Statistical mechanics of the focusing nonlinear Schrödinger equation

Consider a Hamiltonian equation on $\mathbb{R}^d$:

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Hamiltonian, and p, q solve

$$\begin{cases} \partial_t p = J \dfrac{\partial H}{\partial q}, \\ \partial_t q = -J \dfrac{\partial H}{\partial p}, \end{cases}$$

where $J : \mathbb{R}^d \circlearrowleft$ is skew-adj. $J^* = -J$ and $J^2 = -Id$,

We say that a measure μ is <u>conserved by the flow</u> if $\forall\, F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ cont. and bdd.,

$$\int F(p(t), q(t))\, d\mu(p_0, q_0) = \int F(p_0, q_0)\, d\mu(p_0, q_0).$$

The following measures are conserved by Hamiltonian flows:

1. Lebesgue: $dp\, dq$ $\leftarrow$ <u>Liouville's thm.</u>

2. Gibbs measure: $\frac{1}{Z} \exp(-H(p,q))\, dp\, dq$ $\leftarrow$ probability

3. For every $M : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ that is constant along the flow, $\frac{1}{Z_M} \exp(-H(p,q)) \exp(-M(p,q))\, dp\, dq$.
probability $\to$ $\frac{1}{Z_M}$

We call these (generalised) grand-canonical measure

↑ non-standard

In Statistical mechanics, 2. and 3. describe the eq. of the system.

Rk.: If M is conserved, then $\forall \beta : \mathbb{R} \to \mathbb{R}$,

$\beta(M)$ is also conserved.

Important: We want our invariant measures to be probabilities, whenever possible.

Pathological example: $\dot{x} = 1$, or $x(t) = x + t$.

Then dx is invariant, but $x(t) \to \infty$ $\forall x_0$.

Therefore, dx has nothing to do with the equilibrium.

Consider NLS, $u : \mathbb{T}^d \to \mathbb{C}$.

$$\begin{cases} i\partial_t u = -\Delta u \pm \sigma |u|^{p-2} u \\ u(0) = 0. \end{cases}$$

This is Hamiltonian, with $p = \mathrm{Re}\, u$, $q = \mathrm{Im}\, u$,

$$H(p,q) = H(u) = \mp \frac{\sigma}{p} \int_{\mathbb{T}^d} |u|^p + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2 .$$

Moreover, $M(u) = \int_{\mathbb{T}^d} |u|^2$ is also conserved.

Grand - canonical measure: formally

$$d\rho(u) = \frac{1}{Z} \exp\left(-H(u) - \frac{1}{2} M(u)\right) du\, d\bar{u}.$$

Defocusing case: Sign +:

$$d\rho(u) = \underbrace{\frac{1}{Z} \exp\left(-\frac{\sigma}{p} \int_{\mathbb{T}^d} |u|^p\right.}_{\text{density}} \underbrace{\left. - \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2}\int |u|^2\right) du\, d\bar{u}}_{\text{Gaussian}}$$

"Standard" Φ^p_d measure on $\mathbb{T}^d$. Defined:

- $d=1$: $\forall p > 0$.
- $d=2$: $p = 2k$, $k \in \mathbb{N}$. Requires renormalisation
- $d=3$: Only $p=2$ (Gaussian), $p=4$. For $p=4$, Φ^4_3 and Gaussian are mutually singular.

Focusing case: Sign −

We want to build the grand-canonical ensemble:

$$d\rho(u) = \frac{1}{Z} \exp\left(\frac{\sigma}{p} \int_{\mathbb{T}^d} |u|^p - \frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 - \frac{1}{2}\int_{\mathbb{T}^d} |u|^2\right) \mathbb{1}\{M(u) \le K\}\, du$$

Lebowitz - Rose - Speer '88: add mass cutoff.

Heuristic: Gagliardo - Nirenberg - Sobolev: for $2 \le p \le p^* = \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ \infty & \text{if } d \le 2 \end{cases}$,

$$\|u\|^p_{L^p(\mathbb{R}^d)} \le C_{GNS}(p) \|u\|_{L^2}^{p+d-\frac{pd}{2}} \|\nabla u\|_{L^2}^{\frac{pd}{2}-d} \quad p \ne \infty,$$

Therefore, for $\|u\|_{L^2} \leq K^{1/2}$, we expect:

- ρ exists for every K when $\frac{pd}{2} - d < 2 \Leftrightarrow p < \frac{4+2d}{d}$, i.e.

$p < 6$ in $d=1$
$p < 4$ in $d=2$
$p < 10/3$ in $d=3$
$p < 3$ in $d=4$

- When $p = \frac{4+2d}{d}$,

ρ exists (as a probability measure) whenever

$$C_{GNS}(p)\, K^{\frac{p+d}{2} - \frac{pd}{4}} \frac{\sigma}{p} < \frac{1}{2},$$

and maybe also $=$.

Theorem:

($d=1$) (LRS '88) When $d=1$, the measure ρ exists under the above conditions.
(Oh-Sosoe-T. '21): When $p=6$, in the $=$ case, the measure exists as well.

($d=2$) Even under appropriate renormalisation, the measure does not exist when $p=4$, for any value of σ, K.
(Brydges-Slade '06, Oh-Seong-T. '23)
It does exist when $p=3$, for any value of σ, K.

($d=3$) When $p=3$, the measure exists when $\sigma \ll 1$, and does not exist when $\sigma \gg 1$. The value of K is irrelevant

1. Gaussian measure $\mu \sim \frac{1}{Z}\exp\left(-\frac{1}{2}\int|\nabla u|^2 - \frac{1}{2}\int|u|^2\right)$.

We have that

$$\mu = \text{Law}(X), \quad \text{where}$$

$$X = \frac{1}{(2\pi)^{d/2}}\sum_{n\in\mathbb{Z}^d}\frac{g_n}{\sqrt{1+n^2}}e^{in\cdot x} =: \sum_{n\in\mathbb{Z}^d}\frac{g_n}{\langle n\rangle}e_n(x),$$

and $g_n \sim N_{\mathbb{C}}(0,1)$, i.i.d.. We have that (Ex.)

- $\mathbb{E}\|X\|_{H^s}^2 = \mathbb{E}\sum_{n\in\mathbb{Z}^d}\frac{\langle n\rangle^{2s}|g_n|^2}{\langle n\rangle^2} = \sum_{n\in\mathbb{Z}^d}\frac{1}{\langle n\rangle^{2-2s}} < \infty$

  iff $s < 1 - d/2$. This is sharp, i.e.

  $$\|X\|_{H^{1-d/2}}^2 = \infty \quad \text{a.s.} \quad (\text{Ex.})$$

- Similarly,

$$\mathbb{E}\,|(1-\Delta)^{s/2}X(x)|^2 = \mathbb{E}\left|\frac{1}{(2\pi)^{d/2}}\sum_{n\in\mathbb{Z}^d}\frac{\langle n\rangle^s g_n e^{in\cdot x}}{\langle n\rangle^2}\right|^2$$

$$\overset{\text{Ind.}}{=} \sum_{n\in\mathbb{Z}^d}\frac{1}{(2\pi)^d}\frac{1}{\langle n\rangle^{2-2s}} = \frac{1}{(2\pi)^d}\mathbb{E}\|X\|_{H^s}^2.$$

- For $d=1$,

$$\mathbb{E}\,\|X\|_{L^p}^p \overset{\text{Fubini}}{=} \int \mathbb{E}|X(x)|^p\,dx$$

$$\overset{X(x)\text{ Gaussian}}{=} \int c_p\left(\mathbb{E}|X(x)|^2\right)^{p/2}dx$$

$$= \int \frac{c_p}{2\pi}\left(\mathbb{E}\|X\|_{L^2}^2\right)^{p/2}dx$$

$$= c_p\left(\mathbb{E}\|X\|_{L^2}^2\right)^{p/2}$$

$$< \infty.$$

This is enough to define

$$d\rho^+(u) := \frac{\exp\left(-\frac{\sigma}{p}\int |u|^p\right) d\mu}{\int \exp\left(-\frac{\sigma}{p}\int |u|^p\right) d\mu}.$$

<u>Variational formula</u> (easy version)

Let $F : H^{1-d/2-\varepsilon} \to \mathbb{R}$ be a meas. functional, bounded below. Then

$$\log \int \exp(F(u))\, d\mu(u)$$

$$\leq \mathbb{E}\left[\sup_{V \in H^1} F(X+V) - \frac{1}{2}\|V\|_{H^1}^2\right].$$

<u>1-d focusing case</u>:

We build

$$\exp\left(\frac{\sigma}{p}\int_{\mathbb{T}^d} |u|^p \, \mathbb{1}_{\{M(u) \leq K\}}\right) d\mu$$

- GNS on the torus: $\forall \varepsilon > 0$

$$\|V\|_{L^p(\mathbb{T})}^p \leq C_{GNS}(p)(1+\varepsilon)\|V\|_{L^2}^{p/2+1}\|V\|_{H^1}^{p/2-1} + C_\varepsilon \|V\|_{L^2}^p.$$

- Let $u_\varepsilon = e^{\varepsilon^2\Delta} u$. Then $\forall\, 1 < p < \infty$, $\forall\, 0 \leq s \leq 1$

$$\|X - X_\varepsilon\|_{L^p} \lesssim \varepsilon^s \|(1-\Delta)^{s/2} X\|_{L^p},$$

$$\|X_\varepsilon\|_{H^1} \lesssim \varepsilon^{-(1-s)} \|X\|_{H^s}.$$

We just need to estimate

$$\mathbb{E}\left[\sup_{V\in H^1} \frac{\sigma}{p}\int |X+V|^p \mathbb{1}_{\{M(X+V)\le K\}} - \frac{1}{2}\|V\|_{H^1}^2\right]$$

$$\underset{V=-X_\varepsilon+W}{=} \mathbb{E}\left[\sup_{W\in H^1} \frac{\sigma}{p}\int |X-X_\varepsilon+W|^p \mathbb{1}_{\{M(X-X_\varepsilon+W)\le K\}} - \frac{1}{2}\|W-X_\varepsilon\|_{H^1}^2\right]$$

Young

$$\le \mathbb{E}\left[\sup_{W\in H^1} \frac{\sigma(1+\eta)}{p}\int |W|^p \mathbb{1}_{\{\|W\|_{L^2}\le K^{1/2}+M(X-X_\varepsilon)\}} - \frac{1-\eta}{2}\|W\|_{H^1}^2\right]$$

$$+\mathbb{E}\left[C(\eta,\sigma,p)\left(\int |X-X_\varepsilon|^p + \|X_\varepsilon\|_{H^1}^2\right)\right].$$

Pick $\varepsilon^{1/4} = \frac{\eta}{(1+\|X\|_{H^{1/4}})}$, or $\varepsilon = \frac{\eta^4}{(1+\|X\|_{H^{1/4}})^4}$.

Then $M(X - X_\varepsilon) \lesssim \eta$,

$$\|X - X_\varepsilon\|_{L^p}^p \lesssim \|X\|_{L^p}^p ,$$

$$\|X_\varepsilon\|_{H^1}^2 \lesssim \frac{\left(1 + \|X\|_{H^1 \cap L^4}\right)^8}{\eta^6} .$$

We obtain

1.

$$\leq \sup_{W \in H^1} C_{GNS} \left(K^{1/2} + C\eta\right)^{p/2+1} \frac{(1+\eta)\sigma}{p} \|W\|_{H^1}^{p/2-1} - \frac{1-\eta}{2} \|W\|_{H^1}^2$$

$< +\infty$ if

- $p/2 - 1 < 2 \iff p < 6$, or
- $p = 6$ and $C_{GNS}(6) \left(K^{1/2} + C\eta\right)^4 (1+\eta)^2 \frac{\sigma}{p} \leq \frac{1-\eta}{2}$.

Since η is arbitrary, we get

$$\boxed{C_{GNS}(6) K^2 \frac{\sigma}{p} < 1/2}$$

2. Finite $\forall\, \eta, \sigma, p$.

2d: Expected numerology: $p \leq 4$.

Can we normalise the measure

$$\exp\left(\frac{\sigma}{4} \int :|u|^4: \right) \mathbb{1}\{ :M(u): \leq k \} \, d\mu(u)$$

for some σ, k ?

Remark: This is already a σ-finite measure

We try to repeat the same proof.

$$\frac{\sigma}{4} \int |X+V|^4 \lesssim \sigma \|V\|_{H^1}^2 \|V\|_{L^2}^2 + \text{error}.$$

$$k \geq :M(X+V): = \underset{\substack{\uparrow \\ \text{finite}}}{:M(X):} + \underbrace{2 \int X \underset{L^2}{\cancel{\propto}} V}_{\text{bad}: \ \not\lesssim \|V\|_{L^2}} + \underset{\substack{\uparrow \\ \text{good}}}{\|V\|_{L^2}^2}$$

Issue: if $V = -\lambda X + W$, $W \perp_{L^2} X$, then

$$:M(X+V): = :M(X): - 2\lambda \|X\|_{L^2}^2 + \lambda^2 \|X\|_{L^2}^2 + \|W\|_{L^2}^2$$

For $\lambda \ll 1$, $\approx :M(X): - \lambda(2-\lambda)\infty + \|W\|_{L^2}^2$

can be very big.

Q.: How do we make this rigorous?

Thm.: (Boué-Dupuis formula)

Let F be measurable and bounded. Let $X(t)$ be the cylindrical Brownian motion

$$X(t) = \mathrm{Re} \sum_{n \in \mathbb{Z}^d} \frac{1}{\langle n \rangle} W_n(t),$$

where $\{W_n(t)\}$ are i.i.d. Brownian motions on $\mathbb{C}$, with $\mathbb{E}\,|W_n(t)|^2 = t$.

Let

$$\mathbb{H}_a^1 = \{ V \in L^2(\Omega, H^1) : \dot{V} \in L^2(\Omega, H^1), V \text{ prog. measurable} \}$$

Then

$$\mathbb{E}\left[\exp\left(F(X)\right)\right] = \sup_{V \in \mathbb{H}_a^1} \mathbb{E}\left[F(X(1)+V(1)) - \frac{1}{2}\int_0^1 \|\dot{V}(t)\|_{H^1}^2 \, dt\right]$$

Rk.: Removing the condition V prog. measurable, then sup is realised when $\dot{V} = \text{const} = V(1)$. This implies the simplified version

Rk.: We are going to apply this on F not bounded. See exercises.

Thm: $\forall\, \sigma > 0,\ K > 0$, we have that

$$\mathbb{E}\left[\exp\left(\left(\frac{\sigma}{4}\int u^4\right)\mathbb{1}_{\{|\int :u(u)^2: | \le K\}}\right)\right] = \infty$$

$\forall\, k > 0$.

By B.D., $\forall M \gg 1$, we need to find V adapted s.t.

$$\mathbb{E}\left[\frac{\sigma}{4}\int (X+V)^4\, \mathbb{1}_{\{|\int :X^2: + 2\int XV + \int V^2| \le K\}} - \frac{1}{2}\int_0^1 \|\dot{V}\|_{H^1}^2(s)\right] \ge M.$$

Ansatz: Recall the estimate

$$\|V\|_{L^4}^4 \lesssim \|V\|_{L^2}^2\, \|V\|_{H^1}^2.$$

We want V s.t.

- The $\lesssim$ is "almost sharp"
- $\int V(1)^2 \gg 1$
- $\int :X^2: + 2\int XV + \int |V(1)|^2 \ll 1$.
- $\dot{V}$ adapted.

$$\dot{V}(s) = \alpha_\varepsilon f - \dot{\tilde{X}}_\varepsilon(s),$$

where $\tilde{X}_\varepsilon(1)$ is a (Gaussian) smooth approximation of X. Then

$$:M(X+V):$$

$$= \int :X(1)^2: - 2\int X(1)\tilde{X}_\varepsilon(1) + \int \left(\tilde{X}_\varepsilon(1)\right)^2$$

$$+ 2\int \left(X(1) - \tilde{X}_\varepsilon(1)\right) \cdot \alpha_\varepsilon f$$

$$+ \alpha_\varepsilon^2 f^2 .$$

$$= \int :\left(X(1) - \tilde{X}_\varepsilon(1)\right)^2:$$

$$- 2\,\mathbb{E}\left[\int X(1)\tilde{X}_\varepsilon(1)\right] + \mathbb{E}\left[\int X_\varepsilon(1)^2\right]$$

$$+ 2\int \left(X(1) - \tilde{X}_\varepsilon(1)\right) \cdot \alpha_\varepsilon f$$

$$+ \alpha_\varepsilon^2 \int f^2 ,$$

(I): $\lesssim \varepsilon^\theta$ for some $\theta > 0$.

(II) $\sim -\mathbb{E}\left[\int \left(\tilde{X}_\varepsilon(1)\right)^2\right] \sim \log \varepsilon$

(III) $\lesssim \varepsilon^\theta$ for f very localised

(IV) $= \alpha_\varepsilon^2$,

Pick $\theta_\varepsilon^2 = 2\mathbb{E}\left[\int X(1)\tilde{X}_\varepsilon(1)\right] - \mathbb{E}\left[\int \tilde{X}_\varepsilon(1)^2\right]$

Then $\mathbb{E}\left|{:}M(X+V){:}\right|^2 \lesssim \varepsilon^\theta$,

$$\int |X+V|^4 \sim \alpha_\varepsilon^4 \int |f|^4 \sim (\log \varepsilon)^2 \|f\|_{H^1}^2,$$

$$\int \|\dot{V}\|_{H^1}^2 \lesssim \|\tilde{X}_\varepsilon\|_{H^1_t H^1_x}^2 + \alpha_\varepsilon^2 \|f\|_{H^1}^2$$

$$\lesssim g(\varepsilon) + |\log \varepsilon| \, \|f\|_{H^1}^2.$$

Therefore,

$$\mathbb{E}(*) \geq \left(c_1 (\log \varepsilon)^2 - c_2 |\log \varepsilon|\right) \|f\|_{H^1}^2 - g(\varepsilon)$$

Pick $\varepsilon \ll 1$, $\|f\|_{H^1} \gg \varepsilon$, and obtain the blowup.

<u>Exercises:</u> One (non-optimal) way to build $\tilde{X}_\varepsilon(s)$.

<u>Rk.:</u> We automatically get a stronger non-existence result.

Let ρ_ε be some approx. identity as $\varepsilon \to 0$, and consider

$$\rho_\varepsilon = \frac{1}{Z_\varepsilon} \exp\left(\frac{\lambda}{4} \int {:}(\rho_\varepsilon * u)^4{:}\right) \mathbb{1}_{\{|{:}M(u){:}| \leq K\}}\, \mu.$$

Then ρ_ε has no weak limit in $W^{-1/2, 4+\varepsilon}$.

Pf.: Define the measure

$$\nu_\varepsilon = \frac{1}{Z^\nu_\varepsilon} \exp\left(\frac{\sigma}{4} \int :(\rho_\varepsilon * \upsilon)^4: - \|\upsilon\|^M_{W^{-1/2,4+}} \right) \mu .$$

We can show $\nu_\varepsilon \to \nu_0$ weakly, $Z^\nu_\varepsilon \to Z^\nu_0$.

Suppose by contr. $\rho_\varepsilon \to \rho_0$ on $W^{-1/2,4+}$.

Then

$$\begin{aligned}
\nu_0 &= \lim_{\varepsilon \to 0} \nu_\varepsilon \\
&= \lim_{\varepsilon \to 0} \frac{Z_\varepsilon}{Z^\nu_\varepsilon} \exp\left(- \|\upsilon\|^M_{W^{-1/2,4+}} \right) \rho_\varepsilon \\
&= \lim_{\varepsilon \to 0} \frac{\exp\left(- \|\upsilon\|^M_{W^{-1/2,4+}} \right) \rho_\varepsilon}{\int \exp\left(- \|\upsilon\|^M_{W^{-1/2,4+}} \right) d\rho_\varepsilon(\upsilon)} \\
&= \frac{\exp\left(- \|\upsilon\|^M_{W^{-1/2,4+}} \right) \rho_0}{\int \exp\left(- \|\upsilon\|^M_{W^{-1/2,4+}} \right) d\rho_0}
\end{aligned}$$

Therefore,

$$\rho_0 = \Lambda \exp\left(\|\upsilon\|^M_{W^{-1/2,4+}} \right) \nu_0$$

$$\int \Rightarrow 1 = \Lambda \underbrace{\int \exp\left(\|\upsilon\|^M_{W^{-1/2,4+}} \right) d\nu_0}_{= \infty \text{ with the previous proof.}}$$

3d case, Φ^3_3.

Issue: We do not expect $\Phi^3_3 \ll \mu$. We want to build Φ^3_3 as $\lim_{N\to\infty} \rho_N$, where

$$\rho_N = \exp\left(\frac{\sigma}{3}\int :(P_N v)^3:\right) \mathbb{1}_{|:M(v):| \le K}\, d\mu(v),$$

for some $P_N \to Id$.

cube.

We choose $\widehat{P_N v}(n) = \hat{v}(n)\, \mathbb{1}_{\{|v|_\infty \le N\}}$

This way, $\|P_N\|_{L^p \to L^p} \lesssim_p 1$ for $1 < p < \infty$.

Rigourously, we build

$$\nu_N = \frac{1}{Z}\exp\left(\frac{\sigma}{3}\int :(P_N v)^3: - \|v\|^{20}_{W^{-\frac{3}{4},3}}\right) d\mu(v) \to \nu,$$

and then define $\rho(v) = \exp\left(\|v\|^{20}_{W^{-3/4,3}}\right) \mathbb{1}_{\{|:M(v):| \le K\}} \nu$.

We will use the variational formula to build the measure

Prop.: Suppose that F satisfies the hp. of Exercise 5. Let

$$\rho_F = \frac{\exp(F(v))\,\mu}{\int \exp(F(v))\, d\mu(v)}.$$

Then $\forall f$ meas. and bdd.,

$$\int f(v)\, d\rho_F(v) = \lim_{n\to\infty} \mathbb{E}\left[f(X + V^n)\right],$$

where V^n is an opt. sequence for

$$\sup_{V \in H^1_a} \mathbb{E}\left[F(X+V) - \frac{1}{2}\int_0^1 \|\dot{V}\|^2_{H^1}(s) \right].$$

Therefore, $\exists$ measure $\Leftarrow$ decent properties of (almost) optimisers.

By Boué - Dupuis, for $\mathcal{W} = \mathcal{W}^{-3/4,3}$,

$$Z_N = \sup_{V \in H^1} \mathbb{E}\Big[\frac{\sigma}{3} \int :P_N(X(1)+V(1))^3: - \|X(1)+V(1)\|^{20}_{\mathcal{W}}$$
$$- \frac{1}{2}\int_0^1 \|\dot{V}(s)\|^2_{H^s} \Big]$$

Recall $X \in C^{1-d/2-\varepsilon} = C^{-1/2-\varepsilon}$. Let $X_N = P_N X$.

$:X_N(1)^3:$ is bad, but $\boxed{\mathbb{E}\int :X_N(1)^3: = 0}$.

However, $:X^2: = C^{-1-\varepsilon}$, therefore

$:X_N^2:V$ is problematic in the limit.

Trick (Boroshkov - Gubinelli): Complete the square. Since $:X_N^2(s):$ is a martingale,

$$\mathbb{E}\left[\sigma \int :X_N^2(1): V(1) - \frac{1}{2}\int \langle \dot V(s), \dot V(s)\rangle_{H^1}\right]$$

$$\overset{\text{Itô}}{=} \mathbb{E}\left[\sigma \iint :X_N^2(s): \dot V(s) - \frac{1}{2}\int \langle \dot V(s), \dot V(s)\rangle_{H^1}\right]$$

$$= \mathbb{E}\left[\sigma \int \langle (1-\Delta)^{-1} :X_N^2(s), \dot V(s)\rangle_{H^1} - \frac{1}{2}\langle \dot V(s), \dot V(s)\rangle_{H^1}\right]$$

$$= \mathbb{E}\left[-\frac{1}{2}\int \| \dot V - \sigma(1-\Delta)^{-1} :X_N^2(s): \|_{H^1}^2\right]$$

$$+ \underbrace{\frac{1}{2}\mathbb{E}\left[\int \|\sigma(1-\Delta)^{-1} :X_N^2(s): \|_{H^1}^2 \, ds\right]}_{= \sigma_N.}$$

Call

- $\dot Y_N(s) = \sigma(1-\Delta)^{-1} :X_N^2(s): \in C^{1-\varepsilon}$
- $W = V - Y_N$.

Then

$$\log Z_N - \sigma_N$$

$$= \sup_W \mathbb{E}\Big[\sigma \int X_N (P_N(W+Y_N))^2 + \frac{\sigma}{3}\int P_N(W+Y_N)^3$$

$$- \| X + V \|_{L^2}^{20}$$

$$- \frac{1}{2}\int_0^1 \| \dot W(s) \|_{H^1}^2 \, ds\Big]$$

$$= \sup_{w} \Big[Q_N(X_N, Y_N, w) - \| X + Y_N + w \|_{2\nu}^{20} + \frac{\sigma}{3} \int (P_N w)^3 - \frac{1}{2} \int_0^1 \| \dot{w}(s) \|_{H^1}^2 \, ds \Big]$$

Therefore, V is an almost opt. for Z_N

iff $w := V - Y_N$

is an almost opt. for the above.

We have

- Y_N unif. bdd. (in N) in $C^{1-\varepsilon}$,
- X_N " " in $C^{-\frac{1}{2}-\varepsilon}$.

Therefore, one can show:

$$Q_N(X_N, Y_N, w) \leq C_\varepsilon(X_N, Y_N) + \varepsilon \int |w|^3 + \varepsilon \|w\|_{H^1}^2 .$$

Therefore, we can control $\log Z_N - \sigma_N$ and

consequently $\|w\|_{H^1}^2 + \|w\|_{L^3}^3$ iff we show

$$\sup_{w \in H^1_a} \mathbb{E}\Big[\frac{\tilde{\sigma}}{3} \int w^3 - \| X + Y_N + w \|_{2\nu}^{20} - \frac{1}{2} \int_0^1 \|\dot{w}\|_{H^1}^2 \Big]$$

$< \infty$.

OK by interpolation with the choice of 2ν.

What about normalisability, i.e.

$$\int \exp\left(\|v\|^{20}_{??} \right) d\nu \ ?$$

Just need to check

$$\sup_{w \in \mathbb{H}^1_e} \mathbb{E}\left[\frac{\tilde{\sigma}}{3} \left(\int w^3 \right) 1_{\{ |:M(v):| \leq k \}} - \frac{1}{2} \|w\|^2_{H^1} \right].$$

Finite for $\tilde{\sigma}$ in neighborhood of σ $\Rightarrow$ normalisability

Infinite " " " " " $\Rightarrow$ non-normalisability

Looks like the $d=2$ case! Same ansatz?

$w = -X_M + \nu f$. We get

$$\mathbb{E}\left[\frac{\tilde{\sigma}}{3} \int w^3 \right] \approx \nu^3 \mathbb{E}\left[\frac{\tilde{\sigma}}{3} \int f^3 \right] \qquad (\mathrm{I})$$

$$\mathbb{E} \, |:M(X + Y_N + w):|^\gamma \approx \left| \nu^2 \|f\|^2_{L^2} - \mathbb{E} \|X_M\|^2_{L^2} \right|^\gamma \quad (\mathrm{II})$$

$$\mathbb{E} \int_0^1 \|\dot{w}\|^2_{H^1} \approx \mathbb{E} \int_0^1 \|\dot{X}_M\|^2_{H^1} + \nu^2 \|f\|^2_{H^1} \quad (\mathrm{III})$$

$$\approx C M^3 + \nu^2 \|f\|^2_{H^1}.$$

If we choose $\|f\|^2_{L^2} = 1$, $\nu^2 \sim \mathbb{E} \|X_M\|^2_{L^2} \sim M$,

we can make $(\mathrm{II}) = O(1)$.

We obtain

$$\sup \geq C_1 M^{3/2} \tilde{\sigma} \|f\|_{L^3}^3 - C_2 M^3 - C_3 M \|f\|_{H^1}^2.$$

We have

$$\|f\|_{L^3}^3 \lesssim \|f\|_{L^2}^{3/2} \|f\|_{H^1}^{3/2} \lesssim \|f\|_{H^1}^{3/2} \text{ sharp.}$$

Optimising the above, we pick

$$M^{3/2} \|f\|_{L^3}^3 \sim M \|f\|_{H^1}^2 \iff \|f\|_{H^1}^2 \sim M,$$
$$\|f\|_{L^3}^3 \sim M^{3/2}$$

$\Rightarrow$ All the terms have size M^3. If

$\sigma \gg 1$, we obtain the divergence.

What about $\gamma << 1$?

Boils down to showing for $\gamma >> 1$

$$\int W^3 \lesssim |+|:M(X+Y_N+W):|^{\gamma} + \|W\|_{H^1}^2 .$$

$$\approx 1 + \left| 2\int X W + \int W^2 \right|^{\gamma} + \|W\|_{H^1}^2 . \quad (*)$$

Pf.: Define $\widehat{Q_N v}(n) = \hat{v}(n) \mathbb{1}_{\{ N/2 \le |n|_\infty < N \}}$

Write

$$W = \sum \lambda_N Q_N X + W_N ,$$

with $\lambda_N \in \mathbb{R}$, $W_N \perp Q_N X$ in L^2.

Then

$$2\int \left(X W + W^2 \right)$$

$$= 2 \sum \lambda_N \|Q_N X\|_{L^2}^2 + \sum \lambda_N^2 \|Q_N X\|_{L^2}^2$$

$$+ \sum \|W_N\|_{L^2}^2 .$$

2 cases.

1. $\int w^2 \gg -2\int Xw$. Then

$$\left|2\int Xw + w^2\right| \sim \left|\int w^2\right|, \text{ and } \quad (*)$$

follows from

$$\|w\|_{L^3}^3 \lesssim \|w\|_{L^2}^{3/2}\|w\|_{H^1}^{3/2} \lesssim \|w\|_{L^2}^6 + \|w\|_{H^1}^2$$

or

for $\gamma \geq 3$.

2. $\int w^2 \lesssim 2\left|\int Xw\right|$. Then

$$\sum \lambda_N^2 \|Q_N X\|_{L^2}^2 \leq \sum_N \lambda_N^2 \|Q_N X\|_{L^2}^2 + \|w_N\|_{L^2}^2$$

$$\lesssim \sum \lambda_N \|Q_N X\|_{L^2}^2.$$

Fix $N_0 > 0$. We have that

$$\left|\sum_{N \leq N_0} \lambda_N \|Q_N X\|_{L^2}^2\right| \lesssim \left(\sum_{N \leq N_0} \|Q_N X\|_{L^2}^2\right)^{1/2}\left(\sum_N \lambda_N^2 \|Q_N X\|_{L^2}^2\right)^{1/2}$$

$$\left|\sum_{N \geq N_0} \lambda_N \|Q_N X\|_{L^2}^2\right| \sim \sum_{N \geq N_0} \lambda_N \|Q_N X\|_{H^1} \|Q_N X\|_{H^{-1}}$$

$$\lesssim \left(\sum_{N \gtrsim N_0} \|Q_N X\|_{H^{-1}}^2 \right)^{1/2} \left(\sum_{N \gtrsim N_0} \lambda_N^2 \|Q_N X\|_{H^1}^2 \right)^{1/2}$$

$$\lesssim \|P_{\gtrsim N_0} X\|_{H^{-1}} \|W\|_{H^1} .$$

Call $Z = \sum \lambda_N \|Q_N X\|_{L^2}^2$. We obtain

$$|Z| \lesssim \left(\sum_{N \lesssim N_0} \|Q_N X\|_{L^2}^2 \right)^{1/2} |Z|^{1/2} + \|W\|_{H^1} \|P_{\gtrsim N_0} X\|_{H^{-1}}$$

Solving,

$$|Z| \lesssim \sum_{N \lesssim N_0} \|Q_N X\|_{L^2}^2 + \|W\|_{H^1} \|P_{\gtrsim N_0} X\|_{H^{-1}} .$$

Lemma: (Ex.)

$$\sum_{N \leq N_0} \|Q_N X\|_{L^2}^2 \leq C N_0 + B_1(X)$$

$$\sum_{N \geq N_0} \|Q_N X\|_{H^{-1}}^2 \leq C N_0^{-1} + N_0^{-2} B_2(X) .$$

where $\mathbb{E}\left[|B_1|^p + |B_2|^p \right] \leq C_p < \infty$.

Pick $N_0 \sim 1 + \|W\|_{H^1}^{2/3}$.

We obtain

$$\|w\|_{L^2}^2 \lesssim |Z| \lesssim 1 + \|w\|_{H^1}^{4/3}, \text{ or}$$

$$\|w\|_{L^2}^6 \lesssim |Z|^3 \lesssim 1 + \|w\|_{H^1}^2 .$$

The conclusion follows again by

$$\|w\|_{L^3}^3 \lesssim \|w\|_{L^2}^6 + \|w\|_{H^1}^2 .$$

$$\lesssim 1 + |Z|^3 + \|w\|_{H^1}^2 .$$

Bourgain's invariant measure argument:

Let X Polish, and let $\{\Phi_t : X \to X\}_{t\geq 0}$ be a flow map, i.e. $\Phi_{t+s} = \Phi_t \circ \Phi_s$, $\Phi_0 = \mathrm{Id}$.

Suppose that $\Phi_\cdot(\cdot)$ is jointly meas. on $\mathbb{R} \times X$.

Let ρ an invariant meas. for Φ. Suppose:

1. (LWP) $\exists$ "size function" $r : X \to \mathbb{R}$, $r \geq 0$
$\exists$ "LWP time function" $\tau : \mathbb{R}^+ \to \mathbb{R}^+$, and $K > 1$, s.t.

$$r(\Phi_s(u_0)) \leq K\, r(u_0) \qquad \forall\ 0 \leq s \leq \tau(r(u_0)),$$

and $\tau \gtrsim \langle x \rangle^{-\beta}$ for some $\beta \geq 0$.

2. (LDE for ρ)
$\exists\ \gamma > 0$ s.t. $\displaystyle\int \exp\left(r(u_0)^\gamma\right) d\rho(u_0) < \infty .$

Then we have the estimate

$$z(\Phi_t(u_0)) \lesssim \log(2+t)^{1/\gamma} \qquad \rho\text{-a.e.}$$

More precisely, we have the LDE for $M \gg 1$

$$-\log \rho\left(\left\{ z(\Phi_t(u_0)) > M\left(\log(2+t)\right)^{1/\gamma}\right\}\right) \gtrsim M^{\gamma}.$$

__RK.:__ In the stochastic case,

$$\dot{u} = \mathcal{N}_1(u, \xi),$$

with invariant measure ρ for the semigroup, define $\Phi_t(u,\xi) = (u(t), \xi(\cdot - t))$.

Then $\rho \otimes \mathrm{Law}(\xi)$ is invariant.

__RK.:__ Singular case: usually

- Measurable map $u_0 \to \Xi$ enhanced data set
- Unknowns $u_1, \dots, u_N$.
- Solution $u(t) = F_t(\Xi, u_1, \dots, u_N)$.

Define

$$z(u) = \inf\left\{ \|\Xi\|_0 + \|u_1\|_1 + \dots + \|u_N\|_N + 1 + |t|^{\alpha} \; ; \; u = F_t(\Xi, u_1, \dots, u_N)\right\}$$

Tipically, this is a goood size function.