Mathematics for Informatics 4a

José Figueroa-O'Farrill

Lecture 14
9 March 2012

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- Waiting times of Poisson processes are exponentially distributed

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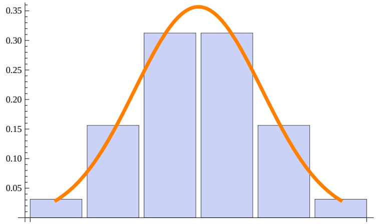
Lectures 6 and 7: this distribution has $\mu = n/2$ and $\sigma^2 = n/4$.



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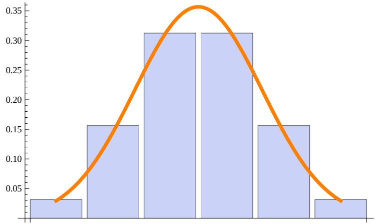

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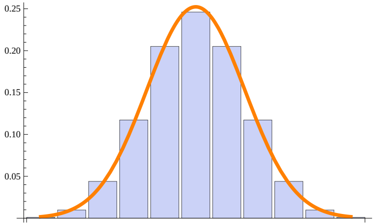

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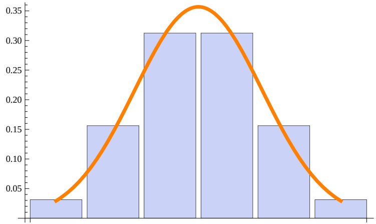

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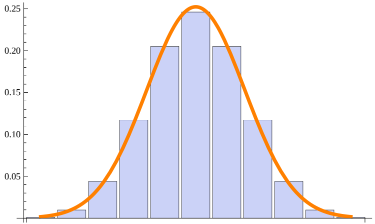

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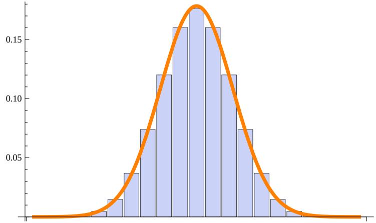

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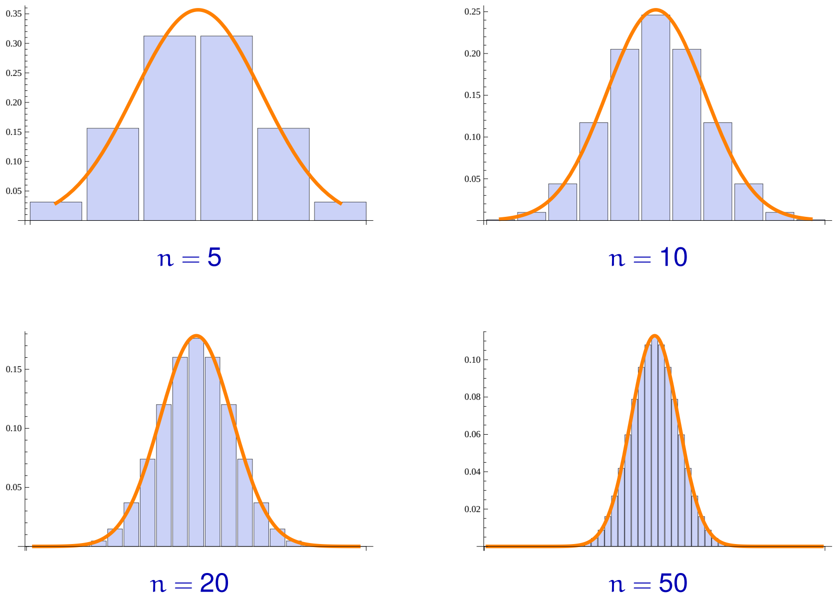
0.35
0.30
0.25
0.20
0.15
0.10
0.05
n = 5
0.25
0.20
0.15
0.10
0.05
n = 10
0.15
0.10
0.05
n = 20
0.10
0.08
0.06
0.04
0.02
n = 50

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which implies that

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!} \simeq 2^n\sqrt{\frac{2}{\pi n}}$$

Proof

Let $k = \frac{n}{2} + x$. Then

$$\binom{n}{k} 2^{-n} = \binom{n}{\frac{n}{2}+x} 2^{-n} = \frac{n! 2^{-n}}{\left(\frac{n}{2}+x\right)! \left(\frac{n}{2}-x\right)!}$$

$$= \frac{n! 2^{-n}}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!} \times \frac{\frac{n}{2}\left(\frac{n}{2}-1\right)\cdots\left(\frac{n}{2}-(x-1)\right)}{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)\cdots\left(\frac{n}{2}+x\right)}$$

$$\simeq \sqrt{\frac{2}{n\pi}} \times \frac{1\left(1-\frac{2}{n}\right)\cdots\left(1-(x-1)\frac{2}{n}\right)\left(\frac{n}{2}\right)^x}{\left(1+\frac{2}{n}\right)\left(1+2\frac{2}{n}\right)\cdots\left(1+x\frac{2}{n}\right)\left(\frac{n}{2}\right)^x}$$

Now we use the exponential approximation

$$1 - z \simeq e^{-z} \qquad \text{and} \qquad \frac{1}{1+z} \simeq e^{-z}$$

(valid for z small) to rewrite the big fraction in the RHS.

Proof – continued.

$$
\begin{aligned}
\binom{n}{k}2^{-n} &\simeq \sqrt{\frac{2}{n\pi}}\exp\left[-\frac{4}{n}-\frac{8}{n}-\cdots-\frac{2(x-1)}{n}-\frac{2x}{n}\right]\\
&= \sqrt{\frac{2}{n\pi}}\exp\left[-\frac{4}{n}(1+2+\cdots+(x-1))-\frac{2x}{n}\right]\\
&= \sqrt{\frac{2}{n\pi}}\exp\left[-\frac{4}{n}\frac{x(x-1)}{2}-\frac{2x}{n}\right]\\
&= \sqrt{\frac{2}{n\pi}}e^{-2x^2/n}
\end{aligned}
$$

which is indeed a normal distribution with $\sigma^2 = \frac{n}{4}$. $\square$

A similar proof shows that the general binomial distribution with $\mu = np$ and $\sigma^2 = np(1-p)$ is also approximated by a normal distribution with the same μ and σ^2.

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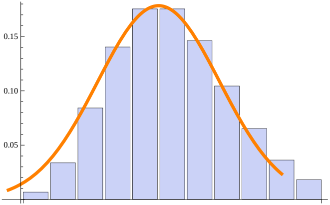
The exact result is

$$\sum_{k=1901}^{2199} \binom{12000}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{12000-k} \simeq 0.992877$$

Normal limit of Poisson distribution



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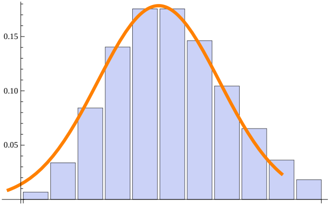

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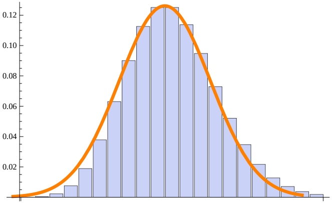

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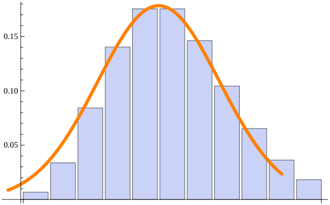

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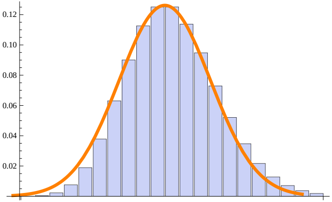

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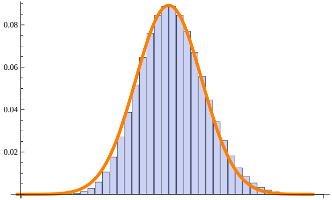

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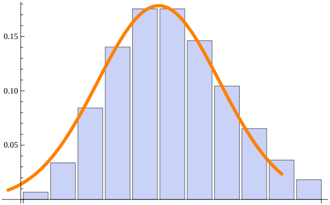

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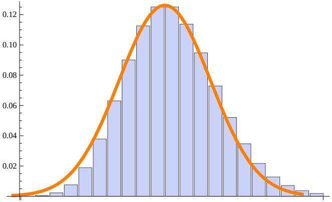

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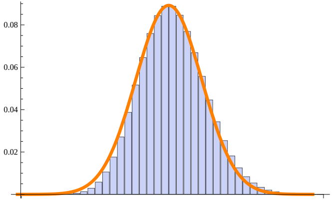

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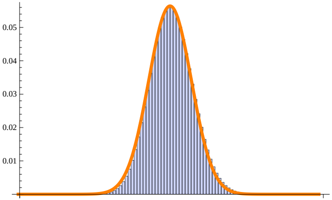

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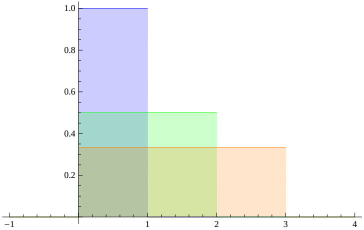

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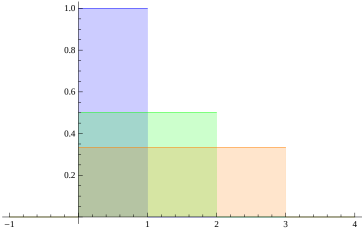

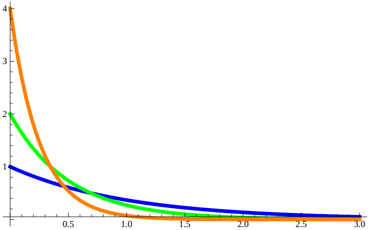

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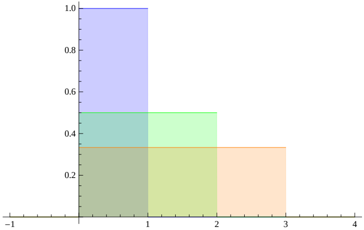

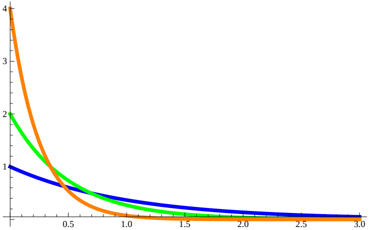

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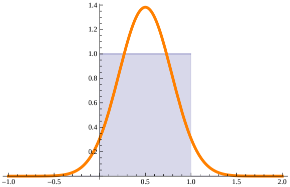
- The situation with the uniform and exponential distributions is different.

Sum of uniformly distributed variables

X_i i.i.d. uniformly distributed on $[0, 1]$: then $X_1 + \cdots + X_n$ is



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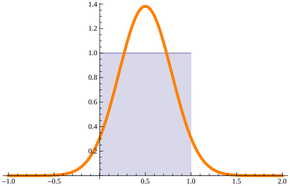

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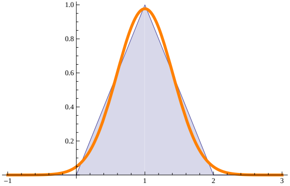

$n = 2$

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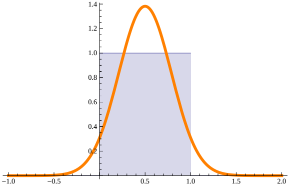

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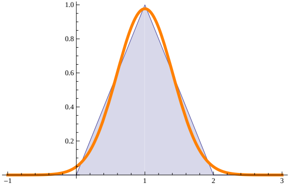
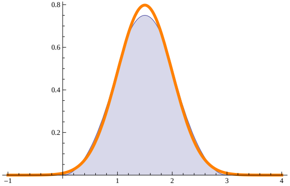


$n = 2$



$n = 3$

Sum of uniformly distributed variables

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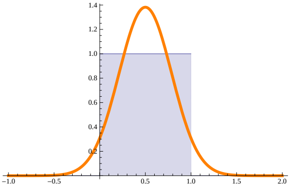

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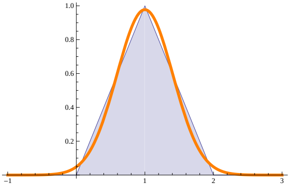

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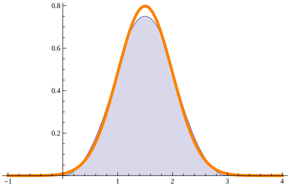

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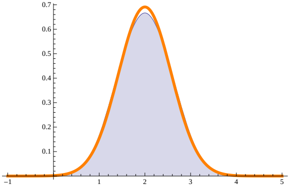

$n = 4$

Sum of exponentially distributed variables

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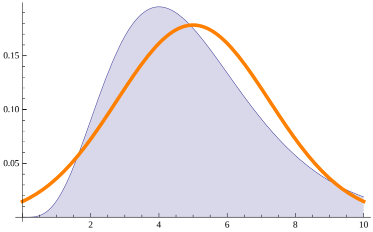
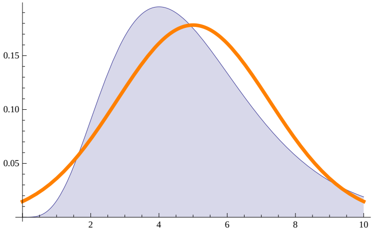
- What happens when we take n large?



$n = 5$



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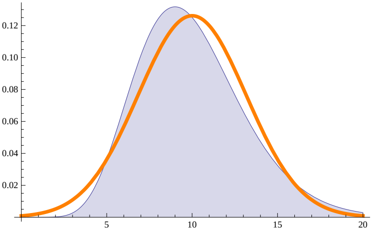
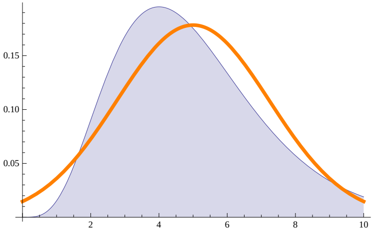
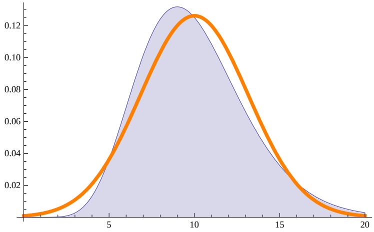


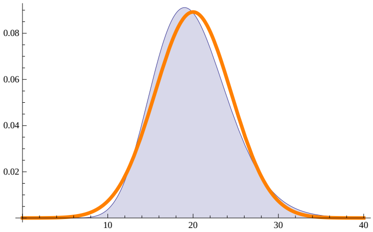
$n = 10$


0.15
0.10
0.05
2
4
6
8
10

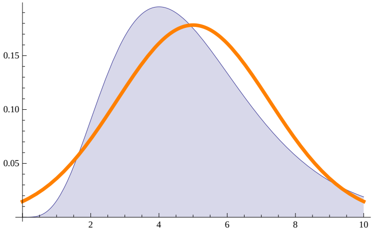
$n = 5$


0.12
0.10
0.08
0.06
0.04
0.02
5
10
15
20

$n = 10$


0.08
0.06
0.04
0.02
10
20
30
40

$n = 20$



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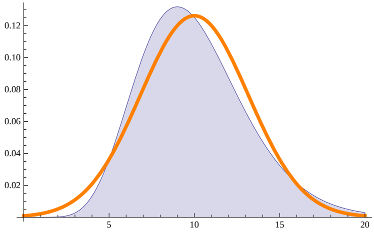

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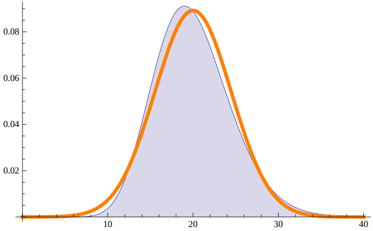
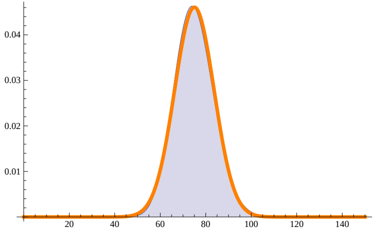


$n = 20$



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The Central Limit Theorem

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In the limit as $n \to \infty$,

$$\mathbb{P}\left(\frac{Z_n - n\mu}{\sqrt{n}\sigma} \leqslant x\right) \to \Phi(x)$$

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In other words, for n large, Z_n is normally distributed.

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- $M_{\frac{Z_n - n\mu}{\sqrt{n}\sigma}}(t) = \left(1 + \frac{\sigma^2 t^2}{2n\sigma^2} + \cdots\right)^n \to e^{t^2/2}$, which is the m.g.f. of a standard normal variable.

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Let us look at a few examples.

Example (Rounding errors)

Suppose that you round off 108 numbers to the nearest integer, and then add them to get the total S. Assume that the rounding errors are independent and uniform on $[-\frac{1}{2}, \frac{1}{2}]$. *What is the probability that S is wrong by more than 3? more than 6?*

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S is wrong by more than 6 iff $\frac{|Z-\mu|}{\sigma} > 2$ and hence

$$\begin{aligned}\mathbb{P}(|Z-\mu| > 2\sigma) &= 1 - \mathbb{P}(|Z-\mu| \leqslant 2\sigma) \\ &= 1 - (2\Phi(2) - 1) = 2(1 - \Phi(2)) \simeq 0.0456\end{aligned}$$

Place your bets!

Example (Roulette)

A roulette wheel has 38 slots: the numbers 1 to 36 (18 black, 18 red) and the numbers 0 and 00 in green.

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You place a $\pounds 1$ bet on whether the ball will land on a red or black slot and win $\pounds 1$ if it does. Otherwise you lose the bet. Therefore you win $\pounds 1$ with probability $\frac{18}{38} = \frac{9}{19}$ and you "win" $-\pounds 1$ with probability $\frac{20}{38} = \frac{10}{19}$.

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After 361 spins of the wheel, what is the probability that you are ahead? (Notice that $361 = 19^2$.)

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$$\sigma^2 = \mathbb{P}(X_i = 1) + \mathbb{P}(X_i = -1) - \mu^2 = 1 - \frac{1}{361} = \frac{360}{361}$$

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Then $Z = X_1 + \cdots + X_{361}$ has mean -19 and variance 360. This means that after 361 spins you are down $\pounds 19$ on average. We are after the probability $\mathbb{P}(Z \geqslant 0)$

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Example (Roulette – continued)

Let X_i denote your winnings on the ith spin of the wheel. Then $\mathbb{P}(X_i = 1) = \frac{9}{19}$ and $\mathbb{P}(X_i = -1) = \frac{10}{19}$. The mean is therefore

$$\mu = \mathbb{P}(X_i = 1) - \mathbb{P}(X_i = -1) = -\frac{1}{19}$$

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So there is about a 16% chance that you are ahead.

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$$\mathbb{P}\left(\left|\frac{Z_n}{n} - d\right| \leqslant 0.5\right) \simeq 0.95$$

Example (Measurements in astronomy – continued)

By the CLT we can assume that $\frac{Z_n - nd}{2\sqrt{n}}$ is standard normal, so we are after

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so choosing $n = 320$ gives $\mathbb{P}\left(\left|\frac{Z_n}{n} - d\right| > 0.5\right) \leqslant 0.05$ or $\mathbb{P}\left(\left|\frac{Z_n}{n} - d\right| \leqslant 0.5\right) \geqslant 0.95$ as desired.

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- We saw some examples on the use of the CLT: rounding errors, roulette game, astronomical measurements.