Mathematics for Informatics 4a

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Lecture 7

8 February 2012

The story of the film so far...

- A **discrete random variable** X in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $X : \Omega \to \mathbb{R}$ which can take only countably many values and such that the subsets $\{X = x\}$ are events.

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- For f_X the binomial distribution with parameters n and p, $\mathbb{E}(X) = np$.

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Suppose that X is a discrete random variable with probability mass function f_X and let $h : \mathbb{R} \to \mathbb{R}$ be a function; e.g., $h(x) = x^2$.

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Proof.

By definition $f_Y(y)$ is the probability of the event $\{\omega \in \Omega | Y(\omega) = y\} = \{\omega \in \Omega | h(X(\omega)) = y\}$, but this is the disjoint union of $\{\omega \in \Omega | X(\omega) = x\}$ for all x such that $h(x) = y$. $\square$

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Proof.

By definition and the previous lemma,

$$\begin{aligned}\mathbb{E}(Y) = \sum_y y f_Y(y) &= \sum_y y \sum_{\substack{x \\ h(x)=y}} f_X(x) \\ &= \sum_y \sum_{\substack{x \\ h(x)=y}} y f_X(x) = \sum_x h(x) f_X(x)\end{aligned}$$

$\square$

Examples

Let a be a constant.

1. Let $Y = X + a$. Then

$$\mathbb{E}(Y) = \sum_x (x + a) f_X(x) = \sum_x x f_X(x) + \sum_x a f_X(x) = \mathbb{E}(X) + a$$

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Lemma

1. $M_X(0) = 1$
2. $\mathbb{E}(X) = M'_X(0)$, *where* $'$ *denotes derivative with respect to* t.

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whence setting $t = 0$, $M'_X(0) = np$, as we obtained before. (This way seems simpler, though.)

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Variance and standard deviation I

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One way in which these three cases differ is by the "spread" of the probability mass function. This is measured by the *variance*.

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One virtue of $\sigma(X)$ is that it has the same units as X.

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$$\operatorname{Var}(X) = 10^{-3}(10^6 - 10^3)^2 + 999 \times 10^{-3}(0 - 10^3)^2 \simeq 10^9$$

   whence $\sigma(X) \simeq \pounds 31,607$.

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Proof.

$$\begin{aligned}
\mathrm{Var}(X) &= \sum_x (x-\mu)^2 f_X(x) = \sum_x (x^2 - 2\mu x + \mu^2) f_X(x) \\
&= \sum_x x^2 f_X(x) - 2\mu \sum_x x f_X(x) + \mu^2 \sum_x f_X(x) \\
&= \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2 = \mathbb{E}(X^2) - \mu^2
\end{aligned}$$

$\square$

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and

$$\operatorname{Var}(X + \alpha) = \mathbb{E}((X + \alpha - (\mu + \alpha))^2) = \mathbb{E}((X - \mu)^2) = \operatorname{Var}(X)$$

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Proof.

Notice that the second derivative with respect to t of $M_X(t)$ is given by

$$\frac{d^2}{dt^2} \sum_x e^{tx} f_X(x) = \sum_x x^2 e^{tx} f_X(x) \; ,$$

whence $M_X''(0) = \mathbb{E}(X^2)$. The result follows from the expression $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mu^2$ and the fact that $\mu = M_X'(0)$. □

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Evaluating at 0, $M_X''(0) = n(n-1)p^2 + np$ and thus

$$\mathrm{Var}(X) = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

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Let X be a discrete random variable with probability mass function given by a Poisson distribution with mean λ. Its moment generating function is

$$M_X(t) = e^{\lambda(e^t - 1)}$$

Differentiating twice

$$M''_X(t) = e^{\lambda(e^t-1)}\lambda e^t + e^{\lambda(e^t-1)}(\lambda e^t)^2$$

Evaluating at 0, $M''_X(0) = \lambda + \lambda^2$ and thus

$$\mathrm{Var}(X) = \lambda + \lambda^2 - \lambda^2 = \lambda$$

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- We rewrite this as

$$\frac{pn(pn-p)\cdots(pn-px+p)}{x!}(1-\tfrac{np}{n})^{n-x}$$

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which, in the limit $n \to \infty$, and using

$$\lim_{n\to\infty}(1-\frac{k}{n}) = 1 \qquad \text{and} \qquad \lim_{n\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}$$

becomes $\frac{\lambda^x}{x!}e^{-\lambda}$, which is the Poisson distribution.

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(Using the binomial distribution the result would be $\simeq 0.425$.)

Example (Overbooking – continued)

Or in fact, exactly

0.424683631192536528200013549116793673026524259040461049452495072968650914837300206
709158040615150407329585535240015120608219272553117981017641384828705922878440370
321524207546996027284835313308829697975143168227319629816601917560644850756341881
742709406993813613377277271057343766544478075676178340690648658612923475894822832
297859172633112693660439822342275313531378295457268742238146456308290233599014111
615480034300074542370402850563940255882870886364953875049514476615747889802955241
921909126317479754644289655961895552129584437472783180772859838984638908099511670
786738177347568229057659219954622594116676934630413343951161190275195407185240714
940186311498218519219119968253677856140792902214787570204845499188084336275774032
5308776995642818675652301492781568473913485123520777596849334453681459063892599

(808 decimal places)

Poisson distribution and the law of rare events I

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We are interested in the question:

how many events take place in a given time interval?

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Therefore the number X of requests arriving in $[0, t]$ has a binomial distribution with parameters n and $p = \lambda t/n$:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^k \approx e^{-\lambda t} \frac{(t\lambda)^k}{k!}$$

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Poisson processes do not only model temporal distributions, but also spatial and spatio-temporal distributions!

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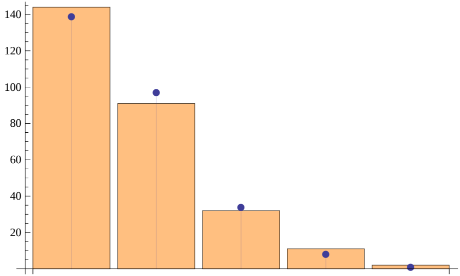

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- Rare events occurring at a constant rate are distributed according to a Poisson distribution.