Mathematics for Informatics 4a

José Figueroa-O'Farrill

Lecture 12

2 March 2012

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **variance**: $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **variance**: $\mathsf{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- **moment generating function**: $M_X(t) = \mathbb{E}(e^{tX})$

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **variance**: $\mathsf{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- **moment generating function**: $M_X(t) = \mathbb{E}(e^{tX})$
- have met uniform, exponential and normal distributions and have computed their mean, variance and m.g.f.

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **variance**: $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- **moment generating function**: $M_X(t) = \mathbb{E}(e^{tX})$
- have met uniform, exponential and normal distributions and have computed their mean, variance and m.g.f.
- if X normally distributed with mean μ and variance σ^2, $Y = \frac{1}{\sigma}(X - \mu)$ has standard normal distribution

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **variance**: $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- **moment generating function**: $M_X(t) = \mathbb{E}(e^{tX})$
- have met uniform, exponential and normal distributions and have computed their mean, variance and m.g.f.
- if X normally distributed with mean μ and variance σ^2, $Y = \frac{1}{\sigma}(X - \mu)$ has standard normal distribution
- The c.d.f. Φ of the standard normal distribution is not an elementary function, but there are tables

The story of the film so far...

- X a c.r.v. with p.d.f. f and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

- **variance**: $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- **moment generating function**: $M_X(t) = \mathbb{E}(e^{tX})$
- have met uniform, exponential and normal distributions and have computed their mean, variance and m.g.f.
- if X normally distributed with mean μ and variance σ^2, $Y = \frac{1}{\sigma}(X - \mu)$ has standard normal distribution
- The c.d.f. Φ of the standard normal distribution is not an elementary function, but there are tables
- **maximum entropy**: normal distribution is the "least biased" among all p.d.f.s with the same mean and variance

Jointly distributed continuous random variables

Definition

Two continuous random variables X and Y are said to be jointly distributed with joint density $f(x, y)$ if for all $a < b$ and $c < d$,

$$\mathbb{P}(a < X < b, c < Y < d) = \int_c^d \int_a^b f(x, y) dx dy$$

Jointly distributed continuous random variables

Definition

Two continuous random variables X and Y are said to be jointly distributed with joint density $f(x, y)$ if for all $a < b$ and $c < d$,

$$\mathbb{P}(a < X < b, c < Y < d) = \int_c^d \int_a^b f(x, y) dx dy$$

It follows that the joint density obeys $f(x, y) \geqslant 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Jointly distributed continuous random variables

Definition

Two continuous random variables X and Y are said to be jointly distributed with joint density $f(x, y)$ if for all $a < b$ and $c < d$,

$$\mathbb{P}(a < X < b, c < Y < d) = \int_c^d \int_a^b f(x, y) dx dy$$

It follows that the joint density obeys $f(x, y) \geqslant 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

and

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) dx dy$$

(provided C is "nice" enough)

Example

Let X and Y have joint density

$$f(x, y) = cxy \qquad 0 \leqslant x, y \leqslant 1.$$

What is c?

Example

Let X and Y have joint density

$$f(x,y) = cxy \qquad 0 \leqslant x, y \leqslant 1.$$

What is c?

From the normalisation condition,

$$1 = \int_0^1 \int_0^1 cxy\, dx\, dy = c \left(\tfrac{1}{2} x^2 \Big|_0^1 \right) \left(\tfrac{1}{2} y^2 \Big|_0^1 \right) = \frac{c}{4} \implies c = 4$$

Example

Let X and Y have joint density

$$f(x,y) = cxy \qquad 0 \leqslant x, y \leqslant 1.$$

What is c*?*

From the normalisation condition,

$$1 = \int_0^1 \int_0^1 cxy \, dx \, dy = c \left(\tfrac{1}{2} x^2 \Big|_0^1 \right) \left(\tfrac{1}{2} y^2 \Big|_0^1 \right) = \frac{c}{4} \implies c = 4$$

What if $0 \leqslant x < y \leqslant 1$*?*

Example

Let X and Y have joint density

$$f(x,y) = cxy \qquad 0 \leqslant x, y \leqslant 1.$$

What is c?
From the normalisation condition,

$$1 = \int_0^1 \int_0^1 cxy \, dx \, dy = c \left(\tfrac{1}{2} x^2 \Big|_0^1 \right) \left(\tfrac{1}{2} y^2 \Big|_0^1 \right) = \frac{c}{4} \implies c = 4$$

What if $0 \leqslant x < y \leqslant 1$?
Since the density is symmetric in $x \leftrightarrow y$, the integral over half the square is half of the previous result, hence c is twice the previous value: $c = 8$.

Uniform joint densities

Let $A \subset \mathbb{R}^2$ be a region with area $|A|$.

Uniform joint densities

Let $A \subset \mathbb{R}^2$ be a region with area $|A|$.

Definition

X and Y are (jointly) uniform in A if

$$f(x, y) = \begin{cases} \frac{1}{|A|}, & (x, y) \in A \\ 0, & \text{elsewhere} \end{cases}$$

Uniform joint densities

Let $A \subset \mathbb{R}^2$ be a region with area $|A|$.

Definition

X and Y are (jointly) uniform in A if

$$f(x,y) = \begin{cases} \frac{1}{|A|}, & (x,y) \in A \\ 0, & \text{elsewhere} \end{cases}$$

Example

Let X, Y be jointly uniform in the unit disk
$D = \{(x,y) \mid x^2 + y^2 \leqslant 1\}$.

Uniform joint densities

Let $A \subset \mathbb{R}^2$ be a region with area $|A|$.

Definition

X and Y are (jointly) uniform in A if

$$f(x,y) = \begin{cases} \frac{1}{|A|}, & (x,y) \in A \\ 0, & \text{elsewhere} \end{cases}$$

Example

Let X, Y be jointly uniform in the unit disk $D = \{(x,y) \mid x^2 + y^2 \leqslant 1\}$. Then $|D| = \pi$, whence

$$f(x,y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$

Marginals

Let X, Y be continuous random variables with joint density $f(x, y)$.

Marginals

Let X, Y be continuous random variables with joint density $f(x, y)$. Then the marginal p.d.f.s $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx$$

Marginals

Let X, Y be continuous random variables with joint density $f(x, y)$. Then the marginal p.d.f.s $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Remark

As in the discrete case there is no need to stop at two random variables, and we can have joint densities $f(x_1, \ldots, x_n)$ for n jointly distributed random variables, with many different marginals.

Example

Let X, Y be jointly uniform on the unit disk D

Example

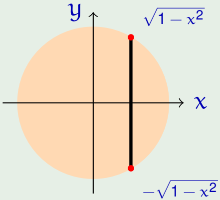
Let X, Y be jointly uniform on the unit disk D:

$$f(x,y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$

Example

Let X, Y be jointly uniform on the unit disk D:

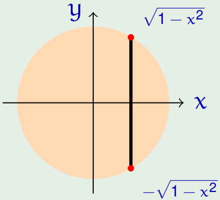
$$f(x, y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$



Example

Let X, Y be jointly uniform on the unit disk D:

$$f(x,y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$



The marginals are given by

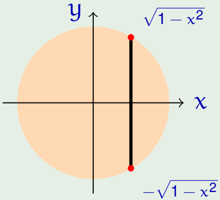
$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}$$

for $-1 \leqslant x \leqslant 1$

Example

Let X, Y be jointly uniform on the unit disk D:

$$f(x,y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$



The marginals are given by

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}$$

for $-1 \leqslant x \leqslant 1$ and, by symmetry,

$$f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$$

for $-1 \leqslant y \leqslant 1$

Joint distributions

Definition

Let X and Y be continuous random variables with joint density $f(x, y)$.

Joint distributions

Definition

Let X and Y be continuous random variables with joint density $f(x,y)$. Their joint distribution is defined as

$$F(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,du\,dv$$

Joint distributions

Definition

Let X and Y be continuous random variables with joint density $f(x,y)$. Their joint distribution is defined as

$$F(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, du\, dv$$

It follows from the fundamental theorem of calculus that

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

Joint distributions

Definition

Let X and Y be continuous random variables with joint density $f(x, y)$. Their joint distribution is defined as

$$F(x, y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) \, du \, dv$$

It follows from the fundamental theorem of calculus that

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

and the marginal distributions are obtained by

$$F_X(x) = F(x, \infty) \qquad \text{and} \qquad F_Y(y) = F(\infty, y)$$

Example

Let X, Y be jointly distributed with $f(x, y) = x + y$ on $0 \leqslant x, y \leqslant 1$.

Example

Let X, Y be jointly distributed with $f(x, y) = x + y$ on $0 \leqslant x, y \leqslant 1$. One checks that indeed $\int_0^1 \int_0^1 (x + y) dx dy = 1$.

Example

Let X, Y be jointly distributed with $f(x, y) = x + y$ on $0 \leqslant x, y \leqslant 1$.
One checks that indeed $\int_0^1 \int_0^1 (x+y)\,dx\,dy = 1$.
The joint distribution is

$$\begin{aligned}
F(x, y) &= \int_0^x \int_0^y (u + v)\,du\,dv \\
&= \int_0^x \left(\int_0^y (u+v)\,dv \right) du \\
&= \int_0^x \left(uy + \tfrac{1}{2}y^2 \right) du \\
&= \tfrac{1}{2}x^2 y + \tfrac{1}{2}xy^2 \qquad \text{for } 0 \leqslant x, y \leqslant 1
\end{aligned}$$

Example

Let X, Y be jointly distributed with $f(x, y) = x + y$ on $0 \leqslant x, y \leqslant 1$.
One checks that indeed $\int_0^1 \int_0^1 (x + y) dx dy = 1$.
The joint distribution is

$$
\begin{aligned}
F(x, y) &= \int_0^x \int_0^y (u + v) du\, dv \\
&= \int_0^x \left(\int_0^y (u + v) dv \right) du \\
&= \int_0^x \left(uy + \tfrac{1}{2} y^2 \right) du \\
&= \tfrac{1}{2} x^2 y + \tfrac{1}{2} x y^2 \qquad \text{for } 0 \leqslant x, y \leqslant 1
\end{aligned}
$$

For $y > 1$, $F(x, y) = \frac{1}{2} x(x + 1)$ and similarly, for $x > 1$, $F(x, y) = \frac{1}{2} y(y + 1)$.

Independence

Definition

Two continuous random variables X and Y are independent if

$$F(x, y) = F_X(x)F_Y(y)$$

Independence

Definition

Two continuous random variables X and Y are independent if

$$F(x, y) = F_X(x)F_Y(y)$$

or, equivalently,

$$f(x, y) = f_X(x)f_Y(y) \; .$$

Independence

Definition

Two continuous random variables X and Y are independent if

$$F(x, y) = F_X(x)F_Y(y)$$

or, equivalently,

$$f(x, y) = f_X(x)f_Y(y) \, .$$

It follows that for X, Y independent

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

Independence

Definition

Two continuous random variables X and Y are independent if

$$F(x, y) = F_X(x)F_Y(y)$$

or, equivalently,

$$f(x, y) = f_X(x)f_Y(y) \ .$$

It follows that for X, Y independent

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

Useful criterion: X and Y are independent iff $f(x, y) = g(x)h(y)$. Then $f_X(x) = cg(x)$ and $f_Y(y) = \frac{1}{c}h(y)$, where $c = \int_{\mathbb{R}} h(y)dy$.

Examples

1. X and Y are jointly uniform on $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$:

$$f(x,y) = \frac{1}{ab} \qquad \text{for } (x,y) \in [0,a] \times [0,b]$$

with marginals $f_X(x) = \frac{1}{a}$ and $f_Y(y) = \frac{1}{b}$. Since $f(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

Examples

1. X and Y are jointly uniform on $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$:

$$f(x,y) = \frac{1}{ab} \qquad \text{for } (x,y) \in [0,a] \times [0,b]$$

with marginals $f_X(x) = \frac{1}{a}$ and $f_Y(y) = \frac{1}{b}$. Since $f(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

2. X and Y are jointly uniform on the disk $0 \leqslant x^2 + y^2 \leqslant a^2$:

$$f(x,y) = \frac{1}{\pi a^2} \qquad \text{for } 0 \leqslant x^2 + y^2 \leqslant a^2$$

with marginals $f_X(x) = \frac{1}{\pi a^2}\sqrt{a^2 - x^2}$ and $f_Y(y) = \frac{1}{\pi a^2}\sqrt{a^2 - y^2}$. Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

Geometric probability

- Geometric probability or "continuous combinatorics" studies geometric objects sharing a common probability space.

Geometric probability

- Geometric probability or "continuous combinatorics" studies geometric objects sharing a common probability space.
- We have already seen some geometric probability problems in the tutorial sheets.

Geometric probability

- Geometric probability or “continuous combinatorics” studies geometric objects sharing a common probability space.
- We have already seen some geometric probability problems in the tutorial sheets.
- For example, in Tutorial Sheet 4 you considered the problem of tossing a coin on a square grid and computing the probability that the coin is fully contained inside one of the squares.

Geometric probability

- Geometric probability or "continuous combinatorics" studies geometric objects sharing a common probability space.
- We have already seen some geometric probability problems in the tutorial sheets.
- For example, in Tutorial Sheet 4 you considered the problem of tossing a coin on a square grid and computing the probability that the coin is fully contained inside one of the squares.
- This game was called *franc-carreau* ("free tile") in France and was studied by Buffon in his treatise *Sur le jeu de franc-carreau* (1733).

Geometric probability

- Geometric probability or "continuous combinatorics" studies geometric objects sharing a common probability space.
- We have already seen some geometric probability problems in the tutorial sheets.
- For example, in Tutorial Sheet 4 you considered the problem of tossing a coin on a square grid and computing the probability that the coin is fully contained inside one of the squares.
- This game was called *franc-carreau* ("free tile") in France and was studied by Buffon in his treatise *Sur le jeu de franc-carreau* (1733).
- In probability, Buffon is perhaps better known for *Buffon's needle*, which is a paradigmatic geometric probability problem.

Buffon's needle I

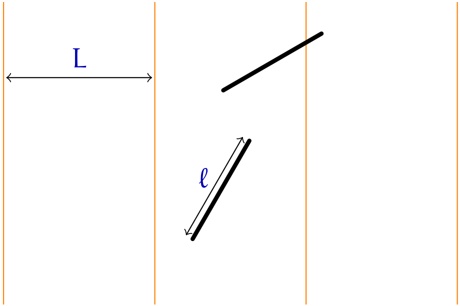
- Drop a needle of length ℓ at random on a striped floor, with stripes a distance L apart.

Buffon's needle I

- Drop a needle of length ℓ at random on a striped floor, with stripes a distance L apart.
- Let $\ell < L$: *short* needles.

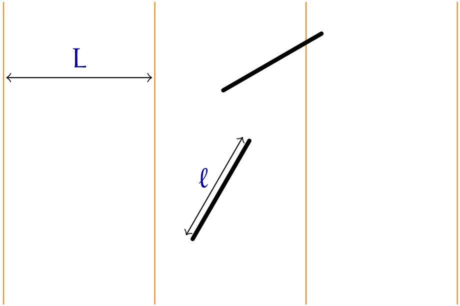
Buffon's needle I

- Drop a needle of length ℓ at random on a striped floor, with stripes a distance L apart.
- Let $\ell < L$: *short* needles.



Buffon's needle I

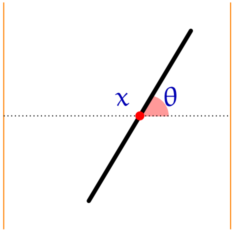
- Drop a needle of length ℓ at random on a striped floor, with stripes a distance L apart.
- Let $\ell < L$: *short* needles.



What is the probability that the needle does not touch any line?

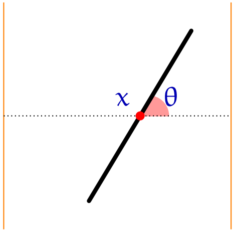
Buffon's needle II

- The needle is described by the midpoint and the angle with the horizontal.



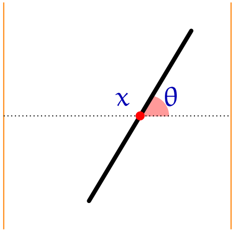
Buffon's needle II

- The needle is described by the midpoint and the angle with the horizontal.
- Symmetry allows us to ignore the vertical component of the midpoint and to assume the horizontal component lies in one of the strips.



Buffon's needle II

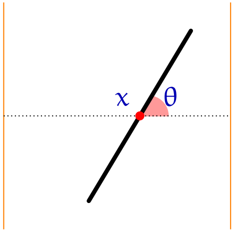
- The needle is described by the midpoint and the angle with the horizontal.
- Symmetry allows us to ignore the vertical component of the midpoint and to assume the horizontal component lies in one of the strips.



- Let X denote the horizontal component of the midpoint. It is uniformly distributed in $[-\frac{L}{2}, \frac{L}{2}]$.

Buffon's needle II

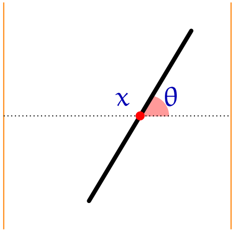
- The needle is described by the midpoint and the angle with the horizontal.
- Symmetry allows us to ignore the vertical component of the midpoint and to assume the horizontal component lies in one of the strips.



- Let X denote the horizontal component of the midpoint. It is uniformly distributed in $[-\frac{L}{2}, \frac{L}{2}]$.
- Let Θ denote the angle with the horizontal, which is uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Buffon's needle II

- The needle is described by the midpoint and the angle with the horizontal.
- Symmetry allows us to ignore the vertical component of the midpoint and to assume the horizontal component lies in one of the strips.



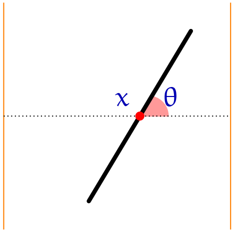
- Let X denote the horizontal component of the midpoint. It is uniformly distributed in $[-\frac{L}{2}, \frac{L}{2}]$.
- Let Θ denote the angle with the horizontal, which is uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- Since X and Θ are independent, the joint probability density function is the product of the two probability density functions and hence is also uniformly distributed.

Buffon's needle III

The needle will touch one of the parallel lines if and only if

$$|x| + \frac{\ell}{2}\cos\theta > \frac{L}{2}$$

for $x \in [-\frac{L}{2}, \frac{L}{2}]$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

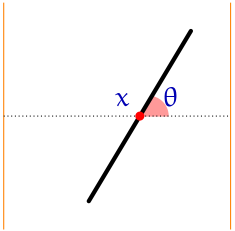


Buffon's needle III

The needle will touch one of the parallel lines if and only if

$$|x| + \frac{\ell}{2}\cos\theta > \frac{L}{2}$$

for $x \in [-\frac{L}{2}, \frac{L}{2}]$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
The complementary probability is



$$\begin{aligned}\mathbb{P}\left(|X| \leqslant \tfrac{1}{2}(L - \ell\cos\Theta)\right) &= \frac{1}{L\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_{-\frac{1}{2}(L-\ell\cos\theta)}^{\frac{1}{2}(L-\ell\cos\theta)} dx\right] d\theta \\ &= \frac{1}{L\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (L - \ell\cos\theta) d\theta \\ &= 1 - \frac{\ell}{L\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = 1 - \frac{2\ell}{L\pi}\end{aligned}$$

Functions of several random variables

- X and Y are continuous random variables with joint density $f(x, y)$

Functions of several random variables

- X and Y are continuous random variables with joint density $f(x, y)$
- $Z = g(X, Y)$, for some function $g : \mathbb{R}^2 \to \mathbb{R}$

Functions of several random variables

- X and Y are continuous random variables with joint density $f(x, y)$
- $Z = g(X, Y)$, for some function $g : \mathbb{R}^2 \to \mathbb{R}$
- How is Z distributed? (assuming it is a c.r.v.)

Functions of several random variables

- X and Y are continuous random variables with joint density $f(x, y)$
- $Z = g(X, Y)$, for some function $g : \mathbb{R}^2 \to \mathbb{R}$
- How is Z distributed? (assuming it is a c.r.v.)
- Its c.d.f. $F_Z(z) = \mathbb{P}(Z \leqslant z)$ is given by

$$F_Z(z) = \iint_{g(x,y) \leqslant z} f(x, y) dx\, dy$$

Functions of several random variables

- X and Y are continuous random variables with joint density $f(x, y)$
- $Z = g(X, Y)$, for some function $g : \mathbb{R}^2 \to \mathbb{R}$
- How is Z distributed? (assuming it is a c.r.v.)
- Its c.d.f. $F_Z(z) = \mathbb{P}(Z \leqslant z)$ is given by

$$F_Z(z) = \iint_{g(x,y) \leqslant z} f(x, y) dx\, dy$$

- Its p.d.f. $f_Z(z) = F'_Z(z)$

Example (The sum of two jointly uniform variables)

- Let X and Y be jointly uniform on $[0, 1]$

Example (The sum of two jointly uniform variables)

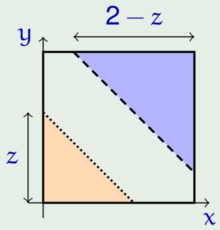
- Let X and Y be jointly uniform on $[0, 1]$
- $f(x, y) = f_X(x) = f_Y(y) = 1$ for $0 \leqslant x, y \leqslant 1$, so X and Y are independent. Let $Z = X + Y$.

Example (The sum of two jointly uniform variables)

- Let X and Y be jointly uniform on $[0, 1]$
- $f(x, y) = f_X(x) = f_Y(y) = 1$ for $0 \leqslant x, y \leqslant 1$, so X and Y are independent. Let $Z = X + Y$.

$$F_Z(z) = \iint_{x+y\leqslant z} dx\, dy$$

$$= \begin{cases} \frac{1}{2}z^2, & z \in [0, 1] \\ 1 - \frac{1}{2}(2 - z)^2, & z \in [1, 2] \end{cases}$$

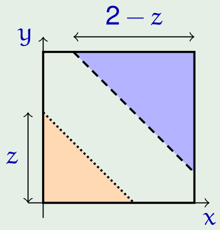


Example (The sum of two jointly uniform variables)

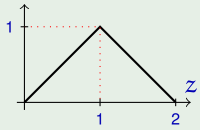
- Let X and Y be jointly uniform on $[0, 1]$
- $f(x, y) = f_X(x) = f_Y(y) = 1$ for $0 \leqslant x, y \leqslant 1$, so X and Y are independent. Let $Z = X + Y$.

$$F_Z(z) = \iint_{x+y \leqslant z} dx\, dy$$

$$= \begin{cases} \frac{1}{2}z^2, & z \in [0, 1] \\ 1 - \frac{1}{2}(2 - z)^2, & z \in [1, 2] \end{cases}$$



$$f_Z(z) = \begin{cases} z, & z \in [0, 1] \\ 2 - z, & z \in [1, 2] \end{cases}$$



The sum of two independent variables: convolution

- Let X and Y be continuous random variables with joint density $f(x, y)$ and let $Z = X + Y$.

The sum of two independent variables: convolution

- Let X and Y be continuous random variables with joint density $f(x, y)$ and let $Z = X + Y$.
- Then $F_Z(z) = \iint_{x+y \leqslant z} f(x, y) dx\, dy$ is given by

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f(x, y) dy \right) dx$$

The sum of two independent variables: convolution

- Let X and Y be continuous random variables with joint density $f(x,y)$ and let $Z = X + Y$.
- Then $F_Z(z) = \iint_{x+y\leqslant z} f(x,y)\,dx\,dy$ is given by

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f(x,y)\,dy \right) dx$$

- Hence $f_Z(z) = F_Z'(z)$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx$$

The sum of two independent variables: convolution

- Let X and Y be continuous random variables with joint density $f(x, y)$ and let $Z = X + Y$.
- Then $F_Z(z) = \iint_{x+y \leqslant z} f(x, y)\,dx\,dy$ is given by

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f(x, y)\,dy \right) dx$$

- Hence $f_Z(z) = F_Z'(z)$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z - x)\,dx$$

- If X and Y be independent, $f(x, y) = f_X(x) f_Y(y)$, whence

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\,dx = (f_X \star f_Y)(z)$$

which defines the convolution product $\star$

Convolution

The convolution product satisfies a number of interesting properties:

- **commutativity**: $f \star g = g \star f$

Convolution

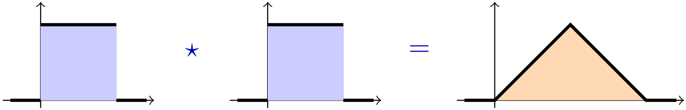
The convolution product satisfies a number of interesting properties:

- **commutativity**: $f \star g = g \star f$
- **associativity**: $(f \star g) \star h = f \star (g \star h)$

Convolution

The convolution product satisfies a number of interesting properties:

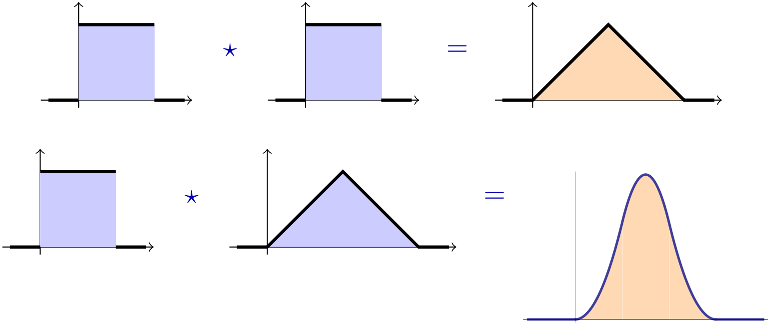
- **commutativity**: $f \star g = g \star f$
- **associativity**: $(f \star g) \star h = f \star (g \star h)$
- **smoothing**: $f \star g$ is a "smoother" function than f or g, e.g.,



Convolution

The convolution product satisfies a number of interesting properties:

- **commutativity**: $f \star g = g \star f$
- **associativity**: $(f \star g) \star h = f \star (g \star h)$
- **smoothing**: $f \star g$ is a "smoother" function than f or g, e.g.,



Summary

- C.r.v.s X and Y have a **joint density** $f(x, y)$ with

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) dx\, dy$$

and a **joint distribution**

$$F(x, y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du\, dv$$

with $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$

Summary

- C.r.v.s X and Y have a **joint density** $f(x, y)$ with

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y)\,dx\,dy$$

  and a **joint distribution**

$$F(x, y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\,du\,dv$$

  with $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
- X and Y **independent** iff $f(x, y) = f_X(x) f_Y(y)$

Summary

- C.r.v.s X and Y have a **joint density** $f(x, y)$ with

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) dx\, dy$$

  and a **joint distribution**

$$F(x, y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du\, dv$$

  with $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
- X and Y **independent** iff $f(x, y) = f_X(x) f_Y(y)$
- Geometric probability is fun! (Buffon's needle)

Summary

- C.r.v.s X and Y have a **joint density** $f(x, y)$ with

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) dx\, dy$$

  and a **joint distribution**

$$F(x, y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du\, dv$$

  with $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
- X and Y **independent** iff $f(x, y) = f_X(x) f_Y(y)$
- Geometric probability is fun! (Buffon's needle)
- We can calculate the c.d.f. and p.d.f. of $Z = g(X, Y)$

Summary

- C.r.v.s X and Y have a **joint density** $f(x, y)$ with

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) dx\, dy$$

  and a **joint distribution**

$$F(x, y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du\, dv$$

  with $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
- X and Y **independent** iff $f(x, y) = f_X(x) f_Y(y)$
- Geometric probability is fun! (Buffon's needle)
- We can calculate the c.d.f. and p.d.f. of $Z = g(X, Y)$
- X, Y independent: $f_{X+Y} = f_X \star f_Y$ (**convolution**)