Mathematics for Informatics 4a

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Lecture 18
23 March 2012

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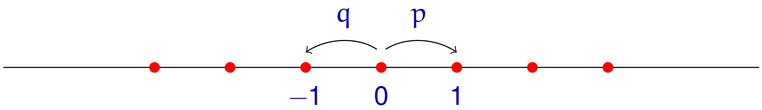
- One example of Markov chains are random walks on graphs: e.g., Google’s **PageRank** is the steady-state distribution of a random walk on the world wide web
- In today’s lecture we will look at random walks on simpler graphs and will focus on a different sort of questions

Random walk revisited

Let us consider again the random walk on the integers:



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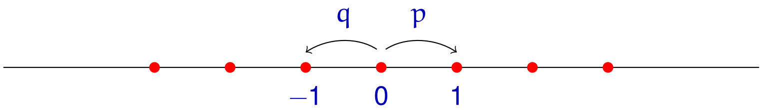

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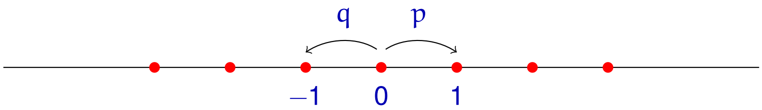

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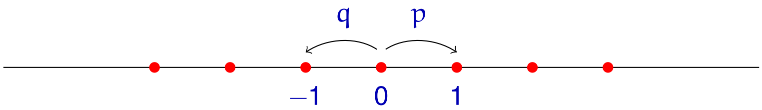

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$$T_r = \begin{cases} \text{number of steps until we visit } r \text{ for the first time,} & r \neq 0 \\ \text{number of steps until we revisit } 0, & r = 0. \end{cases}$$

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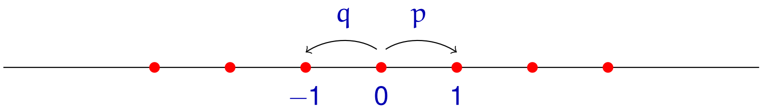

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Question: How the T_r are distributed? i.e., $\mathbb{P}(T_r = n) = ?$

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- $T_r = T_{0,1} + T_{1,2} + \cdots + T_{r-1,r}$ and by independence

$$\mathbb{E}(s^{T_r}) = \mathbb{E}(s^{T_1})^r$$

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$$\mathbb{E}(s^{T_r}) = \mathbb{E}(s^{T_1})^r$$

- Conditioning on the first jump,

$$\begin{aligned}\mathbb{E}(s^{T_1}) &= \mathbb{E}(s^{T_1} \mid J_1 = 1)\mathbb{P}(J_1 = 1) + \mathbb{E}(s^{T_1} \mid J_1 = -1)\mathbb{P}(J_1 = -1) \\ &= sp + s\mathbb{E}(s^{T_{-1,0}+T_{0,1}})q \\ &= sp + sq\mathbb{E}(s^{T_1})^2\end{aligned}$$

which we solve for $\mathbb{E}(s^{T_1})$

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$$\mathbb{E}(s^{T_1}) = \frac{1 - \sqrt{1 - 4pqs^2}}{2sq}$$

- Hence for $r > 0$,

$$\mathbb{E}(s^{T_r}) = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2sq}\right)^r$$

and for $r < 0$ we simply replace $p \leftrightarrow q$

Hitting times for random walks III

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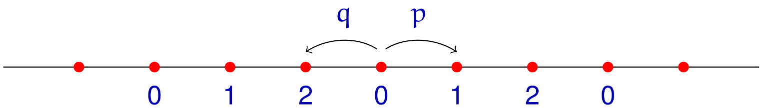
&= s\mathbb{E}(s^{T_{1,0}})p + s\mathbb{E}(s^{T_{-1,0}})q \\
&= sp\mathbb{E}(s^{T_{-1}}) + sq\mathbb{E}(s^{T_1}) \\
&= sp\left(\frac{1-\sqrt{1-4pqs^2}}{2sp}\right) + sq\left(\frac{1-\sqrt{1-4pqs^2}}{2sq}\right) \\
&= 1 - \sqrt{1-4pqs^2}
\end{aligned}$$

$$\therefore \mathbb{E}(T_0) = \frac{4pq}{\sqrt{1-4pq}} \longrightarrow \infty \qquad \text{if } p = q$$

Random walk on a triangle

Let us consider again the random walk on the integers, but this time we only keep track of the position *modulo* 3:



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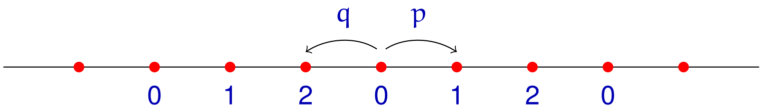
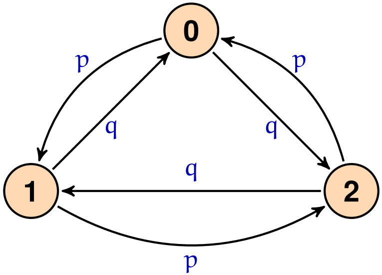


This is equivalent to the following 3-state Markov chain describing a random walk on a triangle:



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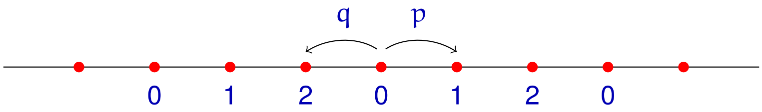

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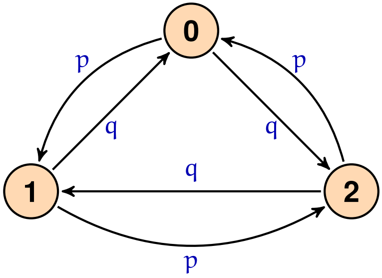

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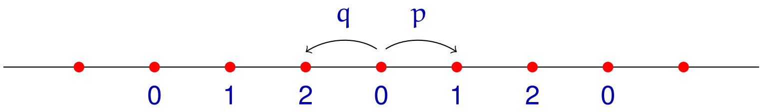

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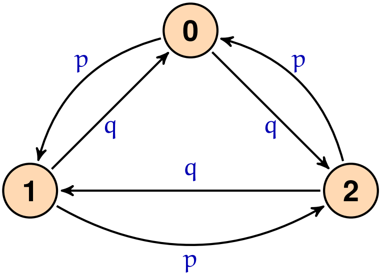

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$$\boldsymbol{\pi} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Mean return times I

A typical quantity of interest is the **mean return time**: say we start at the vertex 0 of the triangle. Let T_0 denote the number of steps until we first revisit 0. *What is* $\mathbb{E}(T_0)$*?*

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Solving the system

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Notice that $\tau_0 = 3 = \frac{1}{\pi_0}$, where (π_0, π_1, π_2) is the steady-state distribution of the Markov chain!

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The steady-state distribution (π_i) of a regular Markov chain is such that $\pi_i = \frac{1}{\tau_i}$, where τ_i is the mean return time to state i.

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(We will not prove it; although it is not hard.)

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In terms of $\gamma_i = \mathbb{E}(s^{T_i})$, we have

$$\gamma_0 = s(p\gamma_1 + q\gamma_2) \qquad \gamma_1 = s(q + p\gamma_2) \qquad \gamma_2 = s(p + q\gamma_1)$$

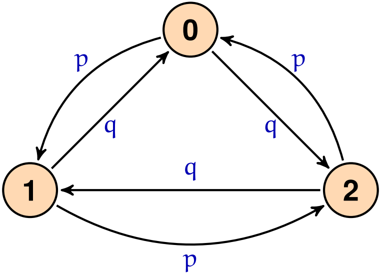
Hitting times distribution II

The unique solution is

$$\mathbb{E}(s^{T_0}) = \frac{s^2(2pq+p^3s+q^3s)}{1-pqs^2} = 2pqs^2 + \left(p^3+q^3\right)s^3 + 2p^2q^2s^4 + \cdots$$

$$\mathbb{E}(s^{T_1}) = \frac{s(q+p^2s)}{1-pqs^2} = qs + p^2s^2 + pq^2s^3 + \cdots$$

$$\mathbb{E}(s^{T_2}) = \frac{s(p+q^2s)}{1-pqs^2} = ps + q^2s^2 + p^2qs^3 + \cdots$$



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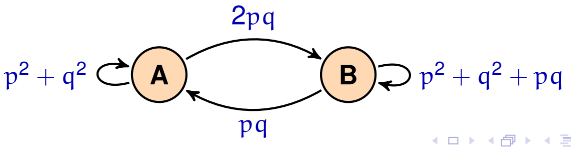

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$$\Longrightarrow \quad \mathbb{E}(U) = \frac{1}{pq} \quad \Longrightarrow \quad \mathbb{E}(T) = 3$$

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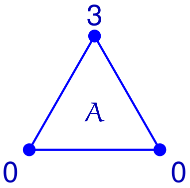

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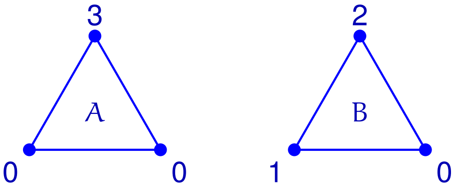

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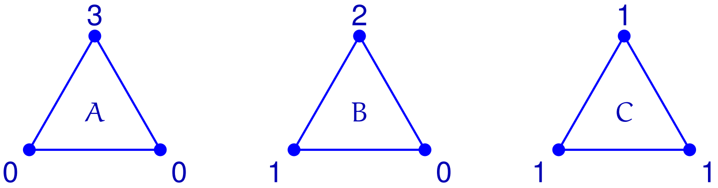

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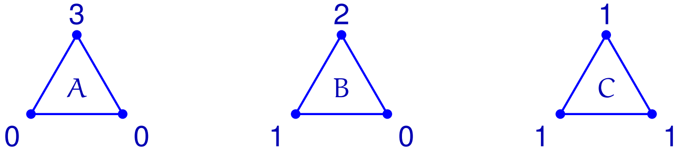

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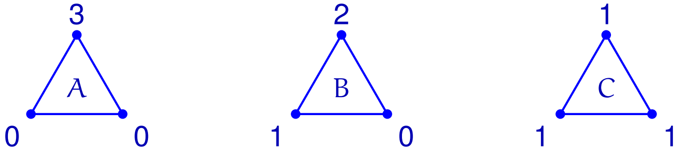

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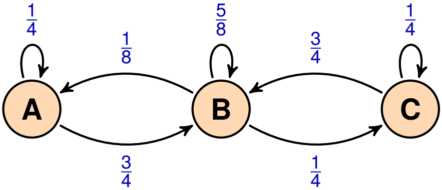

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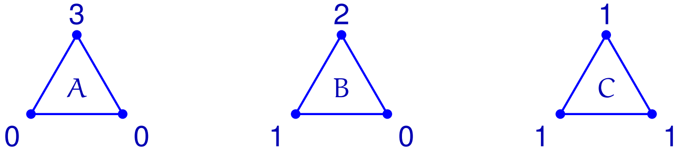

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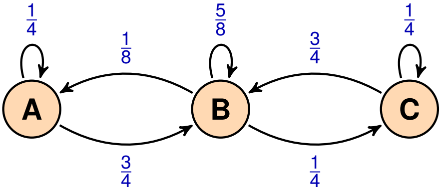

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

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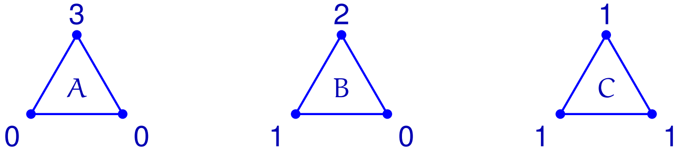

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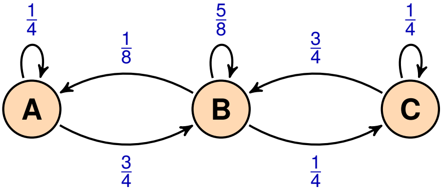

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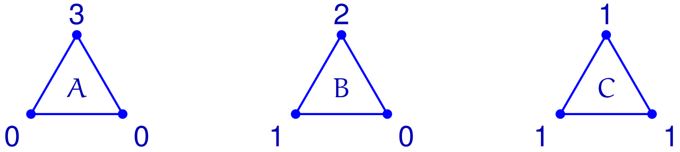

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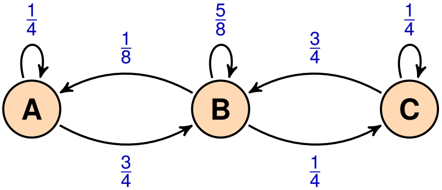

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Therefore $\mathbb{E}(T) = 9$.

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$$\implies \begin{aligned}\mathbb{E}(T) &= 9 \\ \mathbb{E}(U) &= \tfrac{32}{3} \\ \mathbb{E}(V) &= 12\end{aligned}$$

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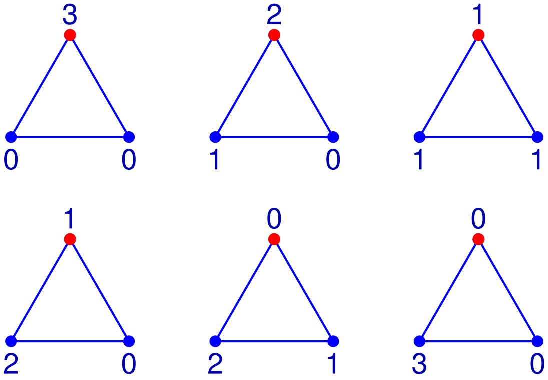
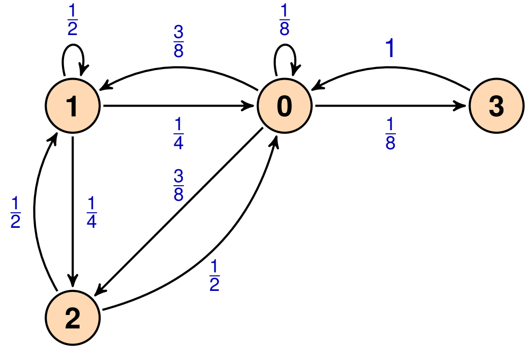


Three independent random walks IV

The corresponding 4-state Markov chain is:



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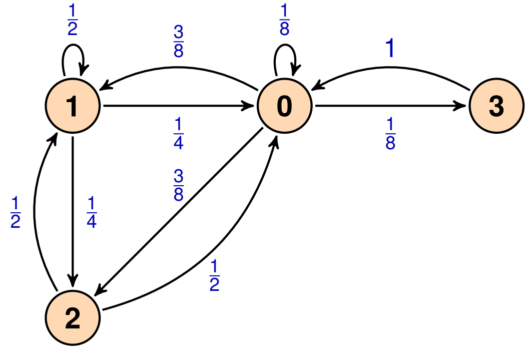

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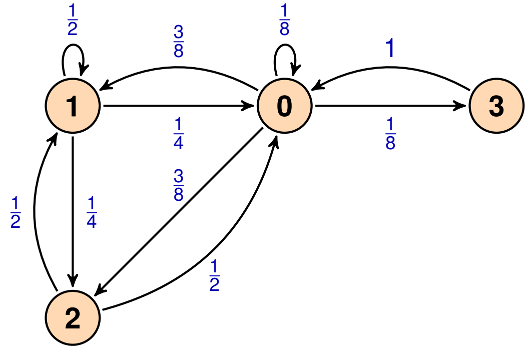

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$$\boldsymbol{\pi} = \left(\frac{8}{27}, \frac{4}{9}, \frac{2}{9}, \frac{1}{27}\right)$$

$$\implies \mathbb{E}(Z) = 27$$

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- Pick your favourite graphs and work out a couple of examples: the last assignment in this course will ask you to do this for K_4