the event* A *that a cure is effected?*

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Let B be the event that **no** side effects occur. We are given that

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Bayes's rule

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Using the partition rule $\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)$ we get a modified version of Bayes's rule:

Theorem (Bayes's rule too)

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So that if half the population is infected ($p = 0.5$), then $\mathbb{P}(V|P) \simeq 90\%$ and the test looks good, but if the virus affects only one person in every thousand ($p = 10^{-3}$), then $\mathbb{P}(V|P) \simeq 1\%$, so not very conclusive at all!

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Notice that the larger the number c, the more likely that the student knew the answer.

Conditional probability in Mendelian genetics I

Basic question

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- A given gene can come in two (or more) mutated forms called **alleles**.

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- Males can therefore be A or a, whereas females can be AA, Aa and aa. (We don't distinguish between Aa and aA.)

Conditional probability in Mendelian genetics III

Example

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Let G_{AA} (resp. G_{Aa}) denote the event that the daughter has genotype AA (resp. Aa) and let S_A denote the event that the daughter's son has genotype A. We want $\mathbb{P}(G_{AA}|S_A)$.

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Bayes's: $\mathbb{P}(G_{AA}|S_A) = \mathbb{P}(S_A|G_{AA})\mathbb{P}(G_{AA})/\mathbb{P}(S_A) = \frac{1}{2}\Big/\frac{3}{4} = \frac{2}{3}$.

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- Bayes's rule allows us to compute $\mathbb{P}(A|B)$ from a knowledge of $\mathbb{P}(B|A)$ via

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}$$