Mathematics for Informatics 4a

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Lecture 10
17 February 2012

Public Service Announcement

Next week it is **Innovative Learning Week**.

There are no lectures or tutorials for **mi4**.

The next tutorial will be **Tuesday 28 February**.

Continuous random variables I

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- We know that $\mathbb{P}([0, 1]) = 1$, but this is *not* a countable disjoint union.

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- This argument also shows that any countable subset of $\mathbb{R}$ has zero probability: rationals, algebraic numbers,...

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$$\{X = x\} = \{X \geqslant x\} \cap \{X \leqslant x\}.$$

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The **distribution function** F of a continuous random variable X is the function $F(x) = \mathbb{P}(X \leqslant x)$. In the above example, $F(x) = x$.

Probability density functions

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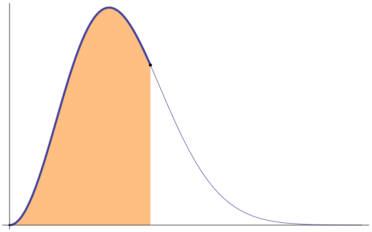

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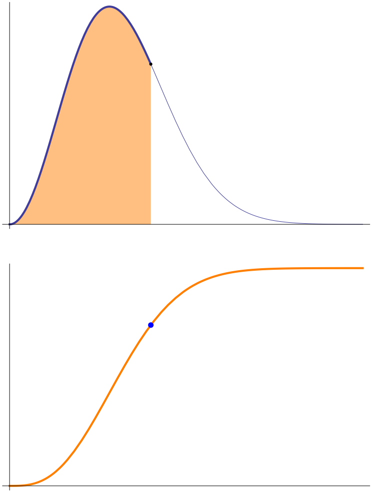

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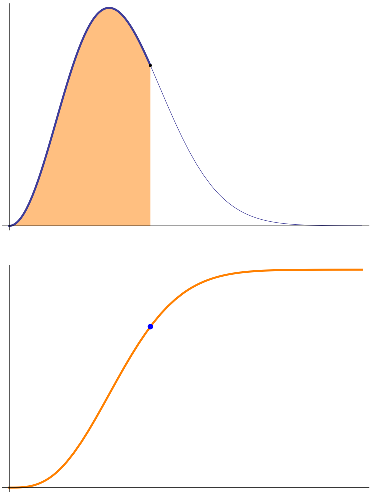
The function f is called a "probability density function" (p.d.f.) and the function F is called a "cumulative distribution function" (c.d.f.).



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Discrete random variables have probability **mass** functions, but **continuous** random variables have probability **density** functions.

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This prompts the following

Question

For which subsets $A \in \mathbb{R}$ is $\{X \in A\}$ an event?

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- and similarly (x, y), $[x, y)$, $(x, y]$,...
- The Borel sets are the smallest σ-field containing the intervals.
- In fact, all subsets of $\mathbb{R}$ you will ever be likely to meet are Borel.

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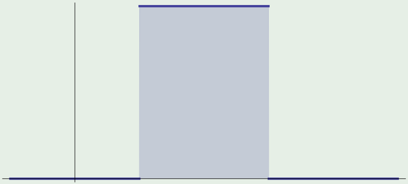

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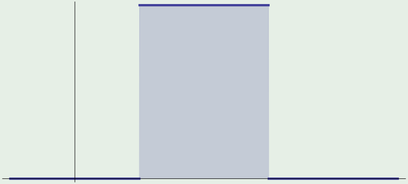

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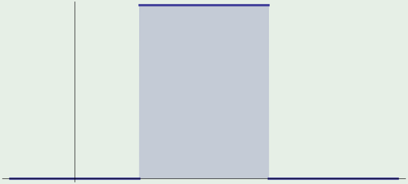

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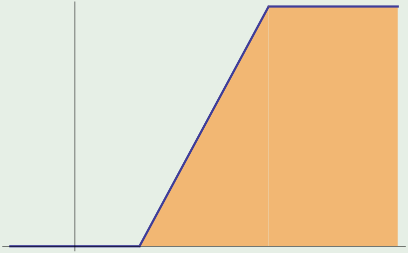

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- You will have to wait at least 5 minutes if you arrive between the time a bus arrives and 10 minutes after that.
- That's 10 minutes in every 15 minutes, so the probability is $\frac{10}{15} = \frac{2}{3}$.

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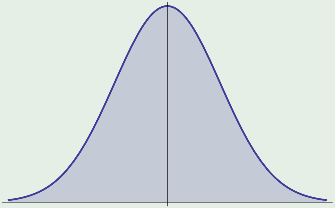

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so that

$$x^2 + y^2 = r^2 \qquad dx\,dy = r\,dr\,d\theta$$

Example (The standard normal distribution – continued)

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r > 0 \qquad 0 \leqslant \theta < 2\pi$$

so that

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Into I^2,

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\iint e^{-(x^2+y^2)/2} dx\,dy \\ &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2} r\,dr\,d\theta \\ &= \int_0^\infty e^{-r^2/2} d(\tfrac{1}{2}r^2) \\ &= \int_0^\infty e^{-u} du = 1 \end{aligned}$$

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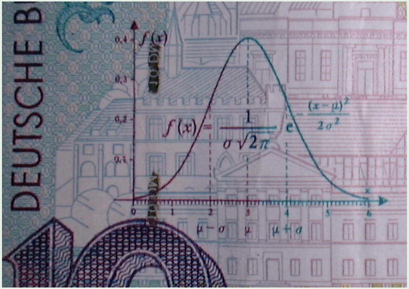

The standard normal distribution has $\mu = 0$ and $\sigma = 1$. We will see that μ and σ^2 are the mean and variance, respectively.

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where erf is the **error function**, defined by

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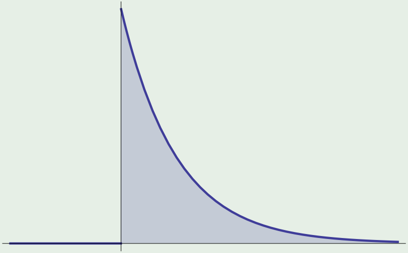

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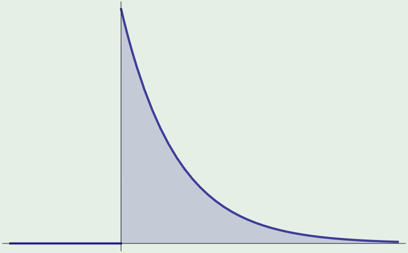

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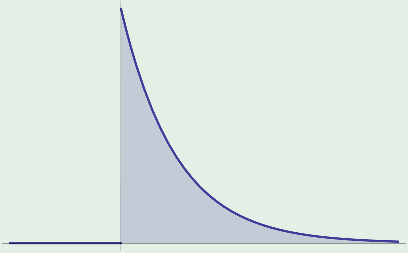

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whence cancelling the $\mathbb{P}(X > x)$ from both sides,

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- i.e., the exponential distribution simply does not remember that fact.

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\mathbb{E}(X) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (y+\mu) e^{-y^2/2\sigma^2} dy \\
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where the first term vanishes because of symmetric integration and the second equals μ by using the normalisation of the normal probability density function.

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