Mathematics for Informatics 4a

José Figueroa-O'Farrill

Lecture 5
1 February 2012

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- Bayes's rule allows us to compute $\mathbb{P}(A|B)$ from a knowledge of $\mathbb{P}(B|A)$ via

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)}$$

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Definition

Let A, B and C be events. We say that A and B are **conditionally independent** (given C), if

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$$

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$$\mathbb{P}(H_1 \cap H_2|C) = \mathbb{P}(H_1|C)\mathbb{P}(H_2|C)$$

yet

$$\mathbb{P}(H_1 \cap H_2) \neq \mathbb{P}(H_1)\mathbb{P}(H_2) .$$

Example (Continued)

Indeed, by the partition rule and letting F denote the event of having picked the fair coin,

$$\mathbb{P}(H_1 \cap H_2) = \mathbb{P}(H_1 \cap H_2|F)\mathbb{P}(F) + \mathbb{P}(H_1 \cap H_2|F^c)\mathbb{P}(F^c) ,$$

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where $\mathbb{P}(H_1 \cap H_2 | F) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ and $\mathbb{P}(H_1 \cap H_2 | F^c) = 1$, whence

$$\mathbb{P}(H_1 \cap H_2) = (\tfrac{1}{4} \times \tfrac{1}{2}) + (1 \times \tfrac{1}{2}) = \tfrac{5}{8} .$$

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On the other hand, the probability of getting a head is $\frac{3}{4}$ since there are four faces in total, three of which are heads, whence

$$\mathbb{P}(H_1)\mathbb{P}(H_2) = \tfrac{3}{4} \times \tfrac{3}{4} = \tfrac{9}{16} .$$

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respectively.

Discrete probability distributions

- Let us consider an experiment whose outcomes X are integers. The **probability distribution of** X is the function $p : \mathbb{Z} \to \mathbb{R}$ defined by $p(x) = \mathbb{P}(X = x)$ for all $x \in \mathbb{Z}$.

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Consider rolling a fair die. The possible outcomes are ⚀, ⚁, . . . , ⚅, which we convert to a numerical outcome $X \in \{1, 2, \ldots, 6\}$ in the obvious way.

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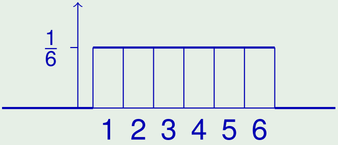

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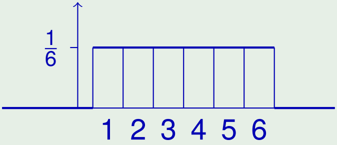

Notice that $\sum_{x \in \mathbb{Z}} p(x) = 1$.

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Generalising the above, we define the **uniform distribution** on $\{1, 2, \ldots, n\}$ by

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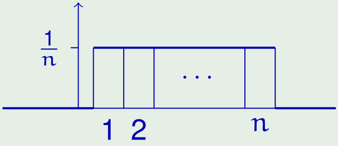

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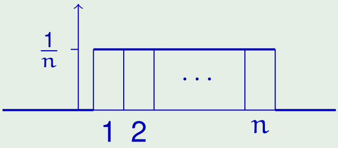

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Example (Bernoulli trials)

Consider a Bernoulli trial with $\mathbb{P}(S) = p$ and $\mathbb{P}(F) = q = 1 - p$. Let $X \in \{0, 1\}$ denote the number of successes, so that $p(0) = q$ and $p(1) = p$ and $p(x) = 0$ for $x \neq 0, 1$. Of course, $\sum_{x \in \mathbb{Z}} p(x) = 1$.

Example (Independent Bernoulli trials)

We could also consider a sequence of n independent Bernoulli trials, each one with $\mathbb{P}(S) = p$ and $\mathbb{P}(F) = q = 1 - p$. We let $X \in \{0, 1, \ldots, n\}$ denote the number of successes.

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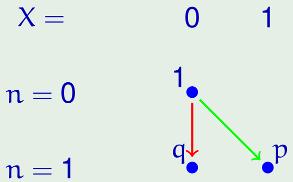

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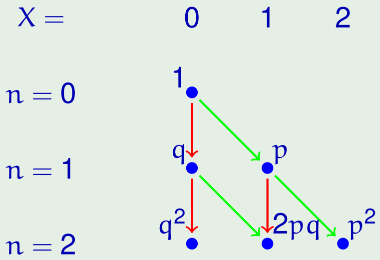

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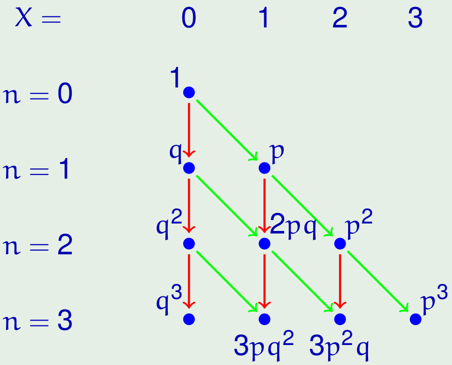

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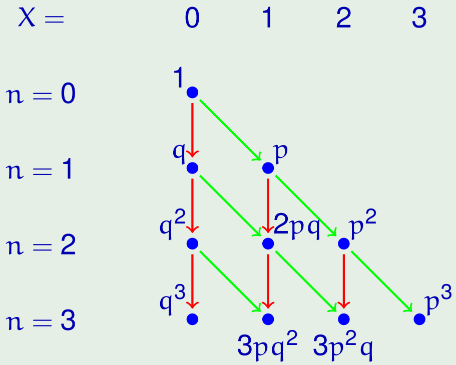

cf. Pascal's triangle!

Example (Binomial distribution)

Continuing with the previous example, it is clear that

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$$\sum_{x \in \mathbb{Z}} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)^n = 1 ,$$

by the binomial theorem.

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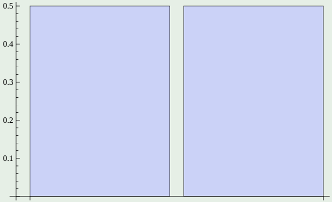

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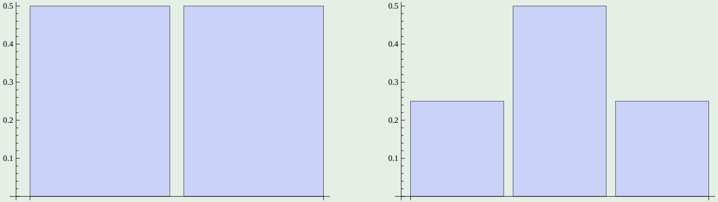

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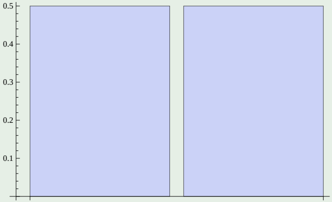

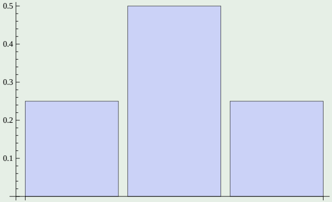

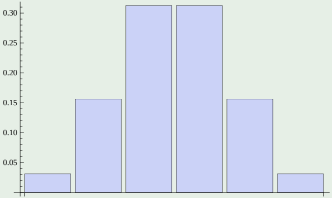

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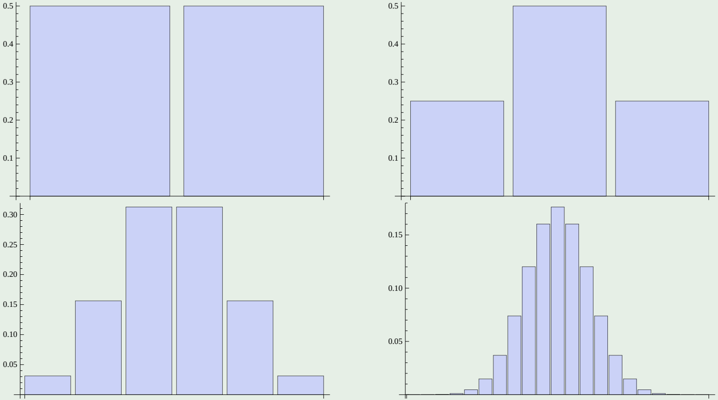

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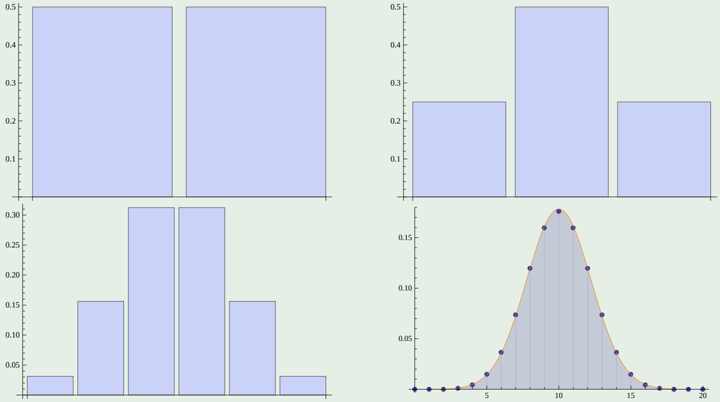

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- Therefore the probability we are after is

$$\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k q^{n+m-1-k}$$

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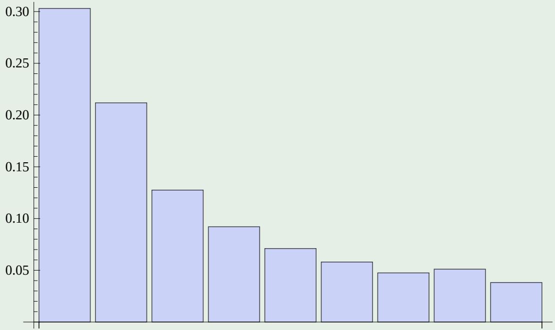
For example, consider the sizes of files (in 512K blocks) in my laptop (excluding directories). It has over 2.5M files and the distribution of significant digits looks like this:



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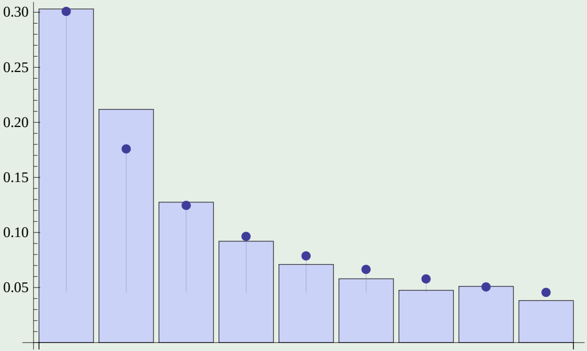

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How about the distribution of the first two significant digits?
Again, it is empirically very close to

$$p(k) = \begin{cases} \log_{10}(1 + \frac{1}{k}), & 10 \leqslant k \leqslant 99 \\ 0, & \text{otherwise} \end{cases}$$

which is **Benford's 2-digit distribution**.

Example (Benford's distribution – continued)

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Should this surprise us? It should not. In fact, if we take our own statistical tables and re-express the entries in base b instead of base 10, we still get a distribution which is close to

$$p(k) = \begin{cases} \log_b(1 + \frac{1}{k}), & 1 \leqslant k \leqslant b - 1 \\ 0, & \text{otherwise.} \end{cases}$$

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- Indeed, Benford's distribution corresponds to the uniform distribution of the first significant digit of the logarithms of the numbers!

General properties of discrete probability distributions

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- and since $\mathbb{P}(\Omega) = 1$, it follows that $\sum_{x \in \mathbb{Z}} p(x) = 1$.

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The only outcomes resulting in a transmission error are those where there are at least two bits in error, whose probability is

$$p^3 + 3p^2(1-p) = p^2(3-2p)$$

Distribution function

Definition

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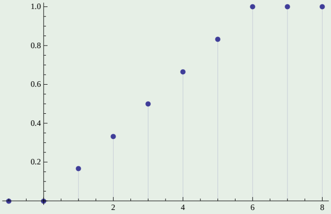
Example (Rolling a fair die)

$$F(x) = \begin{cases} 0, & x \in \{0, -1, -2, \dots\} \\ \frac{x}{6}, & x \in \{1, 2, 3, 4, 5, 6\} \\ 1, & x \in \{7, 8, 9, \dots\}. \end{cases}$$



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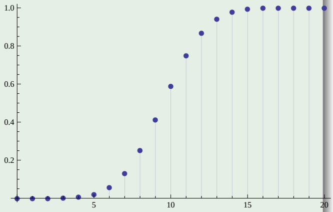

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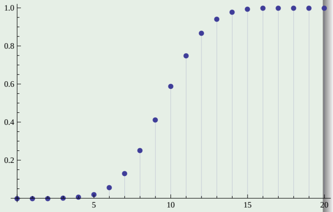

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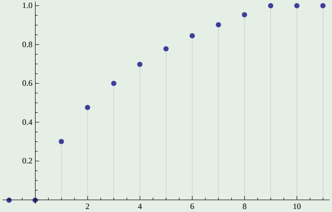

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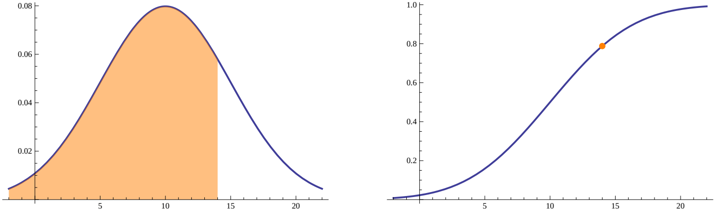
This is not unlike an area $F(x) = \int_{-\infty}^{x} p(y)dy$:



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- Another way to repackage the information in the probability distribution is in the distribution function $F : \mathbb{Z} \to [0, 1]$, defined by $F(x) = \sum_{t \leqslant x} p(t)$