Mathematics for Informatics 4a

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Lecture 20
30 March 2012

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- Poisson process: states $\{0, 1, 2, \ldots\}$, $p_{ij} = 0$ for $j \neq i + 1$ and $p_{i,i+1} = 1$, and all states have equal transition rates

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- The sum $Z = X + Y$ of two independent exponential variables with different rates is **hypoexponential**:

$$f_X(x) = \lambda e^{-\lambda x} \qquad f_Y(y) = \mu e^{-\mu y}$$

$$\implies f_Z(z) = \frac{\lambda\mu}{\mu - \lambda}\left(e^{-\lambda z} - e^{-\mu z}\right)$$

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- By induction, it is enough to show prove it for $n = 2$, so let X, Y be independent exponential variables with rates λ, μ
- With $U = \min(X, Y)$, $\mathbb{P}(U \leqslant u) = 1 - \mathbb{P}(U > u)$, but

$$\begin{aligned}
\mathbb{P}(U > u) = \mathbb{P}(X > u, Y > u) &= \int_u^\infty \int_u^\infty f(x, y)\,dx\,dy \\
&= \int_u^\infty \int_u^\infty \lambda\mu e^{-\lambda x} e^{-\mu y}\,dx\,dy \\
&= \left(\int_u^\infty \lambda e^{-\lambda x}\,dx\right)\left(\int_u^\infty \mu e^{-\mu y}\,dy\right) = e^{-(\lambda+\mu)u}
\end{aligned}$$

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- The final calculation we will need is $\mathbb{P}(X < Y)$ for X, Y exponential with rates λ, μ
- We calculate it by conditioning on X:

$$
\begin{aligned}
\mathbb{P}(X < Y) &= \int_0^\infty \mathbb{P}(X < Y \mid X = x) f_X(x)\, dx \\
&= \int_0^\infty \mathbb{P}(X < Y \mid X = x) \lambda e^{-\lambda x}\, dx \\
&= \int_0^\infty \mathbb{P}(x < Y) \lambda e^{-\lambda x}\, dx \\
&= \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^\infty e^{-(\lambda+\mu)x}\, dx \\
&= \frac{\lambda}{\lambda + \mu}
\end{aligned}
$$

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- Births and deaths are independent and exponentially distributed with rates λ_n and μ_n, respectively, when the population is n
- The parameters $\{\lambda_n \mid n \in \mathbb{N}\}$ and $\{\mu_n \mid n \in \mathbb{N}\}$ are called the **birth rates** and **death rates**, respectively
- A **birth and death process** is a continuous-time Markov process with states $\mathbb{N} = \{0, 1, 2, \dots\}$ for which the allowed transitions are $n \to n+1$ and $n \to n-1$

Birth and death processes (II)

- The transition probabilities are given by $p_{01} = 1$ and

$$p_{n,n+1} = \frac{\lambda_n}{\lambda_n + \mu_n} \qquad p_{n,n-1} = \frac{\mu_n}{\lambda_n + \mu_n} \qquad (n \geqslant 1)$$

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- We argue as follows: $p_{n,n+1}$ is the probability that in a population of n a birth occurs before a death, i.e., $\mathbb{P}(B_n < D_n)$, where B_n and D_n are the exponential variables corresponding to a birth and death, respectively, when the population is n.

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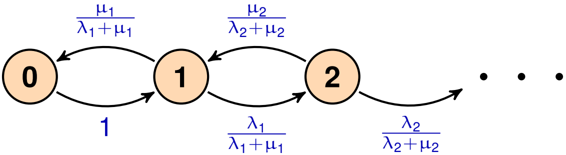
- Since B_n has rate λ_n and D_n has rate μ_n, the results follows from the earlier discussion
- The transition rates are

$$\nu_0 = \lambda_0 \quad \text{and} \quad \nu_n = \lambda_n + \mu_n \quad (n \geqslant 1)$$

since the time to any transition at population n is $\min(B_n, D_n)$, which is exponential with rate $\lambda_n + \mu_n$

Birth and death processes (III)



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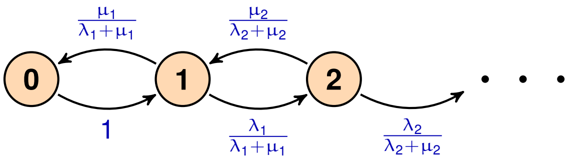

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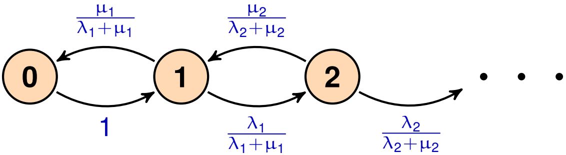

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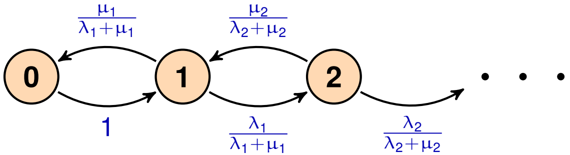

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- Poisson: $\mu_n = 0$ and $\lambda_n = \lambda$ for all $n \geqslant 0$
- **Yule**: $\mu_n = 0$ and $\lambda_n = n\lambda$ for all $n \geqslant 0$, corresponding to a Markov process $\{N(t) \mid t \geqslant 0\}$ where $N(t)$ is the size at time t of a population whose members cannot die, and they give birth to new members independently in an exponentially distributed amount of time with rate λ

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Usually one derives a differential equation that $\mathbb{E}(N(t))$ obeys and solves it to determine $\mathbb{E}(N(t))$.

Steady-state distribution

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- In other words, $\boldsymbol{\pi} = (\pi_n)$, where $\pi_n(t+s) = \pi_n(t)$, so that is constant in time.
- We will not be concerned with the conditions which guarantee the existence and uniqueness of the steady-state distribution.
- We will assume it exists and is unique and we will show how to find it.

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\pi_n(t + \delta t) &= \mathbb{P}(N(t + \delta t) = n \mid N(t) = n)\mathbb{P}(N(t) = n) \\
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- This is the probability that a death occurred in $(t, t+\delta t]$ when the population at time t is $n+1$
- At that population, deaths are exponentially distributed with rate μ_{n+1}, so we want the probability of a death in a time interval of length δt at that rate

- For δt small, this is given by

$$\int_0^{\delta t} \mu_{n+1} e^{-t\mu_{n+1}} dt = 1 - e^{-\mu_{n+1}\delta t} \simeq \mu_{n+1}\delta t$$

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$$\mathbb{P}(N(t+\delta t) = n \mid N(t) = n) \simeq 1 - (\lambda_n + \mu_n)\delta t$$

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- Therefore,

$$\pi_n(t+\delta t) = (1 - \delta t(\lambda_n + \mu_n))\pi_n(t) + \mu_{n+1}\delta t \pi_{n+1}(t) + \lambda_{n-1}\delta t \pi_{n-1}(t)$$

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$$\mathbb{P}(N(t+\delta t) = n \mid N(t) = n) \simeq 1 - (\lambda_n + \mu_n)\delta t$$

- Therefore,

$$\pi_n(t+\delta t) = (1 - \delta t(\lambda_n + \mu_n))\pi_n(t) + \mu_{n+1}\delta t \pi_{n+1}(t) + \lambda_{n-1}\delta t \pi_{n-1}(t)$$

- or, said differently,

$$\frac{\pi_n(t+\delta t) - \pi_n(t)}{\delta t} \simeq \mu_{n+1}\pi_{n+1}(t) + \lambda_{n-1}\pi_{n-1}(t) - (\lambda_n + \mu_n)\pi_n(t)$$

- In the steady state, $\pi_n(t + \delta t) = \pi_n(t)$, whence

$$\mu_{n+1}\pi_{n+1} + \lambda_{n-1}\pi_{n-1} = \lambda_n\pi_n + \mu_n\pi_n \qquad (n \geqslant 1)$$

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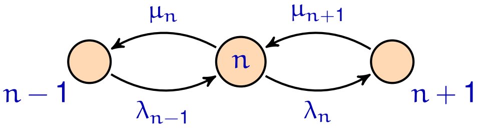

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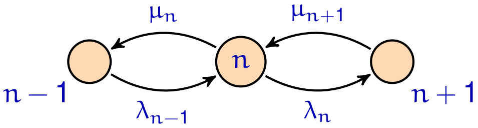

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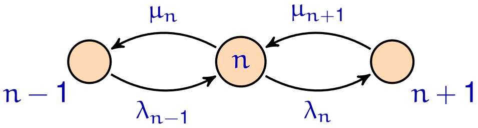

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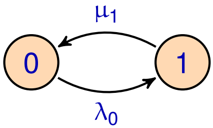

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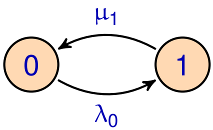

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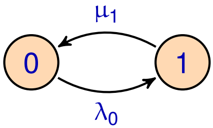

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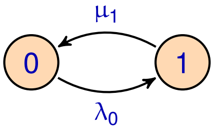

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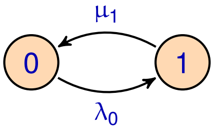

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- which says that the quantity $\lambda_{n-1}\pi_{n-1} - \mu_n \pi_n$ is independent of n
- since it vanishes for $n = 1$, it vanishes for all n, hence the steady state obeys

$$\lambda_n \pi_n = \mu_{n+1}\pi_{n+1} \qquad (n \geqslant 0)$$

- Assuming $\mu_n \neq 0$, we can solve recursively for the π_n in terms of π_0:

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0, \quad \pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0, \quad \dots$$

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- Convergence imposes constraints on the birth and death rates for the existence of a steady state

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- Finally, for all $n \geqslant 1$,

$$\pi_n = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n$$

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- The steady-state probability generating function is

$$G(s) = \sum_n s^n \pi_n = \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{\mu^n}\left(1 - \frac{\lambda}{\mu}\right) = \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{s\lambda}{\mu}} = \frac{\mu - \lambda}{\mu - s\lambda}$$

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- The mean length of the queue is the expectation $\mathbb{E}(N)$, given by

$$\mathbb{E}(N) = \sum_n n\pi_n = G'(1) = \frac{\lambda}{\mu - \lambda}$$

  which grows as $\frac{\lambda}{\mu} \to 1$

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- "Nice" birth and death processes have **steady states** with probabilities (π_n) satisfying the **zero net flow** condition $\lambda_n \pi_n = \mu_{n+1} \pi_{n+1}$ and the normalisation condition $\sum_n \pi_n = 1$