# Mathematics for Informatics 4a

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Lecture 12

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## The story of the film so far...

- $X$ a c.r.v. with p.d.f. $f$ and $g : \mathbb{R} \to \mathbb{R}$: then $Y = g(X)$ is a random variable and

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

- **variance**: $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
- **moment generating function**: $M_X(t) = \mathbb{E}(e^{tX})$
- have met uniform, exponential and normal distributions and have computed their mean, variance and m.g.f.
- if $X$ normally distributed with mean $\mu$ and variance $\sigma^2$, $Y = \frac{1}{\sigma}(X - \mu)$ has standard normal distribution
- The c.d.f. $\Phi$ of the standard normal distribution is not an elementary function, but there are tables
- **maximum entropy**: normal distribution is the "least biased" among all p.d.f.s with the same mean and variance



## Jointly distributed continuous random variables

**Definition**

Two continuous random variables $X$ and $Y$ are said to be jointly distributed with joint density $f(x, y)$ if for all $a < b$ and $c < d$,

$$\mathbb{P}(a < X < b, c < Y < d) = \int_c^d \int_a^b f(x, y)dxdy$$

It follows that the joint density obeys $f(x, y) \geqslant 0$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

and

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y)dxdy$$

(provided $C$ is "nice" enough)



**Example**

Let $X$ and $Y$ have joint density

$$f(x, y) = cxy \qquad 0 \leqslant x, y \leqslant 1.$$

*What is $c$?*

From the normalisation condition,

$$1 = \int_0^1 \int_0^1 cxy\,dx\,dy = c\left(\tfrac{1}{2}x^2\Big|_0^1\right)\left(\tfrac{1}{2}y^2\Big|_0^1\right) = \frac{c}{4} \implies c = 4$$

*What if $0 \leqslant x < y \leqslant 1$?*

Since the density is symmetric in $x \leftrightarrow y$, the integral over half the square is half of the previous result, hence $c$ is twice the previous value: $c = 8$.

## Uniform joint densities

Let $A \subset \mathbb{R}^2$ be a region with area $|A|$.

**Definition**

$X$ and $Y$ are (jointly) uniform in $A$ if

$$f(x,y) = \begin{cases} \frac{1}{|A|}, & (x,y) \in A \\ 0, & \text{elsewhere} \end{cases}$$

**Example**

Let $X$, $Y$ be jointly uniform in the unit disk $D = \{(x,y) \mid x^2 + y^2 \leqslant 1\}$. Then $|D| = \pi$, whence

$$f(x,y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$



## Marginals

Let $X$, $Y$ be continuous random variables with joint density $f(x,y)$. Then the marginal p.d.f.s $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

**Remark**

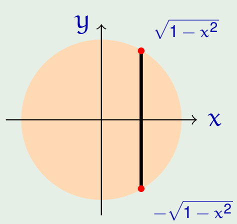
As in the discrete case there is no need to stop at two random variables, and we can have joint densities $f(x_1, \dots, x_n)$ for $n$ jointly distributed random variables, with many different marginals.



**Example**

Let $X$, $Y$ be jointly uniform on the unit disk $D$:

$$f(x,y) = \frac{1}{\pi} \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$




The marginals are given by

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}$$

for $-1 \leqslant x \leqslant 1$ and, by symmetry,

$$f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$$

for $-1 \leqslant y \leqslant 1$



## Joint distributions

**Definition**

Let $X$ and $Y$ be continuous random variables with joint density $f(x,y)$. Their joint distribution is defined as

$$F(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)du\,dv$$

It follows from the fundamental theorem of calculus that

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

and the marginal distributions are obtained by

$$F_X(x) = F(x, \infty) \qquad \text{and} \qquad F_Y(y) = F(\infty, y)$$

## Example

Let $X$, $Y$ be jointly distributed with $f(x,y) = x + y$ on $0 \leqslant x, y \leqslant 1$.
One checks that indeed $\int_0^1 \int_0^1 (x+y)dxdy = 1$.
The joint distribution is

$$\begin{aligned} F(x,y) &= \int_0^x \int_0^y (u+v)du\,dv \\ &= \int_0^x \left( \int_0^y (u+v)dv \right) du \\ &= \int_0^x \left( uy + \tfrac{1}{2}y^2 \right) du \\ &= \tfrac{1}{2}x^2y + \tfrac{1}{2}xy^2 \qquad \text{for } 0 \leqslant x, y \leqslant 1 \end{aligned}$$

For $y > 1$, $F(x,y) = \frac{1}{2}x(x+1)$ and similarly, for $x > 1$, $F(x,y) = \frac{1}{2}y(y+1)$.



## Independence

### Definition

Two continuous random variables $X$ and $Y$ are independent if

$$F(x,y) = F_X(x)F_Y(y)$$

or, equivalently,

$$f(x,y) = f_X(x)f_Y(y) \ .$$

It follows that for $X$, $Y$ independent

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

**Useful criterion**: $X$ and $Y$ are independent iff $f(x,y) = g(x)h(y)$.
Then $f_X(x) = cg(x)$ and $f_Y(y) = \frac{1}{c}h(y)$, where $c = \int_{\mathbb{R}} h(y)dy$.



## Examples

1. $X$ and $Y$ are jointly uniform on $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$:

$$f(x,y) = \frac{1}{ab} \qquad \text{for } (x,y) \in [0,a] \times [0,b]$$

with marginals $f_X(x) = \frac{1}{a}$ and $f_Y(y) = \frac{1}{b}$. Since $f(x,y) = f_X(x)f_Y(y)$, $X$ and $Y$ are independent.

2. $X$ and $Y$ are jointly uniform on the disk $0 \leqslant x^2 + y^2 \leqslant a^2$:

$$f(x,y) = \frac{1}{\pi a^2} \qquad \text{for } 0 \leqslant x^2 + y^2 \leqslant a^2$$

with marginals $f_X(x) = \frac{1}{\pi a^2}\sqrt{a^2 - x^2}$ and $f_Y(y) = \frac{1}{\pi a^2}\sqrt{a^2 - y^2}$. Since $f(x,y) \neq f_X(x)f_Y(y)$, $X$ and $Y$ are not independent.

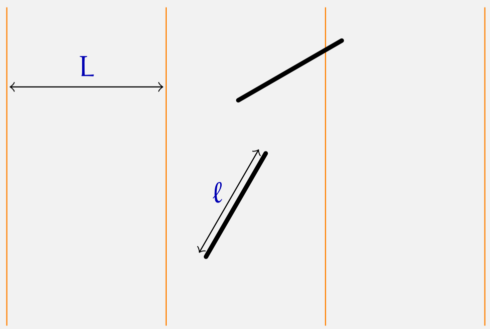


## Geometric probability

- Geometric probability or "continuous combinatorics" studies geometric objects sharing a common probability space.
- We have already seen some geometric probability problems in the tutorial sheets.
- For example, in Tutorial Sheet 4 you considered the problem of tossing a coin on a square grid and computing the probability that the coin is fully contained inside one of the squares.
- This game was called *franc-carreau* ("free tile") in France and was studied by Buffon in his treatise *Sur le jeu de franc-carreau* (1733).
- In probability, Buffon is perhaps better known for *Buffon's needle*, which is a paradigmatic geometric probability problem.

# Buffon's needle I

- Drop a needle of length $\ell$ at random on a striped floor, with stripes a distance $L$ apart.
- Let $\ell < L$: *short* needles.

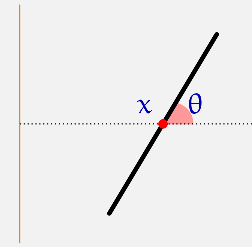



*What is the probability that the needle does not touch any line?*



# Buffon's needle II

- The needle is described by the midpoint and the angle with the horizontal.
- Symmetry allows us to ignore the vertical component of the midpoint and to assume the horizontal component lies in one of the strips.



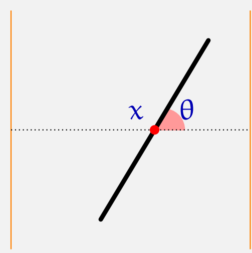

- Let $X$ denote the horizontal component of the midpoint. It is uniformly distributed in $[-\frac{L}{2}, \frac{L}{2}]$.
- Let $\Theta$ denote the angle with the horizontal, which is uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- Since $X$ and $\Theta$ are independent, the joint probability density function is the product of the two probability density functions and hence is also uniformly distributed.



# Buffon's needle III

The needle will touch one of the parallel lines if and only if

$$|x| + \frac{\ell}{2}\cos\theta > \frac{L}{2}$$




for $x \in [-\frac{L}{2}, \frac{L}{2}]$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
The complementary probability is

$$\begin{aligned}\mathbb{P}\left(|X| \leqslant \tfrac{1}{2}(L - \ell\cos\Theta)\right) &= \frac{1}{L\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_{-\frac{1}{2}(L-\ell\cos\theta)}^{\frac{1}{2}(L-\ell\cos\theta)} dx\right] d\theta \\ &= \frac{1}{L\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (L - \ell\cos\theta)\, d\theta \\ &= 1 - \frac{\ell}{L\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta = 1 - \frac{2\ell}{L\pi}\end{aligned}$$



# Functions of several random variables

- $X$ and $Y$ are continuous random variables with joint density $f(x, y)$
- $Z = g(X, Y)$, for some function $g : \mathbb{R}^2 \to \mathbb{R}$
- How is $Z$ distributed? (assuming it is a c.r.v.)
- Its c.d.f. $F_Z(z) = \mathbb{P}(Z \leqslant z)$ is given by

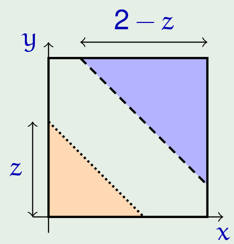
$$F_Z(z) = \iint_{g(x,y) \leqslant z} f(x, y)\, dx\, dy$$

- Its p.d.f. $f_Z(z) = F_Z'(z)$

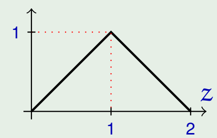
**Example (The sum of two jointly uniform variables)**

- Let $X$ and $Y$ be jointly uniform on $[0,1]$
- $f(x,y) = f_X(x) = f_Y(y) = 1$ for $0 \leqslant x, y \leqslant 1$, so $X$ and $Y$ are independent. Let $Z = X + Y$.

$$F_Z(z) = \iint_{x+y \leqslant z} dx\,dy = \begin{cases} \frac{1}{2}z^2, & z \in [0,1] \\ 1 - \frac{1}{2}(2-z)^2, & z \in [1,2] \end{cases}$$




$$f_Z(z) = \begin{cases} z, & z \in [0,1] \\ 2 - z, & z \in [1,2] \end{cases}$$






## The sum of two independent variables: convolution

- Let $X$ and $Y$ be continuous random variables with joint density $f(x,y)$ and let $Z = X + Y$.
- Then $F_Z(z) = \iint_{x+y \leqslant z} f(x,y)\,dx\,dy$ is given by

$$F_Z(z) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{z-x} f(x,y)\,dy \right) dx$$

- Hence $f_Z(z) = F_Z'(z)$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx$$

- If $X$ and $Y$ be independent, $f(x,y) = f_X(x) f_Y(y)$, whence

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = (f_X \star f_Y)(z)$$

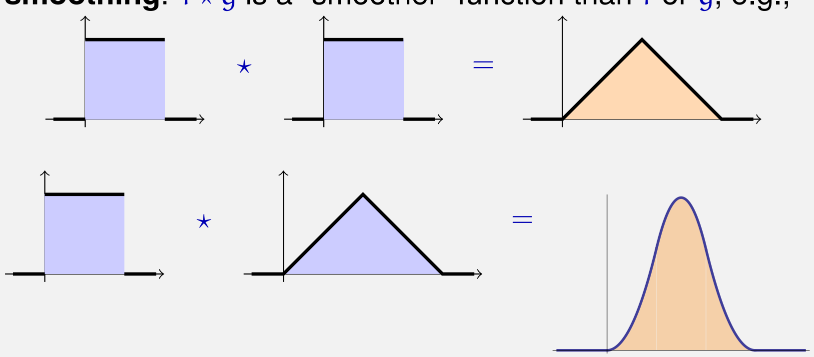
which defines the convolution product $\star$



## Convolution

The convolution product satisfies a number of interesting properties:

- **commutativity**: $f \star g = g \star f$
- **associativity**: $(f \star g) \star h = f \star (g \star h)$
- **smoothing**: $f \star g$ is a "smoother" function than $f$ or $g$, e.g.,





## Summary

- C.r.v.s $X$ and $Y$ have a **joint density** $f(x,y)$ with

$$\mathbb{P}((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$$

  and a **joint distribution**

$$F(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,du\,dv$$

  with $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$
- $X$ and $Y$ **independent** iff $f(x,y) = f_X(x) f_Y(y)$
- Geometric probability is fun! (Buffon's needle)
- We can calculate the c.d.f. and p.d.f. of $Z = g(X,Y)$
- $X$, $Y$ independent: $f_{X+Y} = f_X \star f_Y$ (**convolution**)