Mathematics for Informatics 4a

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Lecture 21
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$$p_{n,n+1} = \frac{\lambda_n}{\lambda_n + \mu_n} \qquad p_{n,n-1} = \frac{\mu_n}{\lambda_n + \mu_n} \qquad (n \geqslant 1)$$

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- Typical examples are **queues**, of which we saw a simple example.

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- This often imposes conditions on the parameters of the process

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- It is convenient to introduce $\rho = \frac{\lambda}{\mu}$

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- so that for $n = 0, 1, \ldots, c$

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- The expression on the RHS is called **Erlang's loss formula**, denoted

$$E(c,\rho) = \frac{\rho^c}{c!\left(1 + \rho + \frac{1}{2}\rho^2 + \cdots + \frac{1}{c!}\rho^c\right)}$$

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- For finite c,

$$\mathbb{E}(N) = \sum_n n\pi_n = \sum_{n=1}^{c} \frac{\rho^n}{(n-1)!}\pi_0$$

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- We can rewrite $\mathbb{E}(N)$ in terms of the Erlang loss formula $E(c, \rho)$

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\begin{aligned}
\mathbb{E}(N) &= \sum_{n=1}^{c} \frac{\rho^n}{(n-1)!}\pi_0 \\
&= \sum_{\ell=0}^{c-1} \frac{\rho^{\ell+1}}{\ell!}\pi_0 \\
&= \rho \frac{\left(\sum_{\ell=0}^{c} \frac{\rho^\ell}{\ell!} - \frac{\rho^c}{c!}\right)}{\sum_{\ell=0}^{c} \frac{\rho^\ell}{\ell!}} \\
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- i.e., $E(c, \rho)$ is the expected fraction of traffic lost

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  - s: the number of servers
  - λ: the arrival rate of calls
  - μ: the service rate
- The birth and death rates are then

$$\lambda_n = \lambda \qquad \text{and} \qquad \mu_n = \begin{cases} n\mu, & n \leqslant s \\ s\mu, & n > s \end{cases}$$

Example (M/M/s queues — continued)

- If there is a steady state, its probabilities are

$$\pi_n = \begin{cases} \frac{\lambda^n}{n!\mu^n}\pi_0, & n = 1, \ldots, s \\ \frac{\lambda^n}{s!s^{n-s}\mu^n}\pi_0, & n > s \end{cases}$$

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- The (geometric) series convergence if $\lambda < s\mu$, which is just the condition that the total service rate of the system exceeds the rate at which calls are entering

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- Finally,

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- Of course, infinitely long queues are an idealisation: in practice, one has a finite buffer in which to store the calls in the queue, with calls being lost if the buffer is full

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  How can we model this as a Markov chain?

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- The transition probabilities are $p_{01} = p_{12} = p_{20} = 1$
- The times T_0, T_1 and T_2 that the system spends on each state before making the transition to the next state, are exponentially distributed with rates $\lambda_0 = \lambda$, $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$, respectively.

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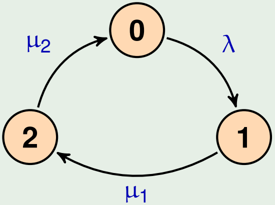

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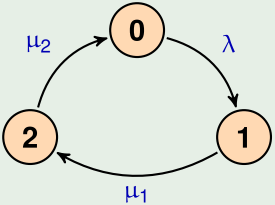

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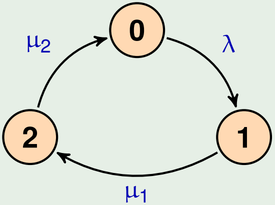

$$\begin{aligned} \lambda\pi_0 &= \mu_2\pi_2 \\ \mu_1\pi_1 &= \lambda\pi_0 \\ \mu_2\pi_2 &= \mu_1\pi_1 \end{aligned}$$

- We can solve for $\pi_{1,2}$ in terms of π_0,

$$\pi_1 = \frac{\lambda}{\mu_1}\pi_0, \quad \pi_2 = \frac{\lambda}{\mu_2}\pi_0 \qquad \Longrightarrow \pi_0\left(1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}\right) = 1$$

Example (A shoe-shine shop — continued)

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$$
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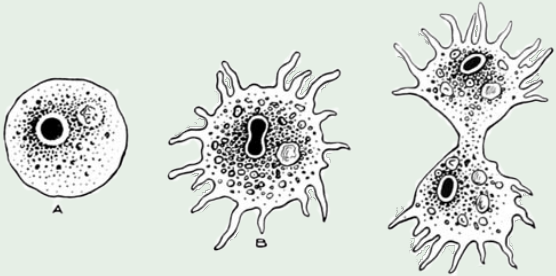
- An interesting example is in the evolution of populations of single-celled organisms

Example (The sex life of the amoeba)



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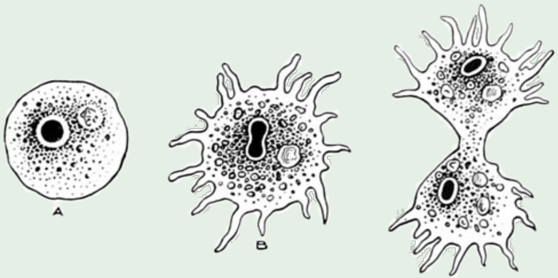

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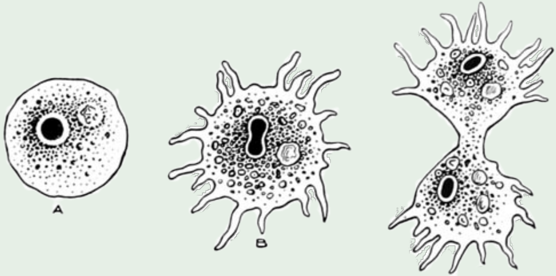

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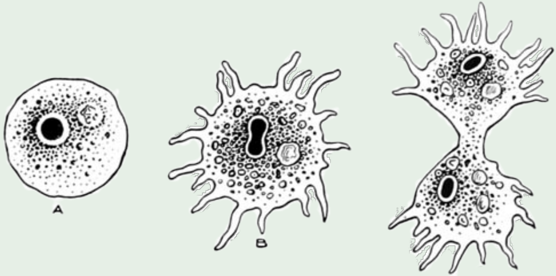

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- By the same argument, the time until the next transition (of either type) is $\min(T_1, T_2)$ which is exponential with rate $n\alpha + m\beta$

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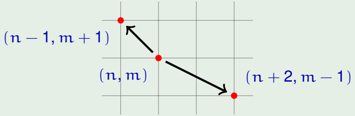

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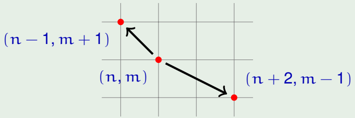

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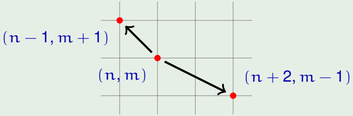

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- the population never decreases, so there is no steady state

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- We have learned how to model them in terms of Markov chains with transition probabilities and with exponential transition rates
- Some of these Markov chains have steady states, whose probabilities can be determined by imposing **zero net proability flow** across the states


That's all Folks!