Mathematics for Informatics 4a

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Lecture 6
3 February 2012

The story of the film so far...

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- We also introduced the distribution function $F : \mathbb{Z} \to [0, 1]$ associated to p, defined by $F(x) = \sum_{t\leqslant x} p(t)$: monotonically increasing from 0 to 1.

The mathematics of waiting

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$$\sum_{k\in\mathbb{Z}} p(k) = \sum_{k=1}^{\infty} (\tfrac{1}{2})^k = \sum_{k=0}^{\infty} (\tfrac{1}{2})^k - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1 \, .$$

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- The resulting probability distribution is now

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  and is called the **truncated geometric distribution** with parameters N and $\frac{1}{2}$.
- Again one has $\sum_k p(k) = \sum_{k=1}^{N-1} (\frac{1}{2})^k + (\frac{1}{2})^{N-1} = 1$.

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The number $p(k)$ is the probability that in independent Bernoulli trials with success probability p, the first success occurs at the kth trial. Notice that

$$\sum_{k \in \mathbb{Z}} p(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{\ell=0}^{\infty} (1-p)^{\ell}$$
$$= p \frac{1}{1-(1-p)} = 1 \, .$$

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$$p_k(n) = \binom{n-1}{k-1} p^{k-1} q^{n-k} \times p = \binom{n-1}{k-1} p^k q^{n-k}$$

for $n \geqslant k$. This is the **negative binomial distribution**.

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2. for every $x_i \in D$, the set $\{\omega \in \Omega | X(\omega) = x_i\} \in \mathcal{F}$.

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Please observe this convention very carefully!!!

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- Being a probability, $0 \leqslant f_X(x) \leqslant 1$ for all $x \in \mathbb{R}$.
- Since the A_x for $x \in \mathbb{Z}$ are a countable partition of Ω, the countable additivity of $\mathbb{P}$ implies that

$$\sum_{x\in\mathbb{Z}} f_X(x) = \sum_{x\in\mathbb{Z}} \mathbb{P}(A_x) = \mathbb{P}\left(\bigcup_{x\in\mathbb{Z}} A_x\right) = \mathbb{P}(\Omega) = 1 \, .$$

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- The probability distributions we have been discussing can play the rôle of probability mass functions.
- One can talk about discrete random variables with uniform, binomial, geometric, Benford,... probability mass functions.

Example (Poisson distribution)

Let $\lambda > 0$ be a positive real number. The **Poisson distribution** with parameter λ is defined by

$$f(x) = \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

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It is clear that $f(x) \geqslant 0$ for all x and that it is nonzero only for a countable subset of $\mathbb{R}$; namely, the natural numbers. Finally

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!}$$

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Let $\lambda > 0$ be a positive real number. The **Poisson distribution** with parameter λ is defined by

$$f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

It is clear that $f(x) \geqslant 0$ for all x and that it is nonzero only for a countable subset of $\mathbb{R}$; namely, the natural numbers. Finally

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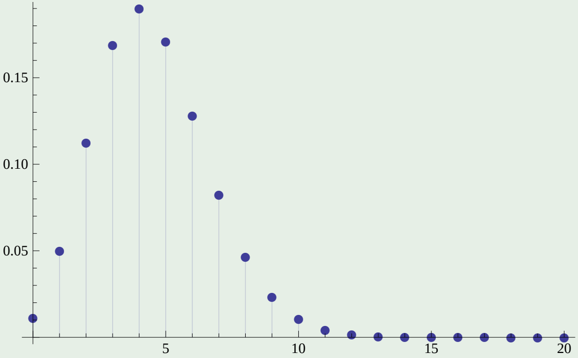
We can therefore talk about discrete random variables whose probability mass function is Poisson with parameter λ.

Example (Poisson distribution – continued)



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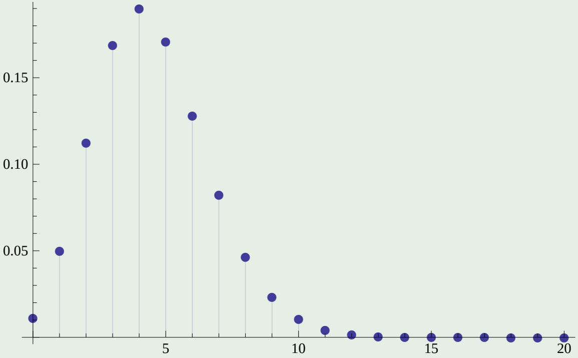

We will see later that the Poisson distribution is a limit of the binomial distribution for large n and small p keeping np fixed. This means that we can use it to approximate the binomial distribution in that limit.

Expectation value as a weighted average

Let X be a discrete random variable with probability mass function f_X. The **expectation value** $\mathbb{E}(X)$ of X is defined by

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Example (Betting)

Consider a betting game based on a Bernoulli trial with success probability p. Every bet costs $\pounds 1$: if you win you get your $\pounds 1$ back and an additional $\pounds 2$, if you lose you get nothing. *How much do you expect to win/lose on average?*

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Introduce the random variable X with values $X(S) = 2$ and $X(F) = -1$ which measures the amount (in $\pounds$) you win: in the case of success you win $\pounds 2$ and in the case of failure you lose $\pounds 1$, which is the same as winning $-\pounds 1$.

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Example (Betting)

Consider a betting game based on a Bernoulli trial with success probability p. Every bet costs $£1$: if you win you get your $£1$ back and an additional $£2$, if you lose you get nothing. *How much do you expect to win/lose on average?*

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Notice that

$$X(S)\mathbb{P}(S) + X(F)\mathbb{P}(F) = 2f_X(2) + (-1)f_X(-1) .$$

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Let X be a discrete random variable with probability mass function f_X given by a Poisson distribution with parameter $\lambda > 0$. *What is its expectation value?* By definition

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But now for $0 < x \leqslant n$,

$$x \binom{n}{x} = x \frac{n!}{(n-x)!x!} = \frac{n!}{(n-x)!(x-1)!} = n \binom{n-1}{x-1}$$

whence

$$\mathbb{E}(X) = \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x q^{n-x} = \sum_{z=0}^{n-1} n \binom{n-1}{z} p^{z+1} q^{n-1-z} = np .$$

Summary

- A **discrete random variable** X in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $X : \Omega \to \mathbb{R}$ which can take only countably many values and such that the subsets $\{X = x\}$ are events.

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- We met the **Poisson distribution** with parameter $\lambda > 0$

$$f(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

for $x \in \mathbb{N}$ and $f(x) = 0$ otherwise. It has expectation value λ.