# Mathematics for Informatics 4a

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**Lecture 8**
**10 February 2012**

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- For binomial $(n, p)$: $\mu = np$ and $\sigma^2 = np(1-p)$
- For Poisson $\lambda$: $\mu = \sigma^2 = \lambda$
- The Poisson distribution with mean $\lambda$ approximates the binomial distribution with parameters $n$ and $p$ in the limit $n \to \infty$, $p \to 0$, but $np \to \lambda$
- "Rare" events occurring at a constant rate are distributed according to a Poisson distribution

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- That is given by their *joint distribution*.

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But also $\sum_{x,y} f(x, y) = 1$, since every outcome $\omega \in \Omega$ belongs to precisely one of the sets $\{X = x\} \cap \{Y = y\}$. In other words, those sets define a partition of $\Omega$, which is moreover countable.

## Examples (Fair dice: scores, max and min)

We roll two fair dice.

1. Let $X$ and $Y$ denote their scores. The joint probability mass function is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{36}, & 1 \leqslant x, y \leqslant 6 \\ 0, & \text{otherwise} \end{cases}$$

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2. Let $U$ and $V$ denote the minimum and maximum of the two scores, respectively. The joint probability mass function is given by

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{36}, & 1 \leqslant u = v \leqslant 6 \\ \frac{1}{18}, & 1 \leqslant u < v \leqslant 6 \\ 0, & \text{otherwise} \end{cases}$$

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and computing $\mathbb{P}$ of both sides:

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A similar story holds for $\{Y = y\}$.

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**Moral**: the marginals do not determine the joint distribution!

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$$\sum_{x_1,\ldots,x_n} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = 1 \ .$$

It has a number of marginals by summing over the possible values of any $k$ of the $X_i$.

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$$\begin{aligned} f_{X,Y}(x,y) &= \mathbb{P}(\{X = x\} \cap \{Y = y\}) \\ &= \mathbb{P}(\{X = x\})\mathbb{P}(\{Y = y\}) \qquad \text{(independent events)} \\ &= f_X(x)f_Y(y) \; . \end{aligned}$$

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**Definition**

Two discrete random variables $X$ and $Y$ are said to be **independent** if for all $x, y$,

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

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We first determine the probability mass functions of $X$ and $Y$.

Conditioning on the value of $N$,

$$f_X(x) = \sum_{n=1}^{\infty} \mathbb{P}(X = x|N = n)\mathbb{P}(N = n) = \sum_{n=x}^{\infty} \binom{n}{x} p^x q^{n-x} e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \frac{(\lambda p)^x}{x!} e^{-\lambda} \sum_{m=0}^{\infty} \frac{q^m}{m!}\lambda^m = \frac{(\lambda p)^x}{x!} e^{-\lambda} e^{\lambda q} = \frac{(\lambda p)^x}{x!} e^{-\lambda p}$$

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$$= \frac{(\lambda p)^x}{x!} e^{-\lambda} \sum_{m=0}^{\infty} \frac{q^m}{m!}\lambda^m = \frac{(\lambda p)^x}{x!} e^{-\lambda} e^{\lambda q} = \frac{(\lambda p)^x}{x!} e^{-\lambda p}$$

So $X$ has a Poisson probability mass function with mean $\lambda p$.

## Example (Bernoulli trials with a random parameter – continued)

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## Example (Bernoulli trials with a random parameter – continued)

One person's success is another person's failure, so $Y$ also has a Poisson probability mass function but with mean $\lambda q$. Therefore

$$f_X(x) f_Y(y) = \frac{(\lambda p)^x}{x!} e^{-\lambda p} \frac{(\lambda q)^y}{y!} e^{-\lambda q} = e^{-\lambda} \frac{\lambda^{x+y}}{x! y!} p^x q^y$$

## Example (Bernoulli trials with a random parameter – continued)

One person's success is another person's failure, so $Y$ also has a Poisson probability mass function but with mean $\lambda q$. Therefore

$$f_X(x)f_Y(y) = \frac{(\lambda p)^x}{x!}e^{-\lambda p}\frac{(\lambda q)^y}{y!}e^{-\lambda q} = e^{-\lambda}\frac{\lambda^{x+y}}{x!y!}p^x q^y$$

On the other hand, conditioning on $N$ again,

$$\begin{aligned} f_{X,Y}(x,y) &= \mathbb{P}(\{X=x\}\cap\{Y=y\}) \\ &= \mathbb{P}(\{X=x\}\cap\{Y=y\}|N=x+y)\mathbb{P}(N=x+y) \\ &= \binom{x+y}{x}p^x q^y e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!} \\ &= e^{-\lambda}\frac{\lambda^{x+y}}{x!y!}p^x q^y = f_X(x)f_Y(y) \end{aligned}$$

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This is the same as saying that for any $2 \leqslant k \leqslant n$ variables $X_{i_1}, \ldots, X_{i_k}$ of the $X_1, \ldots, X_n$,

$$f_{X_{i_1},\ldots,X_{i_k}}(x_{i_1}, \ldots, x_{i_k}) = f_{X_{i_1}}(x_{i_1}) \ldots f_{X_{i_k}}(x_{i_k})$$

for all $x_{i_1}, \ldots, x_{i_k}$.

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The proof is *mutatis mutandis* the same as in the one-variable case.

Let's skip it!

## Proof

The cardinality of the set $Z(\Omega)$ of all possible values of $Z$ is at most that of $X(\Omega) \times Y(\Omega)$, consisting of pairs $(x, y)$ where $x$ is a possible value of $X$ and $y$ is a possible value of $Y$.

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$$f_Z(z) = \sum_{\substack{x,y \\ h(x,y)=z}} f_{X,Y}(x, y) .$$

## Proof – continued

The expectation value is

$$\begin{aligned} f_Z(z) &= \sum_z z f_Z(z) \\ &= \sum_z z \sum_{\substack{x,y \\ h(x,y)=z}} f_{X,Y}(x,y) \\ &= \sum_{x,y} h(x,y) f_{X,Y}(x,y) \end{aligned}$$

□

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The proof is basically the same as the one for two variables and shall be left as an exercise.

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$$\begin{aligned}
\mathbb{E}(X+Y) &= \sum_{x,y} (x+y) f(x,y) \\
&= \sum_x x \sum_y f(x,y) + \sum_y y \sum_x f(x,y) \\
&= \sum_x x f_X(x) + \sum_y y f_Y(y) = \mathbb{E}(X) + \mathbb{E}(Y)
\end{aligned}$$

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# Linearity of expectation II

Together with $\mathbb{E}(\alpha X) = \alpha\mathbb{E}(X)$,... this implies the linearity of the expectation value:

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Let $X_i$, $i = 1, 2$, denote the score of the $i$th die.
We saw earlier that $\mathbb{E}(X_i) = \frac{7}{2}$, hence

$$\mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = \frac{7}{2} + \frac{7}{2} = 7 .$$

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If $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then

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**Important!**

It is important to remember that this is valid for arbitrary discrete random variables **without** the assumption of independence.

## Example (Randomised hats)

A number $n$ of men check their hats at a dinner party. During the dinner the hats get mixed up so that when they leave, the probability of getting their own hat is $1/n$. *What is the expected number of men who get their own hat?*

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- Now let $n = 3$. There are $3! = 6$ possible permutations of the hats: the identity permutation has $X = 3$, three transpositions have $X = 1$ and two cyclic permutations have $X = 0$. Now we get $\frac{1}{6}(3 + 3 \times 1 + 2 \times 0) = 1$... again!

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There has to be an easier way.

## Example (Randomised hats – continued)

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- Notice that $\mathbb{E}(X_i) = \frac{1}{n}$, so that

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On average one (lucky) man gets his own hat!

## Example (The coupon collector problem)

A given brand of cereal contains a small plastic toy in every box. The toys come in $c$ different colours, which are uniformly distributed, so that a given box has a $1/c$ chance of containing any one colour. You are trying to collect all $c$ colours. *How many cereal boxes do you expect to have to buy?*

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## Example (The coupon collector problem – continued)

- $\mathbb{P}(X_i = k) = \left(\frac{i-1}{c}\right)^{k-1} \frac{c-i+1}{c}$ for $k = 1, 2, \ldots$

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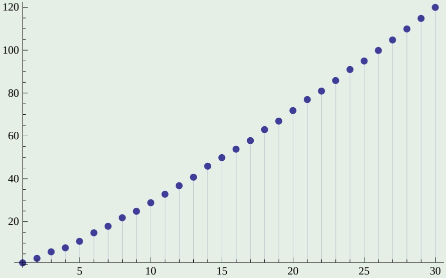
$$\begin{aligned}\mathbb{E}(X) &= \sum_{i=1}^{c} \frac{c}{c-i+1} \\ &= c\left(\frac{1}{c} + \frac{1}{c-1} + \cdots + \frac{1}{2} + 1\right) \\ &= cH_c\end{aligned}$$

where $H_c = 1 + \frac{1}{2} + \cdots + \frac{1}{c}$ is the $c$th **harmonic number**

## Example (The coupon collector problem – continued)

| $c$ | $cH_c$ | $c$ | $cH_c$ |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 3 | 6 | 4 | 8 |
| 5 | 11 | 6 | 15 |
| 7 | 18 | 8 | 22 |
| 9 | 25 | 10 | 29 |



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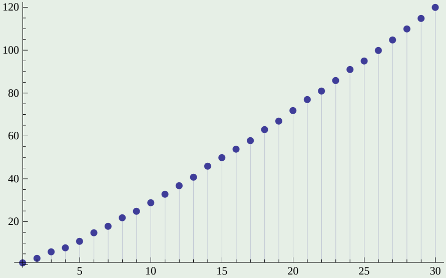


- How many expected tosses of a fair coin until both heads and tails appear?

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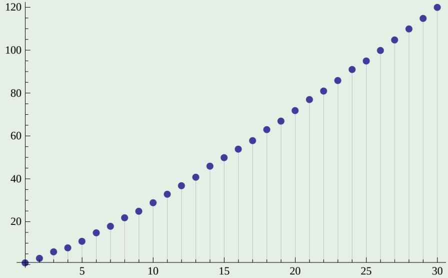


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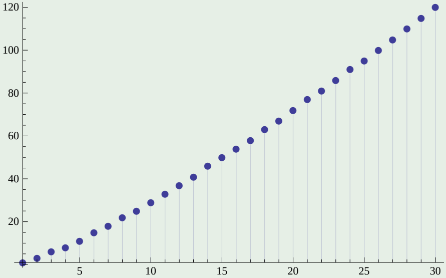


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## Example (The coupon collector problem – continued)

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|---|---|---|---|
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| 3 | 6 | 4 | 8 |
| 5 | 11 | 6 | 15 |
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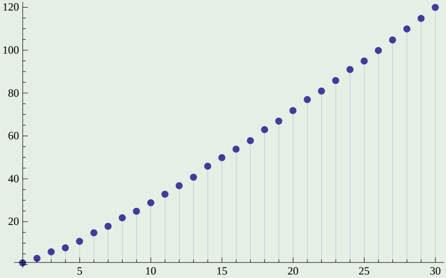


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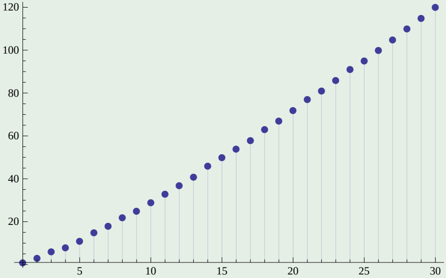


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