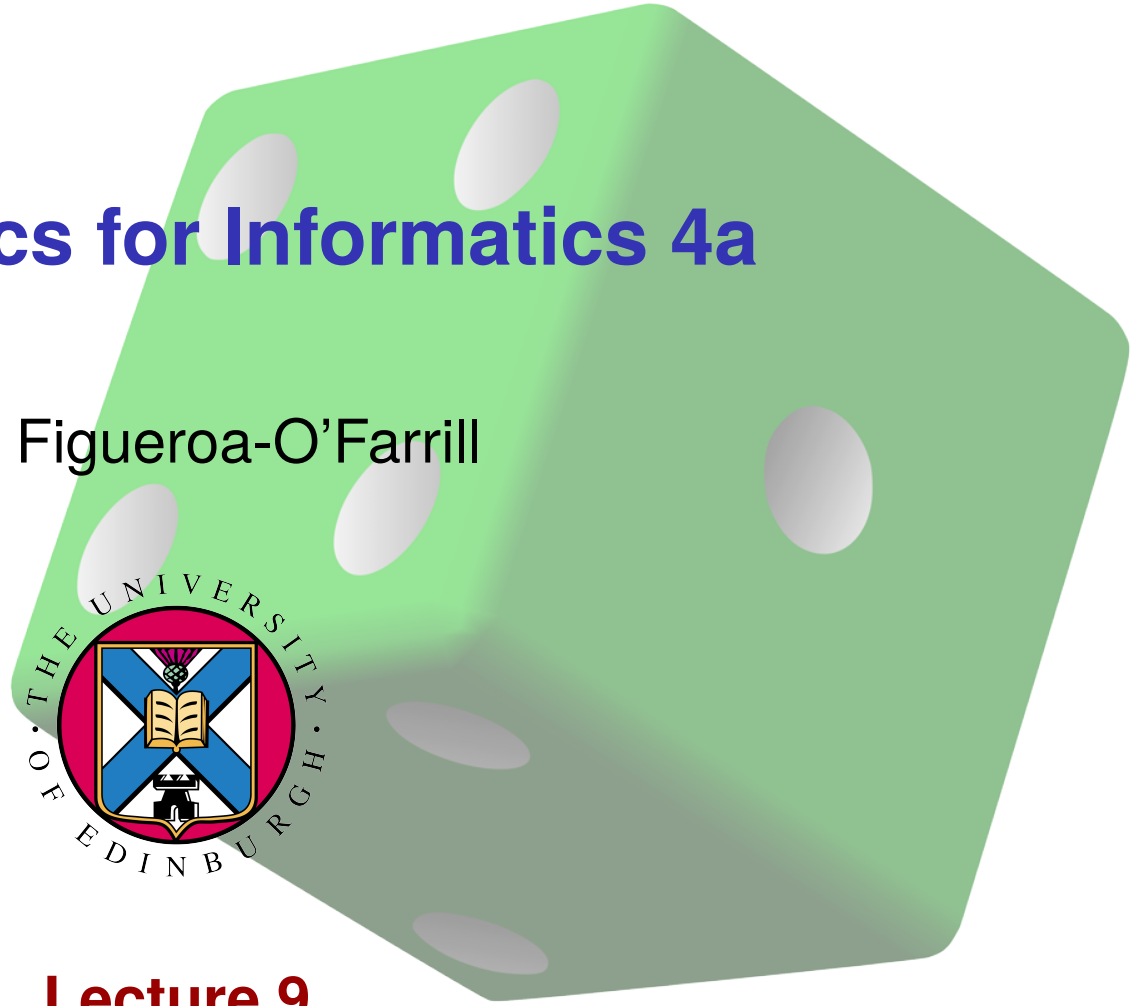
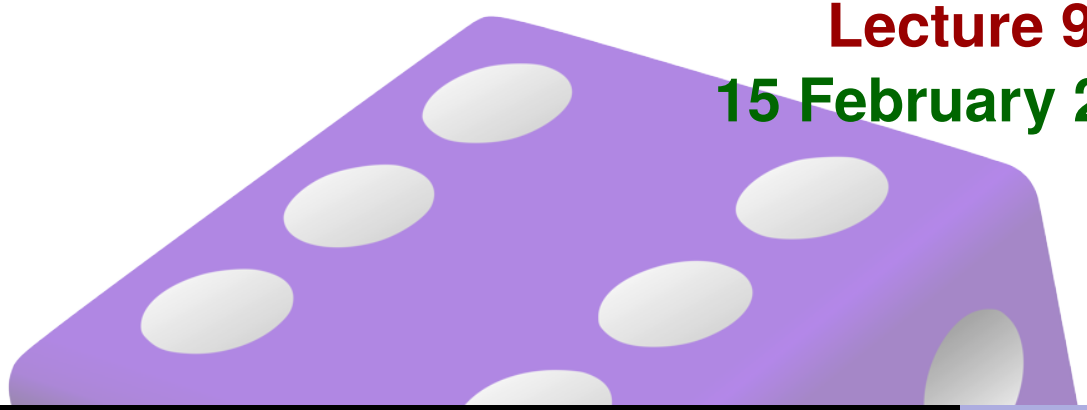
Mathematics for Informatics 4a

José Figueroa-O'Farrill





Lecture 9
15 February 2012

The story of the film so far...

- Discrete random variables $X_1, \ldots, X_n$ on the same probability space have a **joint probability mass function**:

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \mathbb{P}(\{X_1 = x_1\} \cap \cdots \cap \{X_n = x_n\})$$

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- $f_{X_1,\ldots,X_n} : \mathbb{R}^n \to [0, 1]$ and $\sum_{x_1,\ldots,x_n} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = 1$

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- $X_1, \ldots, X_n$ are **independent** if for all $2 \leqslant k \leqslant n$ and $x_{i_1}, \ldots, x_{i_k}$,

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- $h(X_1, \ldots, X_n)$ is a discrete random variable and

$$\mathbb{E}(h(X_1, \ldots, X_n)) = \sum_{x_1,\ldots,x_n} h(x_1, \ldots, x_n) f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

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- Expectation is linear: $\mathbb{E}(\sum_i \alpha_i X_i) = \sum_i \alpha_i \mathbb{E}(X_i)$

Expectation of a product

Lemma

If X *and* Y *are independent,* $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

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If X and Y are independent, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

Proof.

$$\begin{aligned}
\mathbb{E}(XY) &= \sum_{x,y} xy f_{X,Y}(x,y) \\
&= \sum_{x,y} xy f_X(x) f_Y(y) && \text{(independence)} \\
&= \sum_{x} x f_X(x) \sum_{y} y f_Y(y) \\
&= \mathbb{E}(X)\mathbb{E}(Y)
\end{aligned}$$

□

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   - $\langle aX, Y \rangle = \mathbb{E}(aXY) = a\mathbb{E}(XY) = a \langle X, Y \rangle$
   - $\langle X_1 + X_2, Y \rangle = \mathbb{E}((X_1 + X_2)Y) = \mathbb{E}(X_1 Y) + \mathbb{E}(X_2 Y) = \langle X_1, Y \rangle + \langle X_2, Y \rangle$

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3. it is positive-definite: if $\langle X, X\rangle = 0$, then $\mathbb{E}(X^2) = 0$, whence $\sum_x x^2 f(x) = 0$, whence $xf(x) = 0$ for all x. If $x \neq 0$, then $f(x) = 0$ and thus $f(0) = 1$. In other words, $\mathbb{P}(X = 0) = 1$ and hence $X = 0$ almost surely.

Additivity of variance for independent variables

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How about the variance $\mathrm{Var}(X+Y)$?

$$\begin{aligned}
\mathrm{Var}(X+Y) &= \mathbb{E}((X+Y)^2) - \mathbb{E}(X+Y)^2 \\
&= \mathbb{E}(X^2 + 2XY + Y^2) - (\mathbb{E}(X) + \mathbb{E}(Y))^2 \\
&= \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - \mathbb{E}(X)^2 - 2\mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)^2 \\
&= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\left(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\right)
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Theorem

If X and Y are independent discrete random variables

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Covariance

Definition

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Letting μ_X and μ_Y denote the means of X and Y, respectively,

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Indeed,

$$\begin{aligned}\mathbb{E}((X - \mu_X)(Y - \mu_Y)) &= \mathbb{E}(XY) - \mathbb{E}(\mu_X Y) - \mathbb{E}(\mu_Y X) + \mathbb{E}(\mu_X \mu_Y) \\ &= \mathbb{E}(XY) - \mu_X \mu_Y\end{aligned}$$

Example (Max and min for two fair dice)

We roll two fair dice. Let X and Y denote their scores. Independence says that $\mathrm{Cov}(X, Y) = 0$.

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U	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
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$$\mathbb{E}(U) = \frac{91}{36},\ \mathbb{E}(U^2) = \frac{301}{36},\ \mathbb{E}(V) = \frac{161}{36},\ \mathbb{E}(V^2) = \frac{791}{36},\ \mathbb{E}(UV) = \frac{49}{4}$$

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$$\Longrightarrow \mathrm{Var}(U) = \mathrm{Var}(V) = \tfrac{2555}{1296} \qquad \text{and} \qquad \mathrm{Cov}(U, V) = \left(\tfrac{35}{36}\right)^2$$

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and similarly $\mathbb{E}(X) = 0$, whence $\mathbb{E}(X)\mathbb{E}(Y) = 0$.

An alternative criterion for independence

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Two discrete random variables X and Y are independent if and only if

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The proof is not hard, but we will skip it.

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whence

$$\mathrm{Cov}(X, Y)^2 \leqslant \mathrm{Var}(X)\,\mathrm{Var}(Y)$$

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Hence the correlation is a number between -1 and 1:

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- whereas a correlation of -1 suggests a linear relation with negative slope.

Example (Max and min for two fair dice – continued)

Continuing with the ▸ previous example, we now simply compute

$$\rho(U,V) = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}} = \frac{35^2}{36^2} \Big/ \frac{2555}{36^2} = \tfrac{35}{73}.$$

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The funny normalisation of $\rho(X,Y)$ is justified by the following:

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which follows from

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Example (Max and min for two fair dice – continued)

Continuing with the ▸ previous example, we now simply compute

$$\rho(U,V) = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}} = \frac{35^2}{36^2} \Big/ \frac{2555}{36^2} = \tfrac{35}{73}.$$

Remark

The funny normalisation of $\rho(X,Y)$ is justified by the following:

$$\rho(\alpha X + \beta, \gamma Y + \delta) = \mathrm{sign}(\alpha\gamma)\rho(X,Y)$$

which follows from

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and $\sigma_{\alpha X+\beta} = |\alpha|\sigma_X$ and $\sigma_{\gamma Y+\delta} = |\gamma|\sigma_Y$.

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Proof.

$$\begin{aligned}\mathbb{E}(X) &= \sum_{x \geqslant 0} x\mathbb{P}(X = x) = \sum_{0 \leqslant x < a} x\mathbb{P}(X = x) + \sum_{x \geqslant a} x\mathbb{P}(X = x) \\ &\geqslant \sum_{x \geqslant a} x\mathbb{P}(X = x) \geqslant \sum_{x \geqslant a} a\mathbb{P}(X = x) = a\mathbb{P}(X \geqslant a)\end{aligned}$$

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Example

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It has one interesting corollary, though.

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Let X be a discrete random variable with mean μ and variance σ^2. Then for any $\varepsilon > 0$,

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Notice that for $\varepsilon > 0$, $|X - \mu| \geqslant \varepsilon$ if and only if $(X - \mu)^2 \geqslant \varepsilon^2$, so

$$\begin{aligned}\mathbb{P}(|X - \mu| \geqslant \varepsilon) &= \mathbb{P}((X - \mu)^2 \geqslant \varepsilon^2) \\ &\leqslant \frac{\mathbb{E}((X - \mu)^2)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2} \qquad \text{(by Markov's)}\end{aligned}$$

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Back to the factory in the previous example, let the average be $n = 500$ and the variance in a week's production is 100, then what can be said about the probability that this week's production falls between 400 and 600?

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whence

$$\begin{aligned}\mathbb{P}(|X - 500| < 100) &= 1 - \mathbb{P}(|X - 500| \geqslant 100) \\ &\geqslant 1 - \tfrac{1}{100} = \tfrac{99}{100}\ .\end{aligned}$$

So pretty likely!

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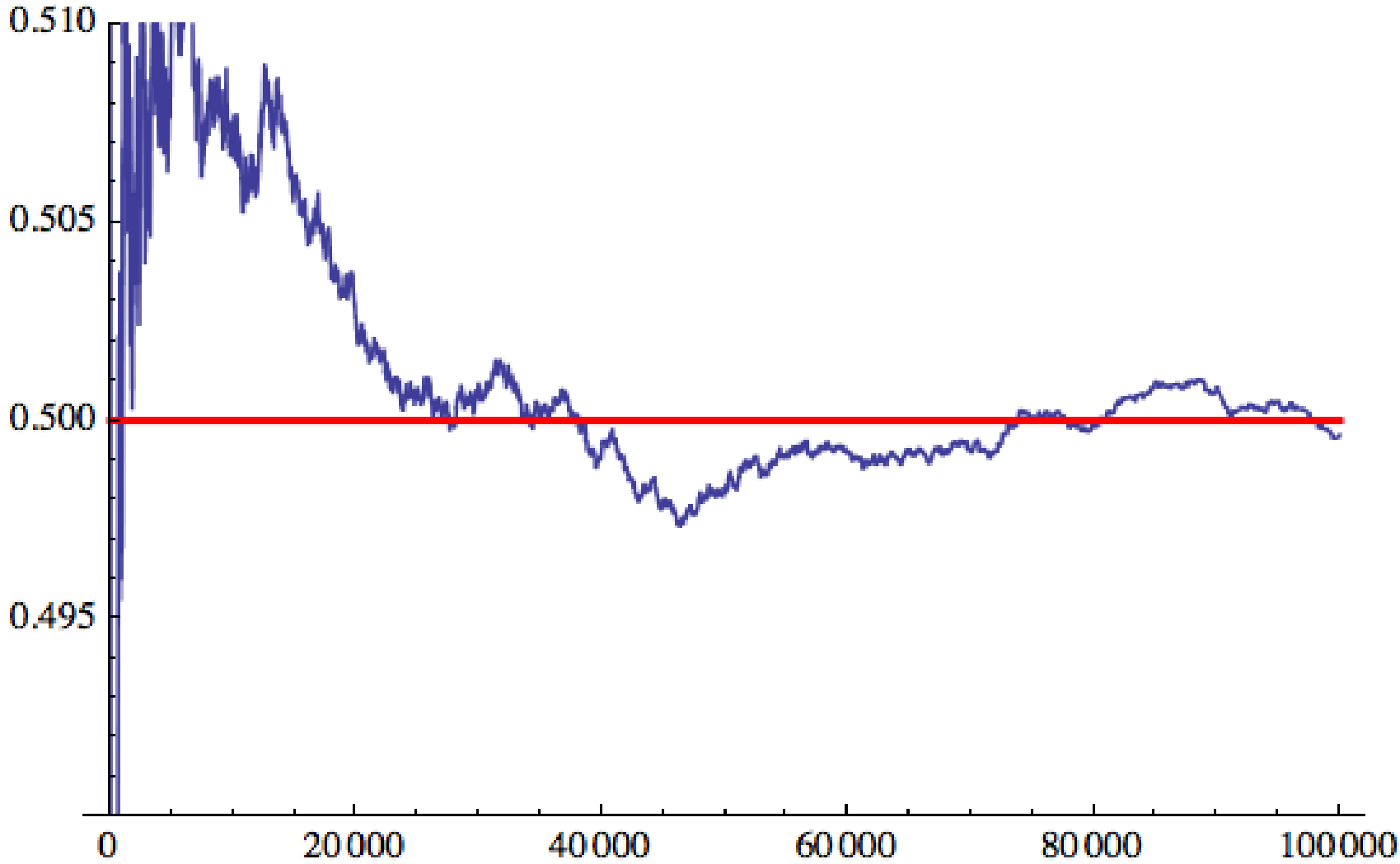
100,000 tosses of a fair (?) coin



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Theorem (The (weak) law of large numbers)

Let $X_1, X_2, \ldots$ *be i.i.d. discrete random variables with mean* μ *and variance* σ^2 *and let* $Z_n = \frac{1}{n}(X_1 + \cdots + X_n)$.

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$$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

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and its variance by

$$\begin{aligned} \sigma^2 = \mathrm{Var}(I_A) &= (0-\mu)^2 f(0) + (1-\mu)^2 f(1) \\ &= \mathbb{P}(A)^2(1-\mathbb{P}(A)) + (1-\mathbb{P}(A))^2\mathbb{P}(A) \\ &= \mathbb{P}(A)(1-\mathbb{P}(A)) \end{aligned}$$

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- This makes precise our initial hand-waving argument of $N(A)/n$ "converging in some way" to $\mathbb{P}(A)$.

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Proof of the Cauchy–Schwarz inequality

The Cauchy–Schwarz inequality says that if $\mathbf{x}, \mathbf{y}$ are any two vectors in a positive-definite inner product space, then

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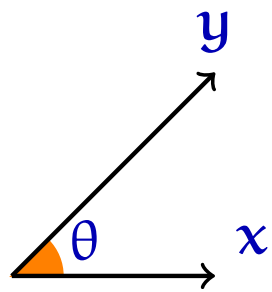
$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta \,,$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Since $|\cos\theta| \leqslant 1$, the inequality follows.



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