Mathematics for Informatics 4a

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Lecture 15
14 March 2012

Determinism *vs* randomness

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  - Random walks
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  - Birth-death processes
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- These are the subject of the last part of this course.

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- The interpretation is that X_t is the state of the system at time t, which for a non-deterministic system is a random variable with some probability distribition.
- There are many kinds of stochastic processes, differing in how the probability of X_t being in a given state depends on the history of the system; that is, in which state the system was in times $t' < t$.

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Definition

A stochastic process $X = \{X_0, X_1, X_2, \dots\}$ is a **Markov chain** if it satisfies the **Markov property**:

$$\mathbb{P}(X_{n+1} = s | X_0 = s_0, \dots, X_n = s_n) = \mathbb{P}(X_{n+1} = s | X_n = s_n)$$

for all $n \geqslant 0$ and $s_0, s_1, \dots, s_n, s \in \mathcal{S}$.

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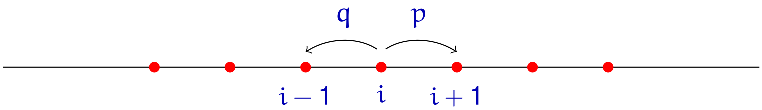
"given the present, the future does not depend on the past"

Random walks

Consider a particle moving on the integer lattice in $\mathbb{R}$:



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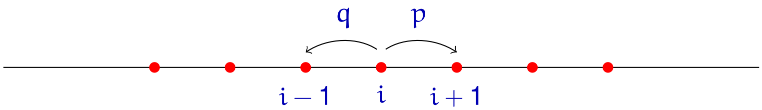

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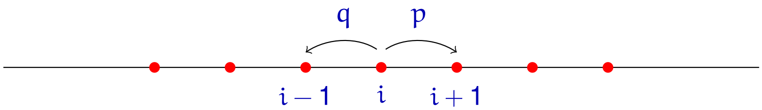

Therefore $\mathcal{S} = \mathbb{Z}$ and J_i are independent random variables with

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Let X_n denote the position of the particle at time n, so that

$$X_n = X_0 + \sum_{i=1}^{n} J_i$$

Proposition

The sequence $\{X_0, X_1, X_2, \dots\}$ exhibits **spatial homogeneity**:

$$\mathbb{P}(X_n = j \mid X_0 = a) = \mathbb{P}(X_n = j + b \mid X_0 = a + b)$$

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Proof.

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$$\mathbb{P}(X_n = j \mid X_0 = a) = \mathbb{P}\left(\sum_{i=1}^{n} J_i = j - a\right)$$

and also

$$\mathbb{P}(X_n = j + b \mid X_0 = a + b) = \mathbb{P}\left(\sum_{i=1}^{n} J_i = j + b - (a + b) = j - a\right)$$

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but the J_i are i.i.d. □

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Proof.

This follows because

$$X_{m+n} = X_m + \sum_{i=m+1}^{n} J_i$$

so X_{m+n} does not depend explicitly on the X_j for $j < m$. □

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$$\mathbb{P}_k(R) = \tfrac{1}{2}\mathbb{P}_{k+1}(R) + \tfrac{1}{2}\mathbb{P}_{k-1}(R)$$

Example (Gambler's ruin – continued)

Letting $p_k = \mathbb{P}_k(R)$, we have the following difference equation:

$$p_k = \frac{1}{2}(p_{k+1} + p_{k-1}) \qquad p_0 = 1 \quad p_N = 0$$

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Therefore $a_k = a_1$ for all k and hence

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Since $p_0 = 1$ and $p_N = 0$, we find $a_1 = -\frac{1}{N}$, whence $p_k = 1 - \frac{k}{N}$.

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$$p_k = c_1\theta_1^k + c_2\theta_2^k$$

for some c_1, c_2 which are determined by $p_0 = 1$ and $p_N = 0$.

Example (Gambler's ruin – continued)

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Therefore the transition probabilities are encoded in a **transition matrix** $\mathbf{P} = (p_{ij})$, where

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1. This is obvious since the p_{ij} are probabilities.
2. $$\sum_j p_{ij} = \sum_j \mathbb{P}(X_{n+1} = j \mid X_n = i) = 1$$

   since X_{n+1} must take *some* value.

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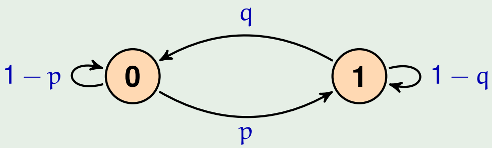
We will answer this naively at first.

Example (Continued)

We often represent Markov chains graphically; e.g.,



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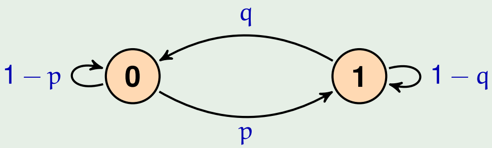

which allows us to read the transition probabilities at a glance and write down the transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 \to 0 & 0 \to 1 \\ 1 \to 0 & 1 \to 1 \end{pmatrix} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

Example (Continued)

$$
\begin{aligned}
\mathbb{P}(X_{n+1} = 0) = \mathbb{P}(X_{n+1} = 0 \mid X_n = 0)\mathbb{P}(X_n = 0) \\
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e.g. $\pi_1(0) = (1-p-q)\pi_0(0) + q$

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\implies \quad \pi_n(0) &= (1-p-q)^n\pi_0(0) + q\sum_{j=0}^{n-1}(1-p-q)^j
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In other words, the probability of finding the machine in any given state on the nth day, depends only on the initial probabilities and the transition probabilities.

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Therefore (see next lecture for a general proof)

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 \underbrace{\mathbf{P} \dots \mathbf{P}}_{n} = \boldsymbol{\pi}_0 \mathbf{P}^n$$

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- Finite-state Markov chains can be represented graphically.