LOCAL COMPLETE INTERSECTION MORPHISMS

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Abstract. We discuss local complete intersection morphisms, with some examples.

1. Local complete intersection morphisms

Let $\imath\colon Z \to X$ be a closed immersion. Then, $\imath$ is *regular* if locally at the target, it can be written $\mathrm{Spec}(A/I) \to \mathrm{Spec}\, A$ and the ideal I is generated by a regular sequence, That is $I = (x_1, \dots, x_r)$ and for any $1 \leq i \leq r$, x_i is not a zero divisor in $A/(x_1, \dots, x_{i-1})$.

Let $f\colon X \to Y$ be a morphism between algebraic varieties. Then, f is called *local complete intersection (l.c.i.)* if f can be written as the composition $f = g \circ h$ of a regular closed immersion g and a smooth morphism h.

The basic examples are morphisms between smooth varieties $f\colon X \to Y$. Indeed, we can write $f = g \circ h$, where $g\colon X \to X \times Y$ is the graph of f and $h\colon X \times Y \to Y$ is the projection.

The *codimension* of an l.c.i. map $f\colon X \to Y$ is by definition $\dim Y - \dim X$. For example, if f is smooth, it is l.c.i. and its codimension is the opposite of the relative dimension.

2. Borel–Moore homology

If $f\colon X \to Y$ is a local complete intersection morphism of codimension d, then there is a morphism of sheaves

$$\mathbb{D}\,\mathbf{Q}_Y \to f_*(\mathbb{D}\,\mathbf{Q}_X)[2d]$$

which gives the virtual pullback in Borel–Moore homology

$$f^!\colon\ \mathrm{H}^{\mathrm{BM}}_*(Y) \to \mathrm{H}^{\mathrm{BM}}_{*-2d}(X)$$

by taking derived global sections. We recall that $\mathrm{H}^{\mathrm{BM}}_i(X) = \mathrm{H}^{-i}(\mathbb{D}\,\mathbf{Q}_X)$.

We do not give the details of the construction of this map. The theory for stacks is presented in [Ols15] but for schemes, it existed earlier.

Here are some elements. Since for f smooth, $f^! \cong f^*[2d]$ where d is the relative dimension of f, we have a morphism $\mathbb{D}\,\mathbf{Q}_Y \to f_*\,\mathbb{D}\,\mathbf{Q}_X[-2d]$ coming from the isomorphism $f^*\,\mathbb{D}\,\mathbf{Q}_Y \cong \mathbb{D}\,\mathbf{Q}_X[-2d]$. Therefore, in virtue of the factorization of l.c.i. morphism as a regular closed immersion followed by a smooth map, it suffices to construct the virtual pullback for regular closed immersions. It is then a theorem that the map obtained is independent of the factorization.

3. Examples

3.1. Section of a smooth map.

Proposition 3.1. *Let $g\colon Y \to X$ be a smooth map and $f\colon X \to Y$ a section of g, that is $g \circ f = \mathrm{id}_X$. Then f is a regular immersion.*

Proof. After writing the more complicated proof below, I understood that one can just simply apply Lemma 37.60.10 of the stacks project to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {}_{\mathrm{id}}\searrow \quad \swarrow_{g} & \\ & X & \end{array}.$$

□

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Proof. Let $x \in X$ and $y = f(x)$. There exists a neighbourhood $U \subset Y$ of Y and a neighbourhood $V \subset X$ of $x = g(y)$ such that $g(U) \subset V$ and the restriction g_U factors as

$$\begin{array}{ccc} U & \xrightarrow{h} & \mathbf{A}^m_V \\ & \overset{g_U}{\searrow} & \downarrow p \\ & & V \end{array}$$

where h is étale and p is the projection. We replace V by $V' = f^{-1}(U) \cap V$ and U by $U' = g^{-1}(V') \cap U$. Then, $g(U') \subset V'$ and $f(V') \subset U$ and, since $gf(V') = V'$, $f(V') \subset g^{-1}(V')$, proving that $f(V') \subset U'$. We therefore obtain a diagram

$$\begin{array}{ccc} U' & \xrightarrow{h'} & \mathbf{A}^m_{V'} \\ & \overset{g_{U'}}{\searrow} & \downarrow p' \\ & & V' \end{array}$$

with p' the projection, U' étale, and $f_{V'} \colon V' \to U'$ gives a section of $g_{U'}$. It follows that $h' \circ f_{V'}$ gives a section of p'. The map $h' \circ f_{V'}$ is clearly l.c.i., as it is even complete intersection. Now, we have $h' \circ f_{V'}$ complete intersection and h' is smooth (since étale) and so, by Lemma 37.60.10 of the stacks project, $f_{V'}$ is l.c.i. □

3.2. **Section of a smooth map and virtual pullback.** We saw in §3.1 that a section $f \colon X \to Y$ of a smooth map (of relative dimension d) is l.c.i. This is actually a very favourable situation among all l.c.i. situations, as the morphism of sheaves

$$\mathbb{D}\,\mathbf{Q}_Y \to f_*(\mathbb{D}\,\mathbf{Q}_X)[2d]$$

comes by adjunction from the isomorphism $f^* \mathbb{D}\,\mathbf{Q}_Y \to (\mathbb{D}\,\mathbf{Q}_X)[2d]$.

Proof. We prove that when $f \colon X \to Y$ is a section of a smooth map $g \colon Y \to X$, then $f^* \mathbb{D}\,\mathbf{Q}_Y \cong (\mathbb{D}\,\mathbf{Q}_X)[2d]$. Indeed, by smoothness of g, we have $g^! = g^*[2d]$. We therefore have

$$\mathbf{Q}_X \cong (\mathrm{id}_X)^! \,\mathbf{Q}_X \cong f^! g^! \,\mathbf{Q}_X \cong f^! \,\mathbf{Q}_Y[2d].$$

By taking Verdier duality, we obtain an isomorphism $\mathbb{D}\,\mathbf{Q}_X \cong f^*(\mathbb{D}\,\mathbf{Q}_Y)[-2d]$. □

As shown in §3.3, this situation is not general. This is nevertheless the kind of situations appearing in geometric representation theory, when constructing cohomological Hall algebra products.

3.3. **The nodal singularity.** We let $Y = \mathbf{C}^2$ and $X = \{xy = 0\} \subset \mathbf{C}^2$. Then, $f \colon X \to Y$ is l.c.i. as X is complete intersection in Y. It is of codimension 1. But we do not have $f^* \mathbb{D}\,\mathbf{Q}_Y \cong (\mathbb{D}\,\mathbf{Q}_X)[2]$. Indeed,

$$f^* \mathbb{D}\,\mathbf{Q}_{\mathbf{A}^2} \cong f^* \,\mathbf{Q}_{\mathbf{A}^2}[4] \cong \mathbf{Q}_X[4]$$

while by using the description of the dualizing sheaf of the nodal singularity given in [Hen22], we can conclude (by looking at the fiber over 0 for example) that it is not isomorphic to $(\mathbb{D}\,\mathbf{Q}_Y)[2]$.

Nevertheless, the morphism $f^* \mathbb{D}\,\mathbf{Q}_Y = \mathbf{Q}_X[4] \to (\mathbb{D}\,\mathbf{Q}_X)[2]$ is given by the composition

$$\mathbf{Q}_X[4] \to \mathbf{Q}_A[4] \oplus \mathbf{Q}_B[4] \to (\mathbb{D}\,\mathbf{Q}_X)[2]$$

of the second morphism in the triangle [Hen22, (0.1)] shifted by 3 with the first morphism of the first triangle in the proof of [Hen22, Proposition 0.2] shifted by 3.

Indeed, by shifting appropriately, we have to describe a morphism of perverse sheaves

$$\mathbf{Q}_X[1] \to \mathbb{D}(\mathbf{Q}_X[1]).$$

In [Hen22], we described $\mathbf{Q}_X[1]$ as an indecomposable perverse sheaf, with $\mathbf{Q}_A[1] \oplus \mathbf{Q}_B[1]$ as maximal semisimple quotient, while $\mathbb{D}(\mathbf{Q}[1])$ is an indecomposable perverse sheaf with $\mathbf{Q}_A[1] \oplus \mathbf{Q}_B[1]$ as maximal semisimple subobject. The statement can then be verified on the complement of the origin of $\mathbf{A}^2$ and is then easy.

References

[Hen22] Lucien Hennecart. “Dualizing sheaf of a nodal singularity”. In: (2022).

[Ols15] Martin Olsson. “Borel–Moore homology, Riemann–Roch transformations, and local terms”. In: *Advances in Mathematics* 273 (2015), pp. 56–123.

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