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On the tails of Pitman–Yor random probability measures: Transport maps and stick-breaking constructions

María F. Gil-Leyva[1] · **Vianey Palacios Ramirez**[2] · **Miguel de Carvalho**[3,4]

Abstract While random probability measures have a long tradition in probability and statistics, little is known about their tails. The few available results are derived using subordinators, and therefore only apply to measures that can be represented as normalized subordinators, such as the Dirichlet process. Our work breaks this barrier, by exploiting the stick-breaking representation to construct a new family of transport maps that preserve the decay of tails. Drawing on recent developments on regular variation and on subordinator theory, the new family of maps allows us to establish that the right tail of a Pitman–Yor process is heavy-tailed if the centering distribution is itself heavy-tailed; the Dirichlet process is the only member of this class that fails to obey this convenient property. Asymptotic envelopes for the tails of the Pitman–Yor processes are also derived. Finally, we discuss some consequences of the main results, including aspects related to the posterior distribution.

Keywords Dirichlet process, Heavy-tail, Pitman–Yor process, Regular variation, Stick-breaking subordinator

1 Introduction

Although random probability measures are well-established (Crauel 2002; Kallenberg 2017), their tail behavior remains poorly understood. This gap is concerning and has been overlooked from the perspective of extreme value theory (Coles 2001; Beirlant et al. 2004; de Haan and Ferreira 2006; Resnick 2007; de Carvalho et al. 2026)—as well as from that of Bayesian nonparametrics (Müller et al. 2015; Ghosal and Van der Vaart 2017), where random probability measures are fundamental components of a wide range of models.

In this paper, we contribute to narrowing this gap by investigating the tails of a large class of random probability measures known as Pitman–Yor processes via the stick–breaking construction; further details on the the stick–breaking construction will be provided in Section 1.1. The class of Pitman–Yor processes, which includes the Dirichlet process as a particular case, has received considerable attention in recent years (Pitman and Yor 1997; Ishwaran and James 2001; Teh 2006; Bassetti et al. 2014; Canale et al. 2017; Arbel et al. 2019; Lijoi et al. 2020). By drawing on regular variation and subordinator theory, we pioneer the study of the tails of this large class of random probability measures. Incidentally, the analysis of the problem of interest leads to a class of functions bounded by regularly varying functions with a common index. Below, we refer to such functions as possessing M-variation, as they exhibit properties analogous to regular variation, including a representation theorem.

A brief overview of subordinators and their connection to the tails of random probability measures is in order. Because distribution functions are nondecreasing, it is natural to model random probability measures using nondecreasing stochastic processes (e.g., Doss and Sellke 1982; Palacios et al. 2025). As discussed in Section 2, subordinators are a canonical example: they are nondecreasing Lévy processes with nonnegative paths and stationary, independent increments; the tail behavior of a random measure admitting a representation in terms of a standardized subordinator can then be characterized through the behavior of the subordinator near the origin. Before turning to a detailed treatment of subordinators, we first introduce the Pitman–Yor process which defines the main class of random probability measures studied in this paper.

✉ Miguel de Carvalho
Miguel.deCarvalho@ed.ac.uk

[1]Department of Probability and Statistics, Universidad Nacional Autónoma de México, México
[2]School of Mathematics, Statistics and Physics; Newcastle University, UK
[3]School of Mathematics, University of Edinburgh, The King's Buildings, Edinburgh, EH9 3FD, UK
[4]Department of Mathematics, University of Aveiro, Portugal

1.1 Background on Pitman–Yor processes

A random probability measure P follows a Pitman–Yor process, denoted as $P \sim \text{PYP}(\sigma, \theta, P_0)$, if it admits a stick-breaking representation of the type,

$$P = \sum_{j=1}^{\infty} \pi_j \delta_{\xi_j}, \qquad \xi_j \overset{\text{iid}}{\sim} P_0. \tag{1.1}$$

Here, δ_ξ assigns a point mass to ξ, and the weights admit a stick-breaking decomposition,

$$\pi_1 = \nu_1, \qquad \pi_j = \nu_j \prod_{k<j} (1 - \nu_k), \qquad j = 2, 3, \ldots, \tag{1.2}$$

with $\nu_j \overset{\text{ind}}{\sim} \mathsf{Be}(1 - \sigma, \theta + j\sigma)$ for $j \in \mathbb{N}$. The parameters are known (e.g., Rigon et al. 2025) as discount (σ), precision (θ), and centering (or baseline) measure (P_0) and must meet the constraints $0 \leq \sigma < 1$, $\theta > -\sigma$ with P_0 being non–atomic. The Dirichlet process is a particular case of the Pitman–Yor process with $\sigma = 0$, and the case $\theta = 0$ is known as the stable law process.

While Pitman–Yor processes are well-defined defined over arbitrary Polish spaces endowed with their Borel σ-algebras, in line with the conventional approach in extreme value theory, we concentrate on the right tail of $P \sim \text{PYP}(\sigma, \theta, P_0)$ defined over $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ and denote its random distribution function by $G(x) \equiv P\{[0, x]\}$. We examine the behavior of the tail function $\bar{G}(x) \equiv 1 - G(x)$ as x approaches the right endpoint, $x^* = \sup\{x : G(x) < 1\}$, which will be mainly assumed to be $x^* = \infty$. It is known that the tails of the Dirichlet process are exponentially much thinner than those of its baseline measure (Ghosal and Van der Vaart 2017, Section 4.3). Hence, although the Dirichlet process is centered around the baseline, in the sense that $\text{E}(P) = P_0$, the tails of P tend to be much lighter than those of P_0. This deficiency has fundamental consequences for modeling and inference. For example, it prevents the Dirichlet process from being a suitable candidate for modeling discrete extreme values and heavy-tailed data. While it is always possible to convolve the process with a continuous heavy-tailed kernel, there are compelling reasons to avoid this approach: I) it may result in posterior inconsistency of the tail index (Li et al. 2019); II) it may yield a super-heavy-tailed distribution apriori (Palacios et al. 2025); III) practical problems often demand modeling discrete extremes.

1.2 Preview of selected consequences of the main results

Figure 1.1 illustrates one of the many implications of our main results. Specifically, it shows that the trajectories of the tails of the process at extreme levels do not concentrate around the centering distribution, but rather around another function of it, as predicted by our main results. Figure 1.1 also illustrates another consequence of our main findings: The right tail of a Pitman–Yor process is heavy-tailed if its centering distribution is heavy-tailed, though with a thinner tail; this applies both *a priori* as well as *a posteriori*. Notably, the Dirichlet process is the only member of this class that fails to exhibit this desirable property (as its tails are always exponentially lighter than those of the centering distribution). Below, we also derive asymptotic envelopes for the corresponding Pitman–Yor process.

Finally, although our primary focus is on the tails of the prior process, we also discuss the implications of our findings in terms of the tails of the posterior.

1.3 On heavy-tails and M-variation

Before presenting our main findings, a few brief remarks on heavy tails are in order. Recall that a distribution function $F(x) = \mathbb{P}(X \leq x)$, with $x^* = \sup\{x : F(x) < 1\} = \infty$, is said to be heavy-tailed if it has a tail heavier than every exponential distribution (Nair et al. 2022, Definition 1.1). Formally,

$$\limsup_{x \to \infty} \frac{1 - F(x)}{e^{-\lambda x}} = \limsup_{x \to \infty} \frac{\bar{F}(x)}{e^{-\lambda x}} = \infty,$$

for all $\lambda > 0$, where $\bar{F}(x) \equiv 1 - F(x)$ is the tail function. Here, we employ two general families of heavy-tailed distributions based on Karamata's theory of regular variation (Bingham et al.

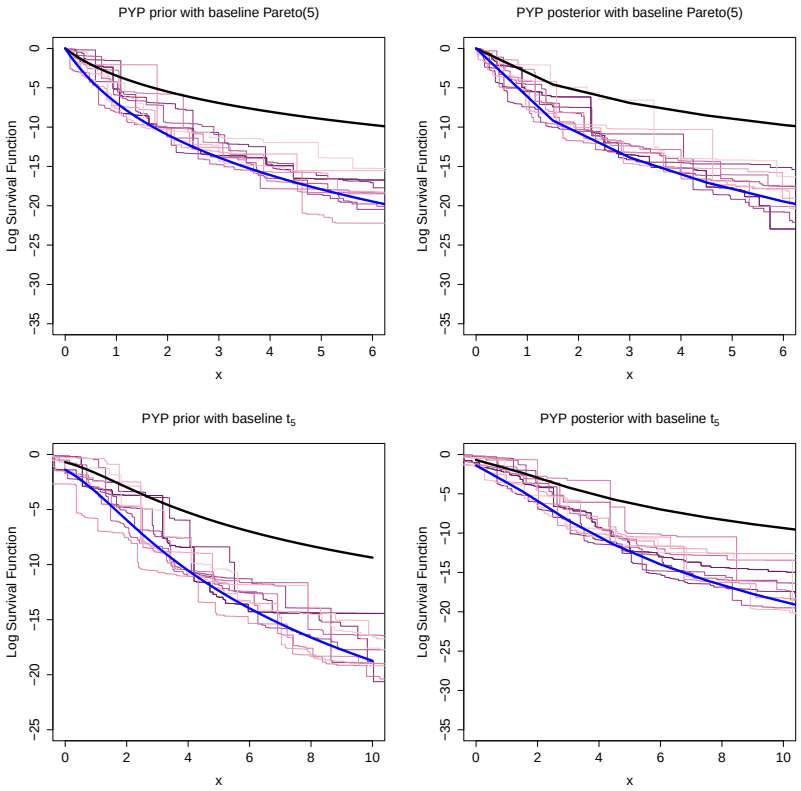

Fig. 1.1 Centering log survival function ($\log \bar{G}_0$; black) vs $\log \bar{G}_0^{1/\sigma}$ (blue) which follows from main results in Section 3; these are plotted alongside random trajectories of log survival functions from PYP($\sigma = 0.5, \theta = 1, P_0$). Top: P_0 is a Pareto distribution with tail index 5. Bottom: P_0 is a t-distribution with 5 degrees of freedom.

1989). First, a tail function $\bar{F}(x)$ is said to be regularly varying with tail index $\alpha > 0$, i.e., $\bar{F} \in \mathrm{RV}_{-\alpha}$, if

$$\lim_{x\to\infty} \frac{\bar{F}(xt)}{\bar{F}(x)} = \frac{1}{t^\alpha}, \tag{1.3}$$

for any $t > 0$; the smaller the tail index α, the slower the decay of the tail function $\bar{F}(x) \equiv 1 - F(x)$ to 0 as $x \to \infty$, indicating a more heavy-tailed distribution. Finally, we recall that the notion of regular variation extends beyond tail functions to general eventually positive functions, and that the notation introduced above will also be used for such functions.

The second class that relates with our main contributions is that of tail functions bounded by two regularly varying functions with the same tail index; a broader class, the M_{RV} class, was recently characterized by Cadena et al. (2017), and while their focus and goals differ from ours, some of their results nicely connect with our framework. Given the similarities between our specific instance of the M_{RV} class and regular variation, we will refer to it as M-variation. Formally, $\bar{F} \in \mathrm{MV}_{-\alpha}(\mathcal{L},\mathcal{U})$, if $\mathcal{L} \in \mathrm{RV}_{-\alpha}$, $\mathcal{U} \in \mathrm{RV}_{-\alpha}$, and

$$\mathcal{L} \preceq \bar{F} \preceq \mathcal{U}, \tag{1.4}$$

where $\alpha > 0$ is the decay rate and $A \preceq B$ stands for $\limsup_{x\to\infty} A(x)/B(x) < \infty$. Here, $\mathcal{L}$ and $\mathcal{U}$ are not required to be distribution functions; however, by the definition of regular variation, they are required to be positive when evaluated at sufficiently large arguments. Trivially, $\mathrm{RV}_{-\alpha} \subset \mathrm{MV}_{-\alpha}$. Tail functions in $\mathrm{RV}_{-\alpha}$ and in $\mathrm{MV}_{-\alpha}$ are heavy-tailed as it is well-known that if $\bar{F} \in \mathrm{RV}_{-\alpha}$, then $\bar{F}(x)/e^{-\lambda x} \to \infty$ as $x \to \infty$ (e.g., Nair et al. 2022, Lemma 2.9), and we prove in Lemma 1 that a similar result holds for M-variation.

1.4 Outline

Section 2 briefly reviews subordinator representations and introduces a novel bound-inducing map which is key to our framework. Section 3 introduces the main results, whereas numerical illustrations are provided in Section 4. We conclude the paper in Section 5. Main proofs are included in the Appendix, whereas additional numerical illustrations are provided in the online supplementary materials.

2 Subordinator-type representations and bounds

2.1 Characterizations based on normalized subordinators

A subordinator, $\{S(t) : t \geq 0\}$, is an increasing stochastic process over the positive real line with independent and stationary increments (Applebaum 2009). Subordinator-type representations of random probability measures facilitate the analysis of tail behavior, as such behavior can be characterized through the short-run properties of subordinators.

According to the Lévy–Khintchine representation (Bertoin 1999), a subordinator is fully characterized by its Laplace exponent:

$$\Phi(\lambda) = \mathtt{k} + \mathtt{d}\lambda + \int_0^\infty (1 - e^{-\lambda u})\, L(\mathrm{d}u), \quad \lambda \geq 0,$$

where $\mathtt{k} \geq 0$ is the killing rate, $\mathtt{d} \geq 0$ is the drift coefficient, and L is a Lévy measure on $(0, \infty)$ that governs the law of the increments. This measure satisfies the condition $\int_0^\infty \min(1, u) L(\mathrm{d}u) < \infty$. The expected exponential of the scaled subordinator is given by $\mathrm{E}[\exp\{-\lambda S(t)\}] = \exp\{-t\Phi(\lambda)\}$ for $t \geq 0$. For any subordinator $S(t)$ with a positive drift coefficient $\mathtt{d}$, the following limit holds:

$$\lim_{t \to 0^+} \frac{S(t)}{t} = \mathtt{d}, \quad \text{a.s.}$$

For the gamma and stable subordinators, which respectively correspond to the Dirichlet process and the stable process, it holds that $\mathtt{k} = \mathtt{d} = 0$; indeed, this applies to the entire Pitman–Yor class. Throughout, we denote the tail function of the centering, P_0, as $\bar{G}_0(x) = 1 - G_0(x)$; with $G_0(x) = P_0\{[0, x]\}$. Both $P \sim \mathrm{PYP}(\sigma, \theta, P_0)$ and P_0 are supported over the same set, and thus the right endpoints of P and P_0 coincide.

Example 1 (Dirichlet process; PYP$(0, \theta, P_0)$) A Dirichlet process P with baseline measure P_0 and total mass parameter θ, can be equivalently defined using its tail function $\bar{G}$ via the following subordinator-type representation

$$\bar{G}(x) = \frac{S(\theta \bar{G}_0(x))}{S(\theta)}, \tag{2.1}$$

where $\{S(t)\}_{0 \leq t \leq \theta}$ is a Gamma process with Lévy intensity $L(\mathrm{d}u) = u^{-1} e^{-u} \mathrm{d}u$.

Example 2 (Stable process; PYP$(\sigma, 0, P_0)$) Another instance of a Pitman–Yor process, P, that has a representation as a normalized subordinator arises when $\theta = 0$ and $\sigma \in [0, 1)$; in this case the tail function of P has a subordinator-type representation as

$$\bar{G}(x) = \frac{S(\bar{G}_0(x))}{S(1)}, \tag{2.2}$$

where $\{S(t)\}_{0 \leq t \leq \theta}$ is a σ-stable process with Lévy intensity $L(\mathrm{d}u) = u^{-(1+\sigma)} \sigma / \Gamma(1 - \sigma)$.

As can can be seen from the subordinator representations (2.1) and (2.2), there is no connection between S and G_0. The intuition behind these representations can be readily understood by considering, for instance, a standard uniform baseline, i.e., $G_0(x) = x$ for $x \in [0, 1]$. In that case, Equation (2.2) becomes $G(x) = S(x)/S(1)$ so that the random distribution function for the stable process is simply given by normalizing the subordinator, with $S(1)$ acting as the random normalizing variable. An analogous interpretation applies to the Dirichlet process.

Relying on these subordinator representations, (Doss and Sellke 1982; Palacios et al. 2025) studied the tails, $\bar{G}(x) = P\{(x, \infty)\}$, of Dirichlet processes and normalized stable subordinators in comparison with those of baseline measure. Explicitly, Doss and Sellke (1982) proved that if P is a Dirichlet process with baseline measure P_0 and total mass parameter θ, with probability one there exists $u \in \mathbb{R}_+$ such that

$$\exp\left\{-\frac{s \log|\log \theta \bar{G}_0(x)|}{\theta \bar{G}_0(x)}\right\} \leq \bar{G}(x) \leq \exp\left\{-\frac{1}{\theta \bar{G}_0(x) |\log \theta \bar{G}_0(x)|^r}\right\}, \quad x \geq u, \tag{2.3}$$

for $s, r > 1$ and where $\bar{G}_0(x) = P_0\{(x, x^*)\}$. Thus the tails of a Dirichlet process will always be exponentially thinner than those of its baseline measure.

For their part, Palacios et al. (2025) showed that if P is a Pitman–Yor process with parameter $\theta = 0$, then with probability one there exists $u \in \mathbb{R}_+$, such that for all $\rho > 0$,

$$\left[\frac{\sigma(1-\sigma)^{(1-\sigma)/\sigma}}{\phi(1)} - \rho\right] \bar{G}_0(x)^{1/\sigma} [\log|\log \bar{G}_0(x)|]^{1-1/\sigma} \leq \bar{G}(x) \leq \bar{G}_0(x)^{1/\sigma} |\log \bar{G}_0(x)|^{r/\sigma}, \tag{2.4}$$

for $x \geq u$ and $r > 1$; here, $\bar{G}_0(x) = P_0\{(x, x^*)\}$, while $\phi(1)$ is a one-sided σ-stable random variable. This in turn yields that if $\bar{G}_0$ is regularly varying with tail index α, i.e.

$$\lim_{x\to\infty} \frac{\bar{G}_0(xt)}{\bar{G}_0(x)} = t^{-\alpha},$$

then $\bar{G}$ will be M-varying with tail index α/σ, a.s. In particular, if the baseline, P_0, is heavy-tailed so will be the corresponding Pitman–Yor process with parameters $\theta = 0$ and $\sigma \in (0, 1)$ a.s.

2.2 Weight transport maps

While the subordinator-type representation of random probability measures has proven to be convenient to analyze the tail behaviour, it is not exhaustive. Meaning that not all random probability measures enjoy such representation. In contrast, every discrete random distribution can be approximated via a stick-breaking decomposition as in (1.2) as a consequence of the full support of the PYP (Bissiri and Ongaro 2014, Proposition 6).

A main goal of this paper is to pioneer the study of tail behaviour via the stick–breaking construction. To this aim we focus on Pitman–Yor processes and a class of bound-inducing maps, T, that arise from the stick–breaking representation and satisfy the following key properties: (i) if P is a Pitman–Yor process, then $T(P)$ is another Pitman–Yor process with different parameters, (ii) the map T preserves the behaviour of the tail. These bound-inducing maps, to be introduced below, then allow us to characterize the tail of a process $T(P)$ in terms of the tail of P.

Fix $\sigma \in (0, 1)$ and $\theta \geq 0$. For each $\tau \in (-\sigma, \sigma)$ and $j \in \mathbb{N}$ define the function $\Upsilon_{\tau,j} : [0, 1] \to [0, 1]$:

$$\Upsilon_{\tau,j} = \mathcal{I}^{-1}_{1-\sigma;\theta+j\sigma} \circ \mathcal{I}_{1-\sigma;\theta+j\sigma+\tau}. \tag{2.5}$$

Here, $\mathcal{I}_{\alpha;\beta}$ stands for the regularized incomplete Beta function, that is

$$\mathcal{I}_{\alpha;\beta}(v) = \frac{1}{\mathcal{B}(\alpha, \beta)} \int_0^v x^{\alpha-1}(1 - x)^{\beta-1} \mathrm{d}x,$$

and $\mathcal{I}^{-1}_{\alpha;\beta}$ its inverse function, i.e. $\mathcal{I}^{-1}_{\alpha;\beta}(u) = v$ if and only if $\mathcal{I}_{\alpha;\beta}(v) = u$.

Some comments on properties of (2.5) are in order. Clearly, if $\nu \sim \mathsf{Be}(1 - \sigma, \theta + j\sigma)$, then $\Upsilon_{\tau,j}(\nu) \sim \mathsf{Be}(1 - \sigma, \theta + \tau + j\sigma)$. Although optimal transport theory is not directly relevant to our work, the function $\Upsilon_{\tau,j}$ is an optimal transport map between the measures

$$\gamma[0, v] \equiv \mathcal{I}_{1-\sigma;\theta+j\sigma}(v), \quad \mu[0, v] \equiv \mathcal{I}_{1-\sigma;\theta+j\sigma+\tau}(v).$$

See, e.g., Santambrogio (2015, Theorem 2.9). Also, the weight transport function $\Upsilon_{\tau,j}$ obeys the properties below.

Proposition 1 (Weight transport)
If $\tau \in [0, \sigma)$, then:

(i) $\Upsilon_{\tau,j}(v) \leq v$ *for all* $j \geq 1$.
(ii) *For every* $m \geq 1$ *there exist* $n \geq 1$ *such that* $\Upsilon_{\tau,j}(v) \geq (1 - 1/m)v$, *for every* $j \geq n$.

If $\tau \in (-\sigma, 0]$, then similar properties hold:

(i) $\Upsilon_{\tau,j}(v) \geq v$ *for all* $j \geq 1$.
(ii) *For every* $m \geq 1$ *there exist* $n \geq 1$ *such that* $\Upsilon_{\tau,j}(v) \leq (1 - 1/m)^{-1}v$, *for every* $j \geq n$.

The following family of closed functions transforms a Pitman–Yor process with positive precision parameter into another element of the class, and plays a key role in our framework:

$$T^{(\tau)}\left(\sum_{j=1}^{\infty} \pi_j \delta_{\xi_j}\right) = \sum_{j=1}^{\infty} \pi_j^{(\tau)} \delta_{\xi_j}, \quad \tau \in (-\sigma, \sigma). \tag{2.6}$$

Here,

$$\pi_1 = \nu_1, \quad \pi_1^{(\tau)} = \Upsilon_{\tau,j}(\nu_1), \quad \pi_j = \nu_j \prod_{i=1}^{j-1}(1 - \nu_i), \quad \pi_j^{(\tau)} = \Upsilon_{\tau,j}(\nu_j) \prod_{i=1}^{j-1}\{1 - \Upsilon_{\tau,j}(\nu_i)\}$$

with

$$\nu_j \sim \mathsf{Be}(1-\sigma, \theta + j\sigma), \qquad \Upsilon_{\tau,j}(\nu_j) \sim \mathsf{Be}(1-\sigma, \theta + j\sigma + \tau).$$

Thus, if $P \sim \mathrm{PYP}(\sigma, \theta, P_0)$ then $T^{(\tau)}(P) \sim \mathrm{PYP}(\sigma, \theta + \tau, P_0)$; for instance, if $\tau \in [0, \sigma)$, then $T^{(\tau)}(P)$ yields a Pitman–Yor process with the same centering and discount parameter, but with a slightly larger precision. We refer to $T^{(\tau)}$ in (2.6) as a bound-inducing map as such functions can be used to bound the tail of any Pitman–Yor process by that of two other Pitman–Yor processes as shown in the next proposition.

Proposition 2 (Weight transport bounds) *Let $P \sim PYP(\sigma, \theta, P_0)$ be a Pitman–Yor process over $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ with parameters $\sigma \in [0, 1)$ and $\theta \geq 0$. For every $0 < \varepsilon < \sigma$ and $0 < \eta < \sigma$ there exist two Pitman–Yor process $P^{(\varepsilon)} \sim PYP(\sigma, \theta + \varepsilon, \theta, P_0)$ and $P^{(-\eta)} \sim PYP(\sigma, \theta - \eta, \theta, P_0)$, such that for every $m \geq 1$, there exist a positive random variable $\xi^{(m)}$, for which it holds*

$$(1 - 1/m)\bar{G}^{(-\eta)}(x) \leq \bar{G}(x) \leq (1 - 1/m)^{-1}\bar{G}^{(\varepsilon)}(x), \quad x \geq \xi^{(m)}, \quad \text{a.s.} \tag{2.7}$$

where $\bar{G}$, $\bar{G}^{(\varepsilon)}$, and $\bar{G}^{(-\eta)}$ are the tail functions of P, $P^{(\varepsilon)}$ and $P^{(-\eta)}$, respectively.

The connection between the bound-inducing map $T^{(\tau)}$ and Proposition 2 arises from the fact that the tail functions of

$$P^{(\varepsilon)} = T^{(\varepsilon)}(P), \quad P^{(-\eta)} = T^{(-\eta)}(P), \tag{2.8}$$

satisfy (2.7). While Proposition 2 explains the role of $T^{(\varepsilon)}$ for $\varepsilon \in (-\sigma, \sigma)$, the next result motivates the definition of $T^{(\varepsilon)}$ for $\varepsilon \in \{-\sigma, \sigma\}$.

Proposition 3 (Pitman–Yor couplings) *Let $P = \sum_{j=1}^{\infty} \pi_j \delta_{\xi_j} \sim PYP(\sigma, \theta, P_0)$ be a Pitman–Yor process over $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$, with discount and precision parameters $\sigma \in [0, 1)$ and $\theta > -\sigma$, and where $\boldsymbol{\pi} = (\pi_j)_{j=1}^{\infty}$ are as in (1.2).*

(a) *There exists a Pitman–Yor process $P^{(\sigma)}$, with parameters $(\sigma, \theta + \sigma, P_0)$ such that*

$$\bar{G}(x) = (1 - \nu_1)\bar{G}^{(\sigma)}(x), \quad x > \xi_1, \quad a.s.$$

where $\bar{G}$ and $\bar{G}^{(\sigma)}$ are the tail functions of P and $P^{(\sigma)}$, respectively, and $\nu_1 = \pi_1$.

(b) *If $\theta > 0$, there exists a Pitman–Yor process $P^{(-\sigma)}$ with parameters $(\sigma, \theta - \sigma, P_0)$ such that*

$$\bar{G}(x) = (1 - \nu_0)^{-1}\bar{G}^{(-\sigma)}(x), \quad x \geq \xi_0, \quad a.s.$$

where $\bar{G}$ and $\bar{G}^{(-\sigma)}$ are the tail functions of P and $P^{(-\sigma)}$, respectively, and where $\xi_0 \sim P_0$, and $\nu_0 \sim \mathsf{Be}(1 - \sigma, \theta)$ are independent random variables.

As a byproduct of Proposition 3, it follows that

$$\begin{cases} T^{(\sigma)}(P) = \frac{P - \pi_1 \delta_{\xi_1}}{1 - \pi_1}, \\ T^{(-\sigma)}(P) = \nu_0\, \delta_{\xi_0} + (1 - \nu_0)P, \end{cases} \quad \text{with } \nu_0 \sim \mathsf{Be}(1 - \sigma, \theta), \quad \xi_0 \sim P_0. \tag{2.9}$$

In words, $T^{(\sigma)}$ is equivalent to removing an atom and normalizing, whereas $T^{(-\sigma)}$ can be understood as adding an atom through a random mixture.

3 Statement of main results

3.1 Right tail

Equipped with the novel bound-inducing framework from Section 2 and the resulting bounds, we are now ready to present the main results; as will be seen in Appendix C, the proofs of these results illustrate an application of the findings from the previous section. The following result establishes asymptotic envelopes for the tails of a random probability measure that follows a Pitman–Yor process with $\sigma > 0$. The case $\sigma = 0$ is well known and has been studied by Doss and Sellke (1982).

Theorem 1 (Asymptotic envelopes for tail of PYP) *Let $P = \sum_{j=1}^{\infty} \pi_j \delta_{\xi_j} \sim PYP(\sigma, \theta, P_0)$ be a Pitman–Yor process over $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ and consider its tail function $\bar{G}$. Then there exists random variable $\kappa(\theta, \sigma)$ taking values in $(0, \infty)$ such that*

$$\liminf_{x\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} = \kappa(\theta,\sigma) \quad \text{a.s.},$$

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & r \le 1, \end{cases} \quad \text{a.s.},$$

with $L(t) = t^{1/\sigma}\{\log|\log t|\}^{1-1/\sigma}$ for $0 < t < e^{-1}$, and $U_r(t) = t^{1/\sigma}|\log t|^{r/\sigma}$ for $0 < t < e^{-r}$.

Theorem 1 extends the bounds of Theorem 2 in Palacios et al. (2025), which were established only for the stable process case $\theta = 0$, to the more general setting $\theta > -\sigma$ of the Pitman–Yor process. In this broader setting, the bounds involve a random variable $\kappa(\sigma, \theta)$ taking values in a subset of $(0, \infty)$; the support of $\kappa(\sigma, \theta)$ changes with the parameters (σ, θ) as revealed in the proof of Theorem 1. The precise distribution of κ is not important here, what matters is simply that such a κ exists to attain the bounds. Since κ does not depend on x (even if random), it plays the role of a constant with respect to x, so the essential behaviour of the bound is determined by the x-dependent terms.

A key takeaway from Theorem 1 is the following result.

Corollary 1 *Let $P = \sum_{j=1}^{\infty} \pi_j \delta_{\xi_j} \sim PYP(\sigma, \theta, P_0)$ be a Pitman–Yor process over $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ and consider its tail function $\bar{G}$. Then, there exist two random variables ξ and $\gamma(\theta, \sigma)$ taking values in $(0, \infty)$ such that*

$$\mathbb{L}(x) \equiv \gamma(\theta,\sigma) L\{\bar{G}_0(x)\} \le \bar{G}(x) \le U_r\{\bar{G}_0(x)\} \equiv \mathbb{U}_r(x), \quad \forall x > \xi \quad a.s. \tag{3.1}$$

with $r > 1$ where L and U_r are as in Theorem 1.

This corollary provides the explicit translation of the lim inf and lim sup bounds in Theorem 1 into concrete upper and lower bounds. It is important to note that the random variable $\gamma(\theta, \sigma)$ appearing here is different from the $\kappa(\sigma, \theta)$ in Theorem 1. Once again, the precise distribution of this random variable is not essential, what matters is only that it exists and is finite. For the purpose of the bounds, it can be regarded as a constant with respect to x, so the key behaviour is driven by the terms depending on x, which dominate as $x \to \infty$.

Theorem 2 (Regularly varying tails) *Let $P = \sum_{j=1}^{\infty} \pi_j \delta_{\xi_j} \sim PYP(\sigma, \theta, P_0)$ be a Pitman–Yor process over $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. Let $\bar{G}$ and $\bar{G}_0$ be the tail functions of P and P_0, and suppose that $\bar{G}_0(x)$, is a regularly varying function with tail index α. If $\sigma > 0$, then $\bar{G}(x)$ is M-varying, that is, $\bar{G} \in \mathrm{MV}_{-\alpha/\sigma}(\mathbb{L}, \mathbb{U}_r)$, for $r > 1$, with $\mathbb{L}$ and $\mathbb{U}_r$ as in (3.1); hence, for any $\varepsilon > 0$,*

$$\lim_{x\to\infty} \frac{\bar{G}(x)}{x^{-\alpha/\sigma+\varepsilon}} = 0, \qquad \lim_{x\to\infty} \frac{\bar{G}(x)}{x^{-\alpha/\sigma-\varepsilon}} = \infty, \qquad \text{a.s.} \tag{3.2}$$

For the case $\sigma = 0$, the tail of P is exponentially thinner than that of P_0.

In essence, Theorem 2 shows that if the centering measure P_0 of a Pitman–Yor process $P \sim \mathrm{PYP}(\sigma, \theta, P_0)$ is regularly-varying (heavy-tailed) and $\sigma > 0$, then P will also be M-varying (heavy-tailed), albeit with a relatively lighter tail; a formal proof that M-varying tails are themselves heavy-tailed is given in Lemma 1 in Appendix A. As noted by Cadena et al. (2017), (3.2) holds if and only if there exist slowly varying functions L_1 and L_2, such that as $x \to \infty$,

$$\frac{\bar{G}(x)}{x^{-\alpha}L_1(x)} \to 0, \qquad \frac{\bar{G}(x)}{x^{-\alpha}L_2(x)} \to \infty.$$

As a byproduct, Theorem 2 provides clarification and a more precise formulation of Corollary 3 in Palacios et al. (2025). While the latter result suggests otherwise, only M-variation (and not regular variation) is generally guaranteed to hold in the context of Normalized Generalized Gamma processes, under assumptions similar to those in Theorem 2; this can be shown using an argument similar to that in Section 2. The oversight in Palacios et al. (2025) is not unwarranted, as the two classes (M-variation and regular variation) share notable similarities, including a comparable representation theorem (Cadena et al. 2017, Theorem 1.2):

- $\bar{F} \in \mathrm{MV}_{-\alpha}$ if and only if $\bar{F}(x) = \exp\{a(x) + \int_u^x b(t)/t\,\mathrm{d}t\}$, for $x \ge u$, where $a(x)/\log x \to 0$ and $b(x) \to -\alpha$; whereas,
- $\bar{F} \in \mathrm{RV}_{-\alpha}$ if and only if $\bar{F}(x) = \exp\{a(x) + \int_u^x b(t)/t\,\mathrm{d}t\}$, for $x \ge u$, where $a(x) \to 0$ and $b(x) \to -\alpha$.

3.2 Tails of posterior

As it will be shown below, all results presented above have consequences for the tails of the posterior of the Pitman–Yor process. From Corollary 20 of Pitman (1996), the posterior law of the Pitman–Yor process is known to behave as a mixture of random measures, where one of the components is itself a Pitman–Yor process. That is, if $P \sim \text{PYP}(\sigma, \theta, P_0)$, and $X_1, \ldots, X_n \mid P \overset{\text{iid}}{\sim} P$, then the posterior distribution of P based on $X_1, \ldots, X_n$ is the distribution of the random measure

$$\mathbb{P}_n = R_n \sum_{j=1}^{K_n} \hat{\pi}_j \delta_{\tilde{X}_j} + (1 - R_n) Q_n. \tag{3.3}$$

Here, $\tilde{X}_1, \ldots, \tilde{X}_{K_n}$ are the distinct values of $X_1, \ldots, X_n$, with multiplicities $N_{1,n}, \ldots, N_{K_n,n}$, that is $N_{j,n} = |\{i \leq n : X_i = \tilde{X}_j\}|$, while

$$R_n \sim \mathsf{Be}(n - K_n\sigma, \theta + K_n\sigma), \quad (\hat{\pi}_1, \ldots, \hat{\pi}_{K_n}) \sim \text{Dir}_{K_n}(N_{1,n} - \sigma, \ldots, N_{K_n,n} - \sigma),$$

and $Q_n \sim \text{PYP}(\sigma, \theta + K_n\sigma, P_0)$ are all independently distributed (Ghosal and Van der Vaart 2017, Theorem 14.37). It follows from (3.3) that

$$\mathbb{P}_n\{(M_n, \infty)\} = (1 - R_n)\, Q_n\{(M_n, \infty)\}, \tag{3.4}$$

where $M_n = \max\{X_1, \ldots, X_n\}$ is the sample maximum. A key takeway from (3.4) is that for any finite n, the tail of $\mathbb{P}_n$ is tantamount to that of the $Q_n \sim \text{PYP}(\sigma, \theta + K_n\sigma, P_0)$; thus, it follows from Theorem 2 and (3.4), that if $P \sim \text{PYP}(\sigma, \theta, P_0)$ and $\bar{G}_0 \in \text{RV}_{-\alpha}$, then for any finite n, it holds that

$$\overline{\mathbb{G}}_n \in \text{MV}_{-\alpha/\sigma}, \tag{3.5}$$

with $\overline{\mathbb{G}}_n(x) := \mathbb{P}_n\{(x, \infty)\}$. This claim is coherent with the well-established asymptotic theory of Pitman–Yor processes (James 2008). For example, if the 'true' random probability measure P^* is discrete and it is not heavy-tailed, then $\mathbb{P}_n$ converges weakly to the non heavy-tailed P^*. However, if $\bar{G}_0 \in \text{RV}_{-\alpha}$ this convergence occurs through a sequence of distributions with M-varying tails, meaning Equation (3.5) holds for any finite n.

4 Numerical illustrations

4.1 Weight transport bounds and couplings

Let $P \sim \text{PYP}(\sigma = 0.70, \theta = 0.35, P_0 = \text{Unif}(0, 1))$. To illustrate the main ideas of Proposition 2 and Proposition 3 we consider the following transformations of P:

1. Weight transport bounds: $T^{(-\eta)}(P)$ and $T^{(\varepsilon)}(P)$, with $\eta = 0.35$ and for $\varepsilon = 0.65$.
2. Coupling: Remove an atom and normalizing, i.e, $T^{(\sigma)}(P) = (P - \pi_1\delta_{\xi_1})/(1 - \pi_1)$.

Figure 4.1 (left) shows that, in line with Proposition 2, $\bar{G}$ is eventually bounded below by $\bar{G}^{(-\eta)}$ and above by $\bar{G}^{(\varepsilon)}$. Figure 4.1 (right) illustrates that $\bar{G}(x) = (1 - \nu_1)\bar{G}^{(\sigma)}(x)$, for $x > \xi_1$, as a expected from Proposition 3. Weight transport and couplings were pivotal to our approach and the proofs of the main results. While Figure 4.1 offers valuable insights into the process's tail behavior, the following section highlights more significant consequences for the tail behavior derived from our theoretical analysis.

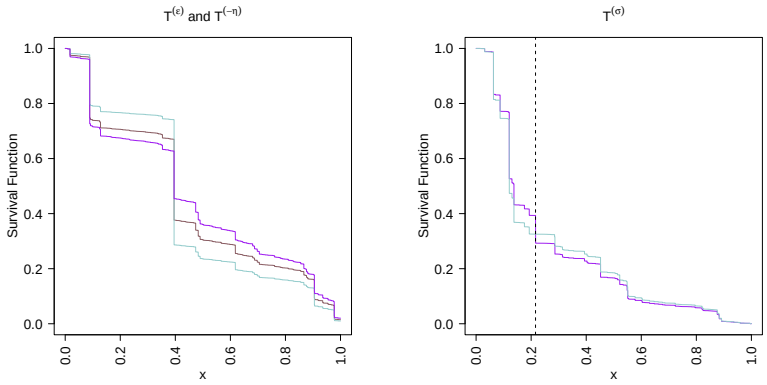

Fig. 4.1 Left: weight transport bounds $T^{(\epsilon)}(P)$ (light blue), $T^{(-\eta)}(P)$ (blue) with $\epsilon = 1-\theta$ and $\eta = \theta$. Right: Coupling $T^{(\sigma)}(P)$ (blue), with a vertical (dashed) line in ξ_1. Both cases with $P \sim \mathrm{PYP}(\sigma = 0.70, \theta = 0.35, P_0 = U(0,1))$ (magenta).

4.2 Tails of prior and posterior process

To illustrate some consequences of Section 3, we consider the following settings where we now assume $P \sim \mathrm{PYP}(\sigma = 0.5, \theta = 1, P_0)$, where P_0 is a:

1. Pareto distribution with $\alpha = 5$ (i.e., $P_0 \in \mathrm{RV}_{-5}$).
2. t-distribution with 5 degrees of freedom (i.e., $P_0 \in \mathrm{RV}_{-5}$).
3. log-gamma distribution with shapelog 1 and ratelog 5 (i.e., $P_0 \in \mathrm{RV}_{-5}$).
4. F-distribution with degrees of freedom $(5, 5)$. (i.e., $P_0 \in \mathrm{RV}_{-5/2}$).

Figures 1.1 and 4.2 depict some illustrations of our main results based on these settings. As anticipated in the Introduction, the prior and posterior trajectories from the log survival functions follow closely the rate of decay of $\log \bar{G}_0^{1/\sigma}$, which is the leading term of the upper and lower bound in Theorem 2 and is indeed $\mathrm{RV}_{-5/\sigma}$. These figures also illustrate that, in line with our theory, centering a PYP (excluding the DP) on a heavy-tailed distribution yields an heavy-tailed random probability measure, albeit with a slightly lighter tail. By contrast, when considering the DP, as shown in the supplementary materials, the tails of the prior and posterior are much lighter than those of the centering distribution.

5 Closing remarks

This paper contributes to a broader understanding of a topic that in our view warrants further attention within the extreme value theory community—namely, the study of the tails of random probability measures.

While several of the aforementioned processes are widely adopted within the Bayesian community, and Bayesian modeling of extremes is by now well-established (e.g., Reich and Shaby 2012; Dombry et al. 2017; Padoan and Rizzelli 2022, 2024), the study of the tail behavior of random probability measures remains largely unexplored. This is not due to a lack of interest, but rather to the technical challenges posed by tools such as regular variation, which are not yet standard in the Bayesian framework.

Based on a novel weight-transport based approach for studying the tails of a random probability measure, we have derived asymptotic envelopes for the entire class of Pitman–Yor processes. In particular, the novel asymptotic envelopes derived from Theorem 1 imply that the right tail of a Pitman–Yor process is heavy-tailed if its centering distribution is also heavy-tailed, as demonstrated in Theorem 2; notably, the Dirichlet Process is the only exception to this property as it is known since Doss and Sellke (1982) that its tails are exponentially lighter than those of the centering. We believe the approach in Section 2.2 may have broader interest, for instance as a framework to study the tails of other random probability measures, including stick-breaking processes. While some results from James (2013) may potentially offer a different route to streamline some arguments in our proofs, our bound-inducing construction was conceived with the tail in mind and hence it may provide a tool for investigating extremal properties of random probability measures in more general settings.

Although not explored in this paper, our findings could have fundamental implications for modeling. There is a scarcity of models for discrete extreme values, as evidenced by some of the

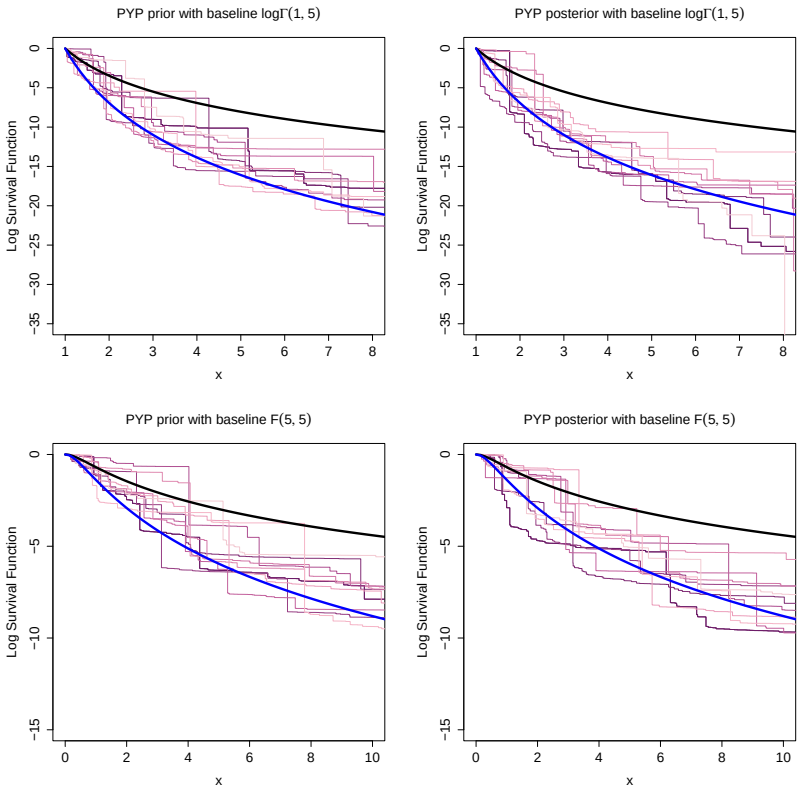

Fig. 4.2 Centering log survival function ($\log \bar{G}_0$; black) vs $\log \bar{G}_0^{1/\sigma}$ (blue) which follows from main results in Section 3; these are plotted alongside random trajectories of log survival functions from PYP($\sigma = 0.5, \theta = 1, P_0$). Top: P_0 is a log-gamma distribution with shapelog 1 and ratelog 5. Bottom: P_0 is an F-distribution with degrees of freedom $(5, 5)$.

most fundamental monographs on extreme value theory—such as those of Coles (2001), Beirlant et al. (2004), de Haan and Ferreira (2006), and Resnick (2007)—which mainly emphasize models for continuous data.

Our main results—combined with the fact that the Pitman–Yor process (PYP) is consistent provided the 'true' limiting measure is itself discrete (James 2008)—suggest that the PYP emerges as a natural candidate for modeling univariate discrete extremes in heavy-tailed settings.

We close the paper with some additional comments on future research. Firstly, a theoretical study of the tails of hierarchical processes, such as the hierarchical PYP (Lim et al. 2016) and the hierarchical DP (Teh 2006), has yet to be conducted; the line of attack proposed here might potentially be helpful in devising similar results for these processes. Secondly, it seems natural to ask whether similar asymptotic envelopes for the tails of the random probability measure in Theorem 1 could also be obtained for other functionals, such as mixtures. Finally, it remains an open question whether similar bound-inducing maps, like those presented herein, can unlock a deeper understanding of the tails of the entire class of stick-breaking processes (Ishwaran and James 2001) or of species sampling processes (Pitman 1996). M-variation, as framed here, may be regarded as a heavy-tailed dual to the class of lower and upper bounds given by Weibull tail distributions considered by Vladimirova et al. (2021); links with O-regular variation (Bingham et al. 1989, Chapter 2) appear natural and, in our view, warrant further investigation. Finally, as noted in Section 2.2, $\Upsilon_{\tau,j}(v)$ itself is an optimal transport, which raises the question of whether transport maps between random probability measures could be broadly interesting for studying their tails. We leave these open problems for future analysis.

Appendix

Appendix A. Auxiliary lemmata

Lemma 1 establishes that tail functions in the M-varying class, as defined in (1.4), are heavy-tailed; note the 'lim' in (5.1) instead of simply 'lim sup', interestingly the same holds for regular

variation (Nair et al. 2022, Lemma 2.9). Lemma 2 gathers two well-known results on lower and upper envelopes of stochastic processes over the short-run which can be found in Bertoin (1999, Theorem 11) and Sato (1999, Proposition 47.16). Lemma 3 is a fundamental result on the regularized incomplete Beta function that plays a central role in our derivations; since it does not appear to be documented in the literature, we provide its proof below.

Lemma 1 *Let $\bar{F} \in \mathrm{MV}_{-\alpha}(\mathcal{L},\mathcal{U})$ be a tail function, with $\alpha > 0$. Then, for any $\lambda > 0$,*

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} = \infty. \tag{5.1}$$

Proof Since $\mathcal{L} \preceq \bar{F}$, and $\bar{F}, \mathcal{L} > 0$, it follows that $0 \leq \inf_{u\geq 0} \sup_{x\geq u} \mathcal{L}(x)/\bar{F}(x) < \infty$. Hence, for every $\varepsilon > 0$ there exist $u > 0$ such that $\mathcal{L}(x) \leq \varepsilon \bar{F}(x)$ for all $x > u$. This together with the fact that $\mathcal{L}$ is heavy-tailed, as by assumption $\mathcal{L} \in \mathrm{RV}_{-\alpha}$, yield

$$\liminf_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} \geq \varepsilon^{-1} \liminf_{x\to\infty} \frac{\mathcal{L}(x)}{e^{-\lambda x}} = \infty. \tag{5.2}$$

That is, $\lim_{x\to\infty} \bar{F}(x)/e^{-\lambda x} = \infty$.

Lemma 2 *The following results hold:*

a) If $\{S(t) : t \geq 0\}$ is a subordinator with Laplace exponent $\Phi \in \mathrm{RV}_\sigma$, with $\sigma \in (0,1)$, then $\liminf_{t\to 0^+} |S(t)|/l(t) = \sigma(1-\sigma)^{(1-\sigma)/\sigma}$, a.s., where $l(t) = \Phi^{-1}(t^{-1}\log|\log t|)$ for $0 < t < e^{-1}$, and Φ^{-1} is the inverse function of Φ.

b) Let $\{S(t)\}$ be a σ-stable process on $\mathbb{R}$ with $0 < \sigma < 2$ and $\nu(0,\infty) > 0$. If $u(t)$ is a real-valued function that is positive, continuous, and increasing on some $(0,\delta)$, with $u(t)/\{t\log\log(1/t)\}^{1/2} \to 0$, as $t \to 0^+$, then

$$\limsup_{t\to 0^+} \frac{S(t)}{u(t)} = \begin{cases} 0, & \int_0^\delta u^{-\sigma}(t)\,\mathrm{d}t < \infty, \\ \infty, & \int_0^\delta u^{-\sigma}(t)\,\mathrm{d}t = \infty. \end{cases} \quad a.s.$$

Lemma 3 *Let $0 < \alpha \leq 1$ and $\beta > 2 - \alpha$. Then for every $n \in \mathbb{N}$ and $v \in [0,1]$*

$$\mathcal{I}_{\alpha;\beta+n}(v) \geq \mathcal{I}_{\alpha;\beta+n+1}(nv/(n+1)).$$

Proof First note that

$$\frac{\partial^2 \mathcal{I}_{\alpha;\beta}(x)}{\partial x^2} = \frac{x^{\alpha-2}(1-x)^{\beta-2}}{\mathcal{B}(\alpha,\beta)}\{(\alpha-1)(1-x) - (\beta-1)x\} \leq 0,$$

i.e. $x \mapsto \mathcal{I}_{\alpha,\beta}(x)$ is concave. Now fix $v \in (0,1)$ and $n \in \mathbb{N}$. By the mean value theorem we know that there exists $u \in (nv/(n+1), v)$ such that

$$\begin{aligned} \left.\frac{\partial \mathcal{I}_{\alpha,\beta+n+1}(x)}{\partial x}\right|_u &= \mathcal{I}'_{\alpha;\beta+n+1}(u) \\ &= \frac{\mathcal{I}_{\alpha;\beta+n+1}(v) - \mathcal{I}_{\alpha;\beta+n+1}(nv/(n+1))}{v - nv/(n+1)}, \end{aligned}$$

and since $x \mapsto \mathcal{I}_x(\alpha,\beta)$ is concave we obtain $\mathcal{I}'_{\alpha;\beta+n+1}(u) \geq \mathcal{I}'_{\alpha;\beta+n+1}(v)$. That is

$$\begin{aligned} \frac{\mathcal{I}_{\alpha;\beta+n+1}(v) - \mathcal{I}_{\alpha;\beta+n+1}(nv/(n+1))}{v(n+1)^{-1}} &\geq \mathcal{I}'_{\alpha;\beta+n+1}(v) \\ &= \frac{v^{\alpha-1}(1-v)^{\beta+n}}{\mathcal{B}(\alpha,\beta+n+1)}, \end{aligned}$$

where $\mathcal{B}$ denotes the beta function. Evidently, $(\alpha+\beta+n) \geq (n+1)$, hence

$$(\alpha+\beta+n)\left\{\frac{\mathcal{I}_{\alpha;\beta+n+1}(v) - \mathcal{I}_{\alpha;\beta+n+1}(nv/(n+1))}{v}\right\} \geq \frac{v^{\alpha-1}(1-v)^{\beta+n}}{\mathcal{B}(\alpha,\beta+n+1)}.$$

Recalling that $\mathcal{B}(a,b+1) = b\mathcal{B}(a,b)/(a+b)$ for $a,b > 0$, the above equation can be written as

$$\mathcal{I}_{\alpha,\beta+n+1}(v) - \mathcal{I}_{\alpha;\beta+n+1}(nv/(n+1)) \geq \frac{v^\alpha(1-v)^{\beta+n}}{(\beta+n)\mathcal{B}(\alpha,\beta+n)}.$$

Finally, noting that $\mathcal{I}_{a;b+1}(v) = \mathcal{I}_{a,b}(v) + \{v^a(1-v)^b\}/\{b\mathcal{B}(a,b)\}$, for $a,b > 0$, we obtain

$$\mathcal{I}_{\alpha;\beta+n}(v) \geq \mathcal{I}_{\alpha;\beta+n+1}(nv/(n+1)).$$

This proves the result for $v \in (0,1)$. The extreme cases $v \in \{0,1\}$ are trivial because $v \mapsto \mathcal{I}_v(a,b)$ is increasing, $\mathcal{I}_v(a,b) = 0$ if and only if $v = 0$ and $\mathcal{I}_1(a,b) = 1$ if and only if $v = 1$.

Appendix B. Proofs on weight transport

Proof of Proposition 1

(i) As proved by Karp (2016), for $v \in [0,1]$ and $\alpha > 0$ fixed, the mapping $\beta \mapsto \mathcal{I}_{\alpha;\beta}(v)$ is log-concave in $(0,\infty)$. Hence, it is quasi concave, i.e. for $\beta_1, \beta_2 > 0$ and $\lambda \in [0,1]$

$$\mathcal{I}_{\alpha;\lambda\beta_1+(1-\lambda)\beta_2}(v) \geq \min\{\mathcal{I}_{\alpha;\beta_1}(v), \mathcal{I}_{\alpha;\beta_2}(v)\}.$$

Putting this together with the well-known property of the regularized incomplete Beta function

$$\mathcal{I}_{\alpha;\beta+1}(v) = \mathcal{I}_{\alpha;\beta}(v) + \frac{v^\alpha (1-v)^\beta}{\beta \mathcal{B}(\alpha,\beta)} > \mathcal{I}_{\alpha;\beta}(v),$$

where $\mathcal{B}(\alpha,\beta)$ denotes the Beta function, we obtain

$$\mathcal{I}_{\alpha;\beta+\varepsilon}(v) \geq \min\{\mathcal{I}_{\alpha;\beta}(v), \mathcal{I}_{\alpha;\beta+1}(v)\} = \mathcal{I}_{\alpha;\beta}(v) \tag{5.3}$$

for each $\varepsilon \in [0,1]$ and $\beta > 0$. The choice $\alpha = 1-\sigma$ and $\beta = \theta + j\sigma$ yields

$$\mathcal{I}_{1-\sigma;\theta+j\sigma}(v) \leq \mathcal{I}_{1-\sigma;\theta+j\sigma+\varepsilon}(v).$$

Finally, being that $v \mapsto \mathcal{I}_{\alpha;\beta}(v)$ increasing so is the inverse mapping, $u \mapsto \mathcal{I}^{-1}_{\alpha;\beta}(u)$, hence we get the result

$$\begin{aligned}\Upsilon_{\varepsilon,j}(v) &= \mathcal{I}^{-1}_{1-\sigma;\theta+j\sigma+\varepsilon}(\mathcal{I}_{1-\sigma;\theta+j\sigma}(v)) \\ &\leq \mathcal{I}^{-1}_{1-\sigma;\theta+j\sigma+\varepsilon}(\mathcal{I}_{1-\sigma;\theta+j\sigma+\varepsilon}(v)) \\ &= v,\end{aligned}$$

for every $v \in [0,1]$, $\varepsilon \in [0,1]$ and $j \geq 1$. In particular if $\tau \in [0,\sigma) \subseteq [0,1]$ we get $\Upsilon_{\tau,j}(v) \leq v$. Now if $\tau \in (-\sigma, 0)$ we have that $\Upsilon^{-1}_{\tau,j} = \mathcal{I}^{-1}_{1-\sigma;\theta+j\sigma} \circ \mathcal{I}_{1-\sigma;\theta+j\sigma+\tau}$ and by the former case $\Upsilon^{-1}_{\tau,j}(v) \leq v$. Since $\Upsilon_{\tau,j}$ is increasing we get

$$v = \Upsilon_{\tau,j}(\Upsilon^{-1}_{\tau,j}(v)) \leq \Upsilon_{\tau,j}(v).$$

(ii) Set $J = \min\{j \geq 1 : (j-2)\sigma > 1\}$ and $\alpha = 1-\sigma$ so that $\theta + J\sigma > 2 - \alpha$ and $J\sigma > 1$. For $j > J$ define $n_j = \lfloor (j-J)\sigma \rfloor$ and $\beta_j = \theta + j\sigma - n_j$, where $\lfloor a \rfloor$ is the floor function. This way, for $j > (J + 1/\sigma)$ we get $n_j \geq 1$ and $\beta_j = \theta + J\sigma + (j-J)\sigma - n_j > 2 - \alpha$, with $\alpha \in (0,1)$, so by Lemma 3

$$\begin{aligned}\mathcal{I}_{\alpha;\theta+j\sigma}(v) &= \mathcal{I}_{\alpha;\beta_j+n_j}(v) \\ &\geq \mathcal{I}_{\alpha;\beta_j+n_j+1}(n_j v/(n_j+1)) \\ &= \mathcal{I}_{\alpha;\theta+j\sigma+1}(n_j v/(n_j+1)),\end{aligned} \tag{5.4}$$

for every $v \in [0,1]$. Now, (5.3) proves that the mapping $\beta \mapsto \mathcal{I}_{\alpha;\beta}(v)$ is monotonically increasing, which yields $\beta \mapsto \mathcal{I}^{-1}_{\alpha;\beta}(v)$ is monotonically decreasing. Thus $\mathcal{I}^{-1}_{\alpha;\theta+j\sigma+\varepsilon}(\hat{v}) \geq \mathcal{I}^{-1}_{\alpha;\theta+j\sigma+1}(\hat{v})$ with $\hat{v} = \mathcal{I}_{\alpha;\theta+j\sigma}(v)$, for every $\varepsilon \in [0,1]$. Moreover, since the function $v \mapsto \mathcal{I}^{-1}_{\alpha;\beta}(v)$ is increasing, by (5.4) we obtain

$$\begin{aligned}\mathcal{I}^{-1}_{\alpha;\theta+j\sigma+\varepsilon}(\mathcal{I}_{\alpha;\theta+j\sigma}(v)) &\geq \mathcal{I}^{-1}_{\alpha;\theta+j\sigma+1}(\mathcal{I}_{\alpha;\theta+j\sigma}(v)) \\ &\geq \mathcal{I}^{-1}_{\alpha;\theta+j\sigma+1}(\mathcal{I}_{\alpha,\theta+j\sigma+1}[n_j v/(n_j+1)]).\end{aligned}$$

That is

$$\Upsilon_{\varepsilon,j}(v) \geq \frac{n_j v}{n_j+1} = \left(1 - \frac{1}{n_j+1}\right) v,$$

for $\varepsilon \in [0,1]$, $v \in [0,1]$ and $j > (J + 1/\sigma)$ with $J = \min\{j \geq 1 : (j-2)\sigma > 1\}$. At this stage, recalling that $n_j = \lfloor (j-J)\sigma \rfloor$, simply take $n = \min\{j > (J+1/\sigma) : n_j + 1 \geq m\}$ so it holds

$$\Upsilon_{\varepsilon,j}(v) \geq \left(1 - \frac{1}{n_j+1}\right) v \geq \left(1 - \frac{1}{m}\right) v,$$

for $j > n$. In particular, this proves (ii) for $\tau \in [0,\sigma)$. The case $\tau \in (-\sigma, 0)$ follows by noting that $\Upsilon^{-1}_{\tau,j} = \mathcal{I}^{-1}_{1-\sigma;\theta+j\sigma} \circ \mathcal{I}_{1-\sigma;\theta+j\sigma+\tau}$ hence, there exist n such that for every $j \geq n$

$$\Upsilon^{-1}_{\tau,j}(v) \geq \left(1 - \frac{1}{m}\right) v = \left(1 - \frac{1}{m}\right) \Upsilon_{\tau,j}(\Upsilon^{-1}_{\tau,j}(v)), \quad v \in [0,1].$$

The choice $v = \Upsilon_{\tau,j}(u)$ yields $u(1-1/m)^{-1} \geq \Upsilon_{\tau,j}(u)$ for $u \in [0,1]$.

□

Proof of Proposition 2

First, we construct $P^{(\varepsilon)}$. Define $\nu_j^{(\varepsilon)} = \Upsilon_{\varepsilon,j}(\nu_j) \sim \mathsf{Be}(1-\sigma, \theta+\varepsilon+j\sigma)$ so that the random probability measure $P^{(\varepsilon)} = \sum_{j=1}^{\infty} \pi_j^{(\varepsilon)} \delta_{\xi_j}$, with $\xi_j \overset{\text{iid}}{\sim} P_0$, $\pi_1^{(\varepsilon)} = \nu_1^{(\varepsilon)}$ and $\pi_j^{(\varepsilon)} = \nu_j^{(\varepsilon)} \prod_{i<j} \left(1 - \nu_i^{(\varepsilon)}\right)$ is a Pitman–Yor process parameters $(\sigma, \theta+\varepsilon, P_0)$. Now, fix $m \geq 1$. Proposition 1 proves that there exists $n_\varepsilon \geq 1$ such that for every $j \geq n_\varepsilon$

$$\nu_j \leq (1-1/m)^{-1} \nu_j^{(\varepsilon)} \quad \text{and} \quad 1 - \nu_j \leq 1 - \nu_j^{(\varepsilon)} \tag{5.5}$$

and the second inequality holds for every $j \geq 1$. Set $\bar{\xi}_{n_\varepsilon} = \max\{\xi_1, \ldots, \xi_{n_\varepsilon}\}$ and note that for $\bar{\xi}_\varepsilon \leq x$, $\mathbb{1}_{\{\xi_j > x\}} = 0$ for every $j > n_\varepsilon$, thus, by (5.5)

$$\begin{aligned}
\bar{G}(x) &= \sum_{j>n_\varepsilon} \pi_j \mathbb{1}_{\{\xi_j > x\}} \\
&= \prod_{i=1}^{n_\varepsilon} (1-\nu_i) \sum_{j>n_\varepsilon} \nu_j \prod_{i=n_\varepsilon+1}^{j-1} (1-\nu_i) \mathbb{1}_{\{\xi_j > x\}} \\
&\leq (1-1/m)^{-1} \prod_{i=1}^{n_\varepsilon} (1-\nu_i) \sum_{j>n_\varepsilon} \nu_j^{(\varepsilon)} \prod_{i=n_\varepsilon+1}^{j-1} (1-\nu_i^{(\varepsilon)}) \mathbb{1}_{\{\xi_j > x\}} \\
&\leq (1-1/m)^{-1} \prod_{i=1}^{n_\varepsilon} \frac{(1-\nu_i)}{(1-\nu_i^{(\varepsilon)})} \sum_{j>n_\varepsilon} \nu_j^{(\varepsilon)} \prod_{i=1}^{j-1} (1-\nu_i^{(\varepsilon)}) \mathbb{1}_{\{\xi_j > x\}} \\
&= (1-1/m)^{-1} \prod_{i=1}^{n_\varepsilon} \frac{(1-\nu_i)}{(1-\nu_i^{(\varepsilon)})} \bar{G}^{(\varepsilon)}(x) \\
&\leq (1-1/m)^{-1} \bar{G}^{(\varepsilon)}(x),
\end{aligned} \tag{5.6}$$

a.s. which proves the upper bound in (2.7). It remains to construct $P^{(-\eta)}$ and prove the lower bound. Define $P^{(-\eta)} = \sum_{j=1}^{\infty} \pi_j^{(-\eta)} \delta_{\xi_j}$, with $\xi_j \overset{\text{iid}}{\sim} P_0$, with $\pi_1^{(-\eta)} = \nu_1^{(-\eta)}$, $\pi_j^{(-\eta)} = \nu_j^{(-\eta)} \prod_{i<j} \left(1 - \nu_i^{(-\eta)}\right)$ and $\nu_j^{(-\eta)} = \Upsilon_{-\eta,j}(\nu_j)$. This time we get $\nu_j^{(-\eta)} \overset{\text{ind}}{\sim} \mathsf{Be}(1-\sigma, \theta-\eta+j\sigma)$ so that $P^{(-\eta)}$ is a Pitman–Yor process with parameters $(\sigma, \theta-\eta, P_0)$. By Proposition 1 there exists $n_\eta \geq 1$ such that for every $j \geq n_\eta$

$$\nu_j^{(-\eta)} \leq (1-1/m)^{-1} \nu_j \quad \text{and} \quad 1 - \nu_j^{(-\eta)} \leq 1 - \nu_j.$$

The second inequality holds for every $j \geq 1$. Hence, similarly to (5.6) we obtain

$$\bar{G}^{(-\eta)}(x) \leq (1-1/m)^{-1} \bar{G}(x),$$

a.s. for every $x \geq \bar{\xi}_{n_\eta} = \max\{\xi_1, \ldots, \xi_{n_\eta}\}$. The final result follows by fixing $\xi^{(m)} = \max\{\bar{\xi}_{n_\varepsilon}, \bar{\xi}_{n_\eta}\} = \max\{\xi_1, \ldots, \xi_{\max\{n_\varepsilon, n_\eta\}}\}$. □

Proof of Proposition 3

(a) We construct $P^{(\sigma)}$ from $P \sim \mathrm{PYP}(\sigma, \theta, P_0)$ by removing the atom ξ_1 and normalizing. This is

$$P^{(\sigma)} = \frac{P - \pi_1 \delta_{\xi_1}}{1 - \pi_1} = (1-\nu_1)^{-1} \sum_{j=2}^{\infty} \pi_j \delta_{\xi_j}, \quad \text{a.s.}$$

and we can write $P^{(\sigma)} = \sum_{j=1}^{\infty} \pi_j^{(\sigma)} \delta_{\xi_j^{(\sigma)}}$ with $\xi_j^{(\sigma)} = \xi_{j+1}$, and

$$\pi_j^{(\sigma)} = \pi_{j+1}(1-\nu_1)^{-1} = \nu_{j+1} \prod_{i=2}^{j} (1-\nu_i), \quad j \geq 1.$$

a.s. Recalling that $\nu_{j+1} \sim \mathsf{Be}(1-\sigma, \theta+\sigma+j\sigma)$, it follows that $P^{(\sigma)}$ is a Pitman–Yor process with parameters $(\sigma, \theta+\sigma, P_0)$. Now, for $x \geq \xi_1$ we get

$$\begin{aligned}\bar{G}^{(\sigma)}(x) &= \sum_{j=1}^{\infty} \pi_j^{(\sigma)} \mathbb{1}_{\{\xi_j^{(\sigma)} > x\}} \\ &= (1-\nu_1)^{-1} \sum_{j=2}^{\infty} \pi_j \mathbb{1}_{\{\xi_j > x\}} \\ &= (1-\nu_1)^{-1} \sum_{j=1}^{\infty} \pi_j \mathbb{1}_{\{\xi_j > x\}} = (1-\nu_1)^{-1} \bar{G}(x), \quad a.s.\end{aligned}$$

and the result follows. (b) We construct $P^{(-\sigma)}$ from P by adding an independent atom, $\xi_0 \sim P_0$, through the mixture random mixture

$$P^{(-\sigma)} = \nu_0\, \delta_{\xi_0} + (1-\nu_0) P$$

where $\nu_0 \sim \mathsf{Be}(1-\sigma, \theta)$ independently. This way, $P^{(-\sigma)} = \sum_{j=1}^{\infty} \pi_j^{(-\sigma)} \delta_{\xi_j^{(-\sigma)}}$ with

$$\pi_1^{(-\sigma)} = \nu_0, \quad \pi_j^{(-\sigma)} = \pi_{j-1}(1-\nu_0) = \nu_{j-1} \prod_{i=0}^{j-2} (1-\nu_i), \quad \xi_j^{(-\sigma)} = \xi_{j-1}, \quad j \geq 2.$$

Recalling that $\nu_{j-1} \overset{\text{ind}}{\sim} \mathsf{Be}(1-\sigma, \theta-\sigma+j\sigma)$ we see $P^{(-\sigma)}$ is a Pitman–Yor process with parameters $(\sigma, \theta-\sigma, P_0)$. Finally, note that for every $x \geq \xi_0 = \xi_1^{(-\sigma)}$ we get

$$\bar{G}'(x) = \sum_{j=1}^{\infty} \pi_j^{(-\sigma)} \mathbb{1}_{\{\xi_j^{(-\sigma)} > x\}} = \sum_{j=2}^{\infty} \pi_j^{(-\sigma)} \mathbb{1}_{\{\xi_j^{(-\sigma)} > x\}} = (1-\nu_0) \sum_{j=1}^{\infty} \pi_j \mathbb{1}_{\{\xi_j > x\}} = (1-\nu_0) \bar{G}(x),$$

and the result follows. □

Appendix C. Proofs of main results

Proof of Theorem 1

Below we focus in the case $\sigma > 0$, as for $\sigma = 0$, P is a Dirichlet process and hence it immediately follows from Doss and Sellke (1982) that the tail of P is exponentially thinner than that of P_0.

The strategy of the proof warrants a brief comment. Cases 1–3 deal with region of the parameter space $\{(\theta, \sigma) \in [0, \sigma] \times (0, 1)\}$, whereas Case 4 uses the latter cases to prove by induction that the result holds for $\{(\theta, \sigma) \in (-\sigma, \infty) \times (0, 1)\}$.

Case 1 ($\theta = 0$, $\sigma \in (0, 1)$)

In this case the random probability measure $P \sim \mathrm{PYP}(\sigma, 0, P_0)$ admits a representation as a normalized stable subordinator, and the result follows from the arguments of Palacios et al. (2025); for completeness, we present a streamlined version of their line of attack. We start with the lower envelope. The Laplace exponent of a stable process $\phi(t)$ with parameter σ, is $\Phi(\lambda) = \lambda^{\sigma}$, is regularly varying at ∞ with index $\sigma \in (0, 1)$, and note also that $\Phi^{-1}(y) = y^{1/\sigma}$. Hence, Lemma 2 a) implies that

$$\liminf_{t \to 0^+} \frac{\phi(t)}{L(t)} = \sigma(1-\sigma)^{(1-\sigma)/\sigma}, \quad \text{with } L(t) = t^{1/\sigma} \{\log |\log t|\}^{1-1/\sigma}. \tag{5.7}$$

Combining (5.7) with the representation of the stable law process in Equation (3) yields

$$\liminf_{x \to \infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} = \liminf_{\bar{G}_0(x) \to 0^+} \frac{\phi\{\bar{G}_0(x)\}/\phi(1)}{L\{\bar{G}_0(x)\}} = \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\phi(1). \tag{5.8}$$

from where the lower envelope follows with $\kappa(\theta, \sigma) = \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\phi(1)$, a.s. Next, we focus on the upper envelope. Consider the following family of functions for $r > 0$,

$$U_r(t) = t^{1/\sigma} |\log t|^{r/\sigma} = t^{1/\sigma} \{\log(1/t)\}^{r/\sigma}, \qquad t \in (0, e^{-r}), \tag{5.9}$$

where $\sigma \in (0,1)$. It follows that $U_r(t)$ is positive and nondecreasing on $(0,\delta)$, with $\delta = \mathrm{e}^{-r}$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\{U_r(t)\} = \sigma^{-1}t^{1/\sigma-1}[\{\log(1/t)\}^{r/\sigma} - r\{\log(1/t)\}^{r/\sigma-1}] > 0, \quad t \in (0,\delta). \tag{5.10}$$

Additionally,

$$\begin{aligned}\lim_{t\to 0^+} \frac{U_r(t)}{\{t\log\log(1/t)\}^{1/2}} &= \lim_{t\to 0^+} \frac{t^{1/\sigma}\{\log(1/t)\}^{r/\sigma}}{\{t\log\log(1/t)\}^{1/2}} \\ &= \lim_{t\to 0^+} t^{1/\sigma-1/2} \times \lim_{t\to 0^+} \frac{\{\log(1/t)\}^{r/\sigma}}{\{\log\log(1/t)\}^{1/2}} \\ &\to 0,\end{aligned} \tag{5.11}$$

given that $1/\sigma - 1/2 > 0$ (recall that $\sigma < 1$), and hence $U_r(t)$ obeys the assumptions of Lemma 2 b). Thus, applying Lemma 2 b), with (5.9), in the representation of the stable law process from Equation (3) yields

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \le 1. \end{cases} \quad \text{a.s.} \tag{5.12}$$

Case 2 ($\theta = \sigma$, $\sigma \in (0,1)$)

Take $P \sim \mathrm{PYP}(\sigma,\sigma,P_0)$. By Proposition 3 (b) we know that there exists a Pitman–Yor process $P^{(-\sigma)}$ with parameters $(\sigma,\theta-\sigma,P_0) = (\sigma,0,P_0)$ such that

$$\bar{G}(x) = (1-\nu_0)^{-1}\bar{G}^{(-\sigma)}(x), \quad x \ge \xi_0$$

a.s. where $\bar{G}(x)$ and $\bar{G}^{(-\sigma)}(x)$ are the tail functions of P and $P^{(-\sigma)}$; $\xi_0 \sim P_0$ and $\nu_0 \sim \mathsf{Be}(1-\sigma,\theta)$ are independent random variables. Since $P^{(-\sigma)}$ has a representation as a normalized stable subordinator, it follows from Equations (5.8) and (5.12)

$$\liminf_{x\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} = \liminf_{x\to\infty} \frac{(1-\nu_0)^{-1}G^{(-\sigma)}(x)}{L\{\bar{G}_0(x)\}} = \frac{\sigma(1-\sigma)^{(1-\sigma)/\sigma}}{(1-\nu_0)\phi(1)}, \tag{5.13}$$

and

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = \limsup_{x\to\infty} \frac{(1-\nu_0)^{-1}G^{(-\sigma)}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \le 1. \end{cases} \quad \text{a.s.} \tag{5.14}$$

This implies the final result with $\kappa(\theta,\sigma) = \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)\phi(1)\}$, a.s.

Case 3 ($\theta \in (0,\sigma)$, $\sigma \in (0,1)$)

Let $P \sim \mathrm{PYP}(\sigma,\theta,P_0)$ with $\theta \in (0,\sigma)$, by Proposition 1, for $\varepsilon = \sigma - \theta$ and $\eta = \theta$ there exist two Pitman–Yor processes $P^{(\varepsilon)}$ and $P^{(-\eta)}$, with parameters $(\sigma,\theta+\varepsilon,P_0) = (\sigma,\sigma,P_0)$ and $(\sigma,\theta-\eta,P_0) = (\sigma,0,P_0)$, respectively, such that for every $m \ge 1$, there exists some random variable $\xi^{(m)}$ taking values in $\mathbb{R}_+$ such that

$$(1-1/m)\bar{G}^{(-\eta)}(x) \le \bar{G}(x) \le (1-1/m)^{-1}\bar{G}^{(\varepsilon)}(x), \quad x > \xi^{(m)}, \quad \text{a.s.} \tag{5.15}$$

where $\bar{G}^{(-\eta)}(x)$ and $\bar{G}^{(\varepsilon)}$ are the tail functions of $P^{(-\eta)}$ and $P^{(\varepsilon)}$, respectively. Putting this together with equations (5.8) and (5.13) we get

$$(1-1/m)\liminf_{x\to\infty} \frac{\bar{G}^{(-\eta)}(x)}{L\{\bar{G}_0(x)\}} \le \liminf_{x\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} \le (1-1/m)^{-1}\liminf_{x\to\infty} \frac{\bar{G}^{(\varepsilon)}(x)}{L\{\bar{G}_0(x)\}},$$

that is

$$(1-1/m)\frac{\sigma(1-\sigma)^{(1-\sigma)/\sigma}}{\phi(1)} \le \liminf_{x\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} \le (1-1/m)^{-1}\frac{\sigma(1-\sigma)^{(1-\sigma)/\sigma}}{(1-\nu_0)\phi(1)}$$

as this holds for every $m \geq 1$, by making $m \to \infty$ we get

$$\liminf_{x\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} =: \kappa(\theta,\sigma), \tag{5.16}$$

with $\kappa(\theta,\sigma) \in [\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\phi(1), \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)\phi(1)\}]$, a.s. Now, for functions $U_r(t)$ as in (5.9), (5.15) yields

$$(1-1/m)\limsup_{x\to\infty} \frac{\bar{G}^{(-\eta)}(x)}{U_r\{\bar{G}_0(x)\}} \leq \limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} \leq (1-1/m)^{-1}\limsup_{x\to\infty} \frac{\bar{G}^{(\varepsilon)}(x)}{U_r\{\bar{G}_0(x)\}}.$$

By (5.12) and (5.14) we have that the superior limits in both, left and right, extremes coincide with 0 if $r > 1$ and diverge for $0 < r \leq 1$. Hence

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \leq 1. \end{cases} \quad \text{a.s.} \tag{5.17}$$

Case 4 ($\theta > \sigma$, $\sigma \in (0,1)$)

Take $P \sim \mathrm{PYP}(\sigma,\theta,P_0)$ with $\theta > \sigma$. At this stage, we already know by the first three cases that the result holds for $\theta \in [0,\sigma]$. We will prove by induction on $n \in \mathbb{N}$ that it holds for $\theta \in ((n-1)\sigma, n\sigma]$. So fix $\theta \in (n\sigma,(n+1)\sigma]$ and assume the result is true in the interval $((n-1)\sigma,n\sigma]$. By Proposition 3 (b) we know that there exist a Pitman–Yor process $P^{(-\sigma)}$ with parameters $(\sigma,\theta-\sigma,P_0)$ with $\theta-\sigma \in ((n-1)\sigma,n\sigma]$ such that

$$\bar{G}(x) = (1-\nu_0)^{-1}\bar{G}^{(-\sigma)}(x), \quad x \geq \xi_0, \tag{5.18}$$

a.s. where $\bar{G}$ and $\bar{G}^{(-\sigma)}$ are the tail functions of P and $P^{(-\sigma)}$, respectively; and $\nu_0 \sim \mathsf{Be}(1-\sigma,\theta-\sigma)$, $\xi_0 \sim P_0$, are independent random variables. By the induction hypothesis $\bar{G}^{(-\sigma)}$ satisfies

$$\liminf_{x\to\infty} \frac{\bar{G}^{(-\sigma)}(x)}{L\{\bar{G}_0(x)\}} = \kappa(\theta-\sigma,\sigma) \quad \text{and} \quad \limsup_{x\to\infty} \frac{\bar{G}^{(-\sigma)}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \leq 1. \end{cases} \quad \text{a.s.} \tag{5.19}$$

for some random variable

$$\kappa(\theta-\sigma,\sigma) \in \left[\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{n-1}\phi(1)\}, \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{n}\phi(1)\}\right], \quad \text{a.s.}$$

Note that by cases Cases 1–3 this holds for $n = 1$. Now by (5.18) and (5.19) we get

$$\liminf_{\bar{x}\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} = \liminf_{x\to\infty} \frac{\bar{G}^{(-\sigma)}(x)}{(1-\nu_0)L\{\bar{G}_0(x)\}} = \frac{\kappa(\theta-\sigma,\sigma)}{1-\nu_0} =: \kappa(\theta,\sigma), \tag{5.20}$$

and

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = \limsup_{x\to\infty} \frac{\bar{G}^{(-\sigma)}(x)}{(1-\nu_0)U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \leq 1. \end{cases} \quad \text{a.s.} \tag{5.21}$$

Implying the final result for some random variable

$$\kappa(\theta,\sigma) \in \left[\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{n}\phi(1)\}, \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{n+1}\phi(1)\}\right], \quad \text{a.s.}$$

Case 5 ($\theta \in (-\sigma, 0)$, $\sigma \in (0,1)$)

Finally let $P \sim \mathrm{PYP}(\sigma, \theta, P_0)$ with $-\sigma < \theta < 0$. By Proposition 3 (a), there exists a Pitman–Yor process, $P^{(\sigma)}$, with discount and precision parameters $(\sigma, \theta+\sigma, P_0)$ with $\theta + \sigma > 0$ such that

$$\bar{G}(x) = (1-\nu_1)\bar{G}^{(\sigma)}(x), \quad x > \xi_1, \tag{5.22}$$

a.s. where $\bar{G}$ and $\bar{G}^{(\sigma)}$ are the tail functions of P and $P^{(\sigma)}$, respectively; and with $\nu_1 = \pi_1 \sim \mathsf{Be}(1-\sigma, \theta+\sigma)$ and $\xi_1 \sim P_0$. By Case 3, $\bar{G}^{(\sigma)}$ satisfies

$$\liminf_{x\to\infty} \frac{\bar{G}^{(\sigma)}(x)}{L\{\bar{G}_0(x)\}} = \kappa(\theta+\sigma, \sigma) \quad \text{and} \quad \limsup_{x\to\infty} \frac{\bar{G}^{(\sigma)}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \le 1. \end{cases} \quad \text{a.s.} \tag{5.23}$$

for some random variable $\kappa(\theta+\sigma,\sigma) \in [\,\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\phi(1), \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{-1}\phi(1)\}\,]$, a.s. By (5.22) and (5.23) we get

$$\liminf_{\bar{x}\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} = \liminf_{x\to\infty} \frac{(1-\nu_1)\bar{G}^{(\sigma)}(x)}{L\{\bar{G}_0(x)\}} = (1-\nu_1)\kappa(\theta+\sigma,\sigma) =: \kappa(\theta,\sigma), \tag{5.24}$$

and

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = \limsup_{x\to\infty} \frac{(1-\nu_1)\bar{G}^{(\sigma)}(x)}{U_r\{\bar{G}_0(x)\}} = \begin{cases} 0, & r > 1, \\ \infty, & 0 < r \le 1. \end{cases} \quad \text{a.s.} \tag{5.25}$$

This implies the final result with

$$\kappa(\theta,\sigma) \in \big[\,(1-\nu_1)\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\phi(1), (1-\nu_1)\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{-1}\phi(1)\}\,\big], \quad \text{a.s.}$$

□

Proof of Corollary 1

First, we focus in the upper bound. By Theorem 1 we have that for $r > 1$ and U_r as in the statement of the same theorexm

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{U_r\{\bar{G}_0(x)\}} = 0, \quad \text{a.s.}$$

This implies that there exists a random variable $\xi' \in (0,\infty)$ such that $\sup_{x\ge\xi} \bar{G}(x)/U_r\{\bar{G}_0(x)\} \le 1$ a.s., that is

$$\bar{G}(x) \le U_r\{\bar{G}_0(x)\}, \quad x \ge \xi', \quad \text{a.s.}$$

We now focus on the lower bound. By Theorem 1 we have

$$\liminf_{x\to\infty} \frac{\bar{G}(x)}{L\{\bar{G}_0(x)\}} = \kappa(\theta,\sigma) > 0, \quad \text{a.s.}$$

Hence, there exists a random variable ξ'' such that $\inf_{x\ge\xi''} \bar{G}(x)/L\{\bar{G}_0(x)\} > \kappa(\theta,\sigma)/2$. Now set

$$\gamma(\theta,\sigma) = \begin{cases} (1-\nu_1)\sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{2\phi(1)\}, & \theta < 0, \\ \sigma(1-\sigma)^{(1-\sigma)/\sigma}/\{(1-\nu_0)^{n-1}2\phi(1)\}, & (n-1)\sigma \le \theta < n,\ n \in \mathbb{N}. \end{cases} \tag{5.26}$$

In the proof of Theorem 1 we noted that $\kappa(\theta,\sigma) \ge 2\gamma(\theta,\sigma)$, thus

$$L\{\bar{G}_0(x)\}(\gamma(\theta,\sigma)) \le \bar{G}(x), \quad x \ge \xi'', \quad \text{a.s.}$$

For $\xi = \max\{\xi', \xi''\}$ we get

$$L\{\bar{G}_0(x)\}(\gamma(\theta,\sigma)) \le \bar{G}(x) \le U_r\{\bar{G}_0(x)\} \quad x \ge \xi, \quad \text{a.s.}$$

□

Proof of Theorem 2

Below we focus on the case $\sigma > 0$ as similarly to the proof of Theorem 1, the case $\sigma = 0$ corresponds to the case of the Dirichlet process, and hence it follows directly from Doss and Sellke (1982).

Fix $\sigma > 0$ and $r > 1$. Recall the definitions of $\mathbb{L}$ and $\mathbb{U}_r$ in (3.1) and L and U_r in (5.7) and (5.9). Next, $\mathbb{L} \in \mathrm{RV}_{-\alpha/\sigma}$ and $\mathbb{U}_r \in \mathrm{RV}_{-\alpha/\sigma}$ follows immediately from the assumption that $\bar{G}_0 \in \mathrm{RV}_{-\alpha}$, the facts that $L \in \mathrm{RV}_{1/\sigma}$ and $U_r \in \mathrm{RV}_{1/\sigma}$, and elementary properties of regularly varying functions (Bingham et al. 1989, Proposition 1.5.7). To show that $\bar{G} \in \mathrm{MV}_{-\alpha/\sigma}(\mathbb{L}, \mathbb{U}_r)$, note first that $\bar{G} \preceq \mathbb{U}_r$, for any $r > 1$, as Theorem 1 proves

$$\limsup_{x\to\infty} \frac{\bar{G}(x)}{\mathbb{U}_r(x)} = 0 < \infty, \quad \text{a.s.}$$

and similarly $\mathbb{L} \preceq G$,

$$\limsup_{x\to\infty} \frac{\mathbb{L}(x)}{\bar{G}(x)} \leq \kappa(\theta,\sigma)^{-1} < \infty, \quad \text{a.s.}$$

Finally, (3.2) follows from Cadena et al. (2019) given $\bar{G} \in \mathrm{MV}_{-\alpha/\sigma}(\mathbb{L}, \mathbb{U}_r)$, if and only if

$$\limsup_{x\to\infty} \frac{\log \bar{G}(x)}{\log x} = -\frac{\alpha}{\sigma}.$$

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