Syzygies of toric varieties

by

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Οὐ παύσομαι τὰς Χάριτας Μούσαις συγκαταμειγνύς,
ἁδίσταν συζυγίαν.

Never will I cease to mingle the Graces and the Muses,
sweetest union.

(Theban Elders. Euripides, Heracles [675])

Für Mama

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CHAPTER I

Introduction

This is a very short introduction to the results presented in this thesis. The definition of N_p is contained in Appendix A and for a more leisurely introduction to property N_p, we refer to Chapter II. The technical tools we need are treated in Chapter III and the applications to toric varieties in Chapter IV. The results in sections 3.2, 4.1, 4.2, 4.3 and 4.5 are the contents of joint work with H. Schenck and G. Smith in [29]. The results in sections 3.3, 4.4 and 4.6 have not appeared elsewhere.

Our most straightforward application is the following theorem.

Theorem I.1. *Let L be an ample line bundle on a toric variety X of dimension $n \geq 1$. Then*

$$L^{n-1+p} \textit{ satisfies } N_p \textit{ for } p \geq 0.$$

The case $p = 0$ is proved in [14, 41, 5] via combinatorial methods (see also Proposition IV.12). Bruns, Gubeladze and Trung also prove in [5] that for $m \geq n$, the section ring R associated to L^m is Koszul, which implies N_1. Recently, Ogata [49] proved that for an ample line bundle L on a toric variety of dimension $n \geq 3$, L^{n-2+p} satisfies N_p for $p \geq 1$.

In fact, using some more information of the Hilbert polynomial, we obtain the following variant of Theorem I.1.

Theorem I.2. *Let $L \ncong \mathcal{O}_{\mathbb{P}^n}(1)$ be a globally generated line bundle on a toric variety X, let r be the number of distinct integer roots of the Hilbert polynomial h of L and let d be the degree of h. Then*

$$L^{d-r-1+p} \text{ satisfies } N_p \text{ for } p \geq 1.$$

Our main application is the following criterion for adjoint line bundles to satisfy N_p.

Theorem I.3. *Let X be a Gorenstein projective toric variety of dimension n that is not isomorphic to $\mathbb{P}^n$. Let $B_1, \ldots, B_r$ be generators of the semigroup of nef divisors, let A be an ample line bundle such that $A \otimes B_j^{-1}$ is nef for all j and let N be nef. Then*

$$A^{n+p} \otimes K_X \otimes N \text{ satisfies } N_p.$$

This is motivated by a theorem of this form for very ample line bundles on smooth projective varieties due to Ein and Lazarsfeld, see Theorem II.4.

Exploiting the correspondence of ample line bundles on toric varieties with lattice polytopes, we obtain the following criterion for a lattice polytope to be normal (see Definition IV.8).

Theorem I.4. *Let P be a lattice polytope of dimension n and let r be the largest integer such that rP does not contain any lattice points in its interior. Then $(n-r)P$ is normal.*

In fact, Bruns, Gubeladze and Trung [5, Theorem 1.3.3] show that $(n-1)P$ is normal.

Another application of our methods yields criteria for the section ring $R(L)$ of a line bundle L to be Koszul.

Theorem I.5. *Let L be a globally generated line bundle on a projective toric variety X. Let r be the number of distinct integer roots of the Hilbert polynomial of L and let d be the degree of the Hilbert polynomial of L. Then for $m \geq d - r$, $R\left(L^m\right)$ is Koszul.*

When $r = 0$, this result is due to Bruns, Gubeladze and Trung [5, Theorem 1.3.3].

Similarly, we obtain a criterion for the section ring associated to an adjoint line bundle to be Koszul along the lines of Pareschi's theorem II.6.

Theorem I.6. *Let X be a Gorenstein projective toric variety of dimension n not isomorphic to $\mathbb{P}^n$ and let A be an ample line bundle on X. Let $B_1, \ldots, B_r$ be the generators of the nef cone of X and suppose that $A \otimes B_j^{-1}$ is nef for all $1 \leq j \leq r$. Let N be a nef line bundle and let $L = K_X \otimes A^d \otimes N$. Then $R(L)$ is Koszul for $d \geq n + 1$.*

Notation

We assume that X is a projective algebraic variety over an algebraically closed field of characteristic zero. The more general cases are discussed in Appendix D.

CHAPTER II

An overview of property N_p

2.1 Curves

It is a classical question whether a given very ample line bundle L on a projective variety X induces a projectively normal embedding and whether the ideal of the image is generated by quadratic equations. When X is a curve of genus g, Castelnuovo [8], Mattuck [44] and Mumford [46] proved that a line bundle L of degree $d \geq 2g+1$ is normally generated. Similarly Fujita [17] and St. Donat [63] proved that when $d \geq 2g+2$, the ideal of the embedding is generated by quadratic equations. Mumford showed in [46] that sufficiently high powers of ample line bundles on projective varieties give rise to a projectively normal embedding whose ideal is generated by quadratic equations. Green realized that these results naturally generalize to higher syzygies in the form of property N_p (see Appendix A.1). Then the classical results on curves generalize to the following theorem due to Green.

Theorem II.1. *[26] Let X be a smooth curve of genus g and let L be a line bundle on X of degree $d \geq 2g+1+p$. Then L satisfies N_p.*

Green and Lazarsfeld [25] show that a line bundle L of degree $2g+p$ satisfies N_p unless X is hyperelliptic or the induced embedding has a $(p+2)$-secant p-plane.

When X is a canonical curve, Green conjectures that the Clifford index associated to X is the least integer p for which N_p does not hold (see Chapter 1.8.D in [40] for an overview). Recently Voisin [70, 69] proved this conjecture for a curve that is general in the moduli space.

2.2 Adjoint line bundles

Adjoint bundles of the form $A^m \otimes K_X$, where A is an ample line bundle and K_X the canonical line bundle have been the objects of extensive study and conjectures. A famous example is

Fujita's conjecture. *Let A be an ample line bundle on a smooth projective variety X. Then*

(i) $A^{n+1} \otimes K_X$ is globally generated and

(ii) $A^{n+2} \otimes K_X$ is very ample.

The case $X \cong \mathbb{P}^n$ shows that this conjecture is sharp.

Questions like Fujita's conjecture tend to be easier to tackle when A is globally generated in addition to being ample, as in this case we can exploit the regularity of A by applying Kodaira vanishing.

Proposition II.2 ([40, Example 1.8.23.]). *Let X be a smooth complex projective variety of dimension n and let A be an ample and globally generated line bundle on X. Let N be a nef line bundle and let*

$$L_k = A^k \otimes K_X \otimes N.$$

Then for $k \geq n+1$, L_k is globally generated and for $k \geq n+2$, L_k is very ample.

When X is a smooth surface, Reider's theorem [59] relates the failure for $A \otimes K_X$ to be globally generated (or very ample) to the self intersection of $c_1(A)$ and to the existence of certain *extremal* curves. In particular, his theorem implies Fujita's conjecture for surfaces. Ein and Lazarsfeld prove Fujita's conjecture for threefolds in [9]. In general, Fujita's conjecture is far from being solved. For an overview of the current state of the art, we refer to [40, Section 10.4.B].

Fujita's conjecture generalizes to questions on the higher syzygies of embeddings induced by adjoint line bundles. Mukai observed that we can interpret Green's theorem II.1 as follows: For an ample line bundle A on a smooth curve X of genus g, $A^{3+p} \otimes K_X$ satisfies N_p. Generalizing this to higher dimensional varieties, we arrive at the following question.

Question II.3 (See [10, Section 4]). *Does $A^{n+2+p} \otimes K_X$ satisfy N_p?*

Again, the case when $A = \mathcal{O}_{\mathbb{P}^n}(1)$ shows that this is sharp for $p = 0$. Moreover, Butler [7] exhibits an example of an n-dimensional variety X and an ample line bundle A such that $A^{n+2} \otimes K_X$ does not satisfy N_1.

On the other hand, when A is very ample, Ein and Lazarsfeld have given a positive answer to Question II.3.

Theorem II.4 (See [10]). *Let A be a very ample line bundle on a smooth projective variety X not isomorphic to $\mathbb{P}^n$, and let N be a nef line bundle on X. Then*

$$A^{n+p} \otimes K_X \otimes N \text{ satisfies } N_p \text{ for } p \geq 1.$$

Question II.3 for surfaces is referred to as Mukai's conjecture. Gallego and Purnaprajna [20, 22, 19, 23, 58] have solved this conjecture in many cases or have given effective bounds in the sense of Mukai's conjecture, but in general this conjecture is not known.

In general, sufficiently high powers of ample line bundles satisfy N_p (cf. [26]). The question which powers suffice as well as Question II.3 has also been studied by

Kempf [32], Rubei [60], Pareschi and Popa [52, 53, 54] for abelian varieties (see also Section 1.8.D in [40]), by Manivel [43] for homogeneous spaces and by Hà [28] for regular varieties. The case of Segre-Veronese embeddings is treated in 4.5.

Remark II.5. The results on curves ([26]), surfaces (for example [59], [22], [23]) and toric varieties ([47], [16], [55], see also Sections 4.4 and 4.1) suggest that there should be numerical criteria for the property N_p to hold. In fact, there exists a numerical version of Fujita's freeness conjecture for varieties of arbitrary dimension (cf. Kollár [36, Conjecture 5.4]).

2.3 Koszul rings

Given a globally generated line bundle L with section ring $R(L)$, the question whether $R(L)$ is Koszul is closely related to N_1 (see also Appendix A.3). If $R(L)$ is Koszul, then L satisfies N_1 (Proposition A.10). Sturmfels [67, Theorem 3.1.] gives an example of a smooth projectively normal curve of genus 7 in $\mathbb{P}^5$ whose coordinate ring is presented by quadrics that is not Koszul. Nevertheless, in general it seems to be true that whenever a line bundle has a natural reason to satisfy N_1, then the corresponding section ring should be Koszul (cf. [10] and [22]). For example, in the setup of Question II.3, we might ask whether the section ring associated to a line bundle of the form $K_X \otimes A^{n+3} \otimes N$ is Koszul.

In the vein of Theorem II.4, Pareschi proves the following

Theorem II.6 ([51, Theorem C]). *Let X be a smooth complex projective variety of dimension n, let A be a very ample line bundle on X and let N be a nef line bundle on X. Assume that $A \not\cong \mathcal{O}_{\mathbb{P}^n}(1)$. Then for $d \geq n+1$, the section ring of $A^d \otimes K_X \otimes N$ is Koszul.*

In general, $R(L^m)$ is Koszul for m sufficiently large (cf. [1, 12]). The Koszul property has also been studied for canonical curves by Vishik and Finkelbert [68], Polishchuk [56] as well as Gallego and Purnaprajna [21], for abelian varieties by Mumford [45], Kempf [32] and Rubei [60] and for homogeneous spaces by Brion and Kumar [3, Theorem 3.5.3].

CHAPTER III

Property N_p and regularity

3.1 Regularity and N_p

Let B be a globally generated line bundle on a projective variety X. In this section we review a criterion for B^ℓ to satisfy N_p in terms of the regularity of B, and we generalize this criterion to the multigraded case in Section 3.2. For a review of N_p, we refer to Appendix A.

Definition III.1. A sheaf $\mathcal{F}$ is called $\mathcal{O}_X$-regular (with respect to B) if

$$H^i\left(X, \mathcal{F} \otimes B^{-i}\right) 0 \text{ for all } i > 0.$$

The following lemma due to Gallego and Purnaprajna is a very useful criterion for property N_p.

Lemma III.2 ([22, Theorem 1.2]). *If B^m is $\mathcal{O}_X$-regular, then B^{m+p} satisfies N_p for $p \geq 1$.*

For a proof, we refer to the case $r = 1$ in the proof of Lemma III.8. In [46] Mumford proves similar criteria for the case $p = 0$ and 1.

For example, the following folklore lemma gives a criterion for the regularity of a globally generated line bundle.

Lemma III.3. *Let X be a smooth complex projective variety of dimension n and let B be an ample and globally generated line bundle on X. Suppose that $B^m \otimes K_X^{-1}$ is ample. Then B^{m+n} is $\mathcal{O}_X$-regular. In particular, $B^{m+n-1+p}$ satisfies N_p for $p \geq 1$.*

Proof. The required vanishing follows from Kodaira vanishing C.12:

$$H^i\left(X, B^{m+n-i}\right) = H^i\left(X, B^{m+n-i} \otimes K_X^{-1} \otimes K_X\right) = 0 \text{ for } i \leq n. \qquad \square$$

Remark III.4. In fact, it suffices to assume that $B^m \otimes K_X^{-1}$ is big and nef. Then the Kawamata-Viehweg vanishing theorem gives the desired result.

This lemma gives some nice criteria for N_p when K_X is trivial or K_X^{-1} is ample.

Corollary III.5 ([22, Corollary 1.6]). *Let X be a smooth variety of dimension n with K_X trivial, and let L be an ample and globally generated line bundle on X. Then L^{n+p} satisfies N_p for $p \geq 1$.*

Corollary III.6. *Let X be a smooth Fano variety of dimension n and let L be an ample and globally generated line bundle on X. Then L^{n-1+p} satisfies N_p.*

Gallego and Purnaprajna [23, Section 3] study N_p more carefully when X is Fano.

3.2 Multigraded regularity and N_p

It is a natural question whether products of different ample line bundles satisfy N_p. Some results along these lines have appeared in [22]. We will use multigraded regularity as introduced by Maclagan and Smith in [42] to give criteria for products of globally generated line bundles to satisfy N_p. See also Appendix B for an introduction to multigraded regularity.

Let $B_1, \ldots, B_r$ be globally generated line bundles on a projective variety X. For $u = (u_1, \ldots, u_r) \in \mathbb{Z}^r$, let $B^u := B_1^{u_1} \otimes \cdots \otimes B_r^{u_r}$ and $|u| = u_1 + \cdots + u_r$.

Definition III.7. A sheaf $\mathcal{F}$ on a projective variety X is called $\mathcal{O}_X$-regular (with respect to $B_1, \ldots, B_r$) if

$$H^i\left(X, \mathcal{F} \otimes B^{-u}\right) = 0 \text{ for all } i > 0 \text{ and } u \in \mathbb{N}^r \text{ with } |u| = i.$$

Let $\mathcal{B} = \{B^u \mid u \in \mathbb{N}^r\} \subset \operatorname{Pic}(X)$ be the semigroup generated by $B_1, \ldots, B_r$.

Lemma III.8 ([29, Theorem 1.1]). *Let $m_1, m_2, \ldots$ be a sequence in $\mathbb{N}^r$ such that $B^{m_i} \otimes B_j^{-1} \in \mathcal{B}$ for all i and j. If B^{m_1} is $\mathcal{O}_X$-regular with respect to $\mathcal{B}$, then $B^{m_1+\cdots+m_p}$ satisfies property N_p for $p \geq 1$.*

Proof of Lemma III.8. Let $L = B^{m_1+\cdots+m_p}$. Recall that to L is associated a vector bundle M_L defined by the following short exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0. \tag{2.1}$$

We first prove by induction on q that $M_L^{\otimes q}$ is $(B^{m_1+\cdots+m_q})$-regular for all $q \geq 1$.

Observe that the condition $B^{m_i - e_j} = B^{m_1} \otimes B_j^{-1} \in \mathcal{B}$ for all j means that $B^{m_i - e_j}$ is linearly equivalent to some B^u for $u \in \mathbb{N}^r$.

Let $q = 1$. Since B^{m_1} is $\mathcal{O}_X$-regular, it follows that $H^0(X, L) \otimes \mathcal{O}_X$ is B^{m_1}-regular by Remark B.2, that L is $(B^{m_1 - e_j})$-regular by Theorem B.3.1, and that the map of global sections

$$H^0(X, L) \otimes H^0(X, B^{m_1 - e_j}) \to H^0(X, L \otimes B^{m_1 - e_j})$$

is surjective by Theorem B.3.2 for all j. By Lemma B.5 it follows that M_L is (B^{m_1})-regular.

For the induction step, we apply a similar argument to (2.1) tensored with $M_L^{\otimes(q-1)}$:

$$0 \to M_L^{\otimes q} \to H^0(X, L) \otimes M_L^{\otimes(q-1)} \to L \otimes M_L^{\otimes(q-1)} \to 0$$

Since $M_L^{\otimes(q-1)}$ is $(B^{m_1+\cdots+m_{q-1}})$-regular by the induction hypothesis, it also is $(B^{m_1+\cdots+m_q-e_j})$-regular for all j. Hence the map of global sections

$$H^0(X,L)\otimes H^0\left(X,M_L^{\otimes(q-1)}\otimes B^{m_1+\cdots+m_q-e_j}\right)$$
$$\to H^0\left(X,L\otimes M_L^{\otimes(q-1)}\otimes B^{m_1+\cdots+m_q-e_j}\right)$$

is surjective by Theorem B.3.2. Moreover, by Theorem B.3.1 $L\otimes M_L^{\otimes(q-1)}$ is $(B^{m_1+\cdots+m_q-e_j})$-regular for all j and $M_L^{\otimes(q-1)}$ is $(B^{m_1+\cdots+m_q})$-regular. Lemma B.5 now implies that $M_L^{\otimes q}$ is $(B^{m_1+\cdots+m_q})$-regular.

To prove the lemma, it suffices to show that

$$H^1\left(X,M_L^{\otimes q}\otimes L^\ell\right)=0 \text{ for } 1\le q\le p+1 \text{ and } \ell\ge 1$$

by Corollary A.4. By B.3.1, $M_L^{\otimes q}$ is $\left(L^\ell\otimes B^{e_j}\right)$-regular for $1\le q\le p$, $\ell\ge 1$ and some j and hence

$$H^1\left(X,M^{\otimes q}\otimes L^\ell\right)=0 \text{ for } q\le p.$$

When $q=p+1$, we twist (2.1) by $M_L^{\otimes p}\otimes L^\ell$ to obtain

$$0\to M_L^{\otimes(p+1)}\otimes L^\ell\to H^0(X,L)\otimes M_L^{\otimes p}\otimes L^\ell\to M_L^{\otimes p}\otimes L^{\ell+1}\to 0.$$

By Theorem B.3.2, the induced map of global sections

$$H^0(X,L)\otimes H^0\left(X,M_L^{\otimes p}\otimes L^\ell\right)\to H^0\left(X,M_L^{\otimes p}\otimes L^{\ell+1}\right)$$

is surjective. Since $H^1\left(X,M_L^{\otimes p}\otimes L^\ell\right)=0$, the vanishing follows from the long exact sequence of cohomology. □

3.3 Regularity and Koszul rings

In this section we give a criterion for the section ring $R(L)=\bigoplus_{m\ge 0}H^0(X,L^m)$ of a line bundle L to be Koszul in terms of the (multigraded) regularity of L. We use the same notation as in Section 3.2.

Lemma III.9. *Let $B_1, \ldots, B_r$ be globally generated line bundles on a projective algebraic variety X generating a semigroup $\mathcal{B}$. Let $L = B^u$ be an ample line bundle such that $L \otimes B^{-e_j} \in \mathcal{B}$ for all j. Suppose that L is $\mathcal{O}_X$-regular. Then the section ring $R(L)$ is Koszul.*

Proof. Recall that to a vector bundle E that is generated by its global sections is associated a vector bundle M_E via the short exact sequence

$$0 \to M_E \to H^0(X, E) \otimes \mathcal{O}_X \to E \to 0. \tag{3.2}$$

To L are associated vector bundles $M^{(h,L)}$ defined inductively by letting $M^{(0,L)} = L$ and $M^{(h,L)} = M_{M^{(h-1,L)}} \otimes L$ provided that $M^{(h-1,L)}$ is globally generated. We claim by induction on h that $M^{(h,L)}$ is $\mathcal{O}_X$-regular. In particular, by Theorem B.3.3., $M^{(h,L)}$ is globally generated and the inductive definition makes sense.

Tensoring (3.2) for $E = M^{(h-1,L)}$ with L, we obtain the following short exact sequence

$$0 \to M^{(h,L)} \to H^0\left(X, M^{(h-1,L)}\right) \otimes L \to M^{(h-1,L)} \otimes L \to 0.$$

Then $H^0\left(X, M^{(h-1,L)}\right) \otimes L$ is $\mathcal{O}_X$ regular. Since $L \otimes B^{-e_j} \in \mathcal{B}$ it follows that $L \otimes B^{-e_j}$ is linearly equivalent to $B^{u'}$ for $u' \in \mathbb{N}^r$. By the induction hypothesis and Theorem B.3.1., $M^{(h-1,L)}$ is $L \otimes B^{-e_j}$-regular for all j, and so $M^{(h-1,L)} \otimes L$ is B^{-e_j}-regular for all j. Similarly by Theorem B.3.2., the natural map $H^0\left(X, M^{(h-1,L)}\right) \otimes H^0\left(X, L \otimes B^{-e_j}\right) \to H^0\left(X, M^{(h-1,L)} \otimes L \otimes B^{-e_j}\right)$ is surjective for all $1 \le j \le r$. Applying Lemma B.5 we see that $M^{(h,L)}$ is $\mathcal{O}_X$-regular.

The regularity implies that $H^1\left(X, M^{(h,L)} \otimes L^\ell\right) = 0$ for all $h \ge 0$ and $\ell \ge 0$ (see Lemma B.4). By Lemma A.11, $R(L)$ is Koszul. $\square$

Corollary III.10. *Let L be an ample and globally generated line bundle on a projective algebraic variety X. Suppose that L^m is $\mathcal{O}_X$-regular. Then $R(L^m)$ is Koszul.*

CHAPTER IV

Property N_p for toric varieties

4.1 Adjoint line bundles

When X is a smooth toric variety of dimension n, then every ample line bundle is very ample (C.3) and so if X is not isomorphic to $\mathbb{P}^n$, Theorem II.4 implies that $A^{n+p} \otimes K_X$ satisfies N_p for $p \geq 0$.

It remains to study adjoint line bundles on possibly singular toric varieties. In fact, a more general form of Fujita's conjecture has been proved by Fujino and Payne for Gorenstein toric varieties. As for the toric Nakai criterion C.7, for toric varieties there exist numerical criteria for adjoint line bundles to be nef or very ample just in terms of the intersections with curves. In this case, $\mathbb{P}^n$ is the only toric variety where the bounds in Fujita's conjecture are sharp.

Let X be a toric variety and let $D_1, \ldots, D_s$ denote the torus invariant prime divisors on X. Then $K_X \sim -\sum_{i=1}^{s} D_i$ (cf. [18, Section 4.3] and Appendix C.5).

Theorem IV.1. *Let A, D be $\mathbb{Q}$-Cartier divisors on a toric variety X of dimension n such that $0 \geq D \geq K_X$ and $A + D$ is Cartier. Assume that X is not isomorphic to $\mathbb{P}^n$.*

(i) (Fujino [16]) Suppose $A \cdot C \geq n$ for all torus invariant curves C. Then $A + D$ is globally generated.

(ii) (Payne [55]) Suppose $A \cdot C \geq n+1$ for all torus invariant curves C. Then $A + D$ is very ample.

This raises the question whether in the setup of Theorem IV.1, the condition for an ample line bundle A to satisfy $A \cdot C \geq n+1+p$ for all torus invariant curves C implies that $A + K_X$ satisfies N_p. This is known for toric surfaces, see Theorem IV.24.

In general, we obtain a partial answer to Question II.3 using multigraded regularity.

Theorem IV.2. *Let X be a projective toric variety of dimension n that is not isomorphic to $\mathbb{P}^n$. Let $B_1, \ldots, B_r$ be the generators of the semigroup of nef divisors and let $w_1, w_2, \ldots$ be a sequence in $\mathbb{N}^r$ such that*

$$B^{w_i} \otimes B_j^{-1} \text{ is nef for all } j \in \{1, \ldots, r\}.$$

Let $D_1, \ldots, D_t$ be distinct torus invariant prime divisors so that $D = -D_1 - \cdots - D_t$ is a Cartier divisor. Then for $p \geq 1$

$$B^{w_1} \otimes \cdots \otimes B^{w_{n+p}} (D) \text{ satisfies } N_p.$$

Proof. Since X is projective, for a given torus invariant curve C, there exists a $j\,(C)$ such that $c_1 \left(B_{j(C)}\right) \cdot C > 0$. Since we can write $B^{w_i} \sim B' \otimes B_j$ for some nef line bundle B' for all $j \in \{1, \ldots, r\}$, B^{w_i} is ample for all i by (C.7). Similarly, for $|u| = i$, we can write $B^{w_1 + \cdots + w_i} \sim B^u \otimes B'$ and therefore

$$B^{w_1 + \cdots w_{n+1}} \otimes B^{-u} \sim B^u \otimes B' \otimes B^{w_{n+1}} \otimes B^{-u}$$

is ample for $|u| \leq n$. It follows from the toric Kodaira vanishing Theorem C.13 (i) that $B^{w_1 + \cdots + w_{n+1}} (D)$ is $\mathcal{O}_X$-regular and therefore globally generated by B.3.3. Hence there exists $m_1 \in \mathbb{N}^r$, such that $B^{m_1} \sim B^{w_1 + \cdots + w_{n+1}} (D)$.

Moreover, given a torus invariant curve C, we can let $u = ne_{j(C)}$, so we have $B^{w_1+\cdots+w_{n+1}} \sim B^n_{j(C)} \otimes B_j \otimes B'$. Then Fujino's Theorem IV.1 (i) implies that $B^{m_1} \otimes B_j^{-1}$ is nef for all j. So we can apply Lemma III.8 with m_1 and $m_i := w_{n+i}$ for $i \geq 2$ and the claim follows. $\square$

Corollary IV.3. *Let X be a Gorenstein projective toric variety of dimension n that is not isomorphic to $\mathbb{P}^n$. Let $B_1, \ldots, B_r$ be generators of the semigroup of nef divisors, let A be an ample line bundle such that $A \otimes B_j^{-1}$ is nef for all j and let N be nef. Then*

$$A^{n+p} \otimes K_X \otimes N \text{ satisfies } N_p \text{ for } p \geq 1.$$

4.2 The regularity of a line bundle on a toric variety

The following theorem is an easy illustration of how regularity and N_p interacts for line bundles on toric varieties.

Theorem IV.4. *Let A be an ample line bundle on a toric variety of dimension n. Then*

$$A^{n-1+p} \text{ satisfies } N_p.$$

Proof. The case $p = 0$ is treated below, see Corollary IV.14. By Theorem C.10, A^n is $\mathcal{O}_X$-regular and so Lemma III.2 implies that A^{n-1+p} satisfies N_p for $p \geq 1$. $\square$

For $p = 0$, this result is sharp by Example IV.13, but we expect that for n large, A^{n-1} satisfies N_p for $p > 0$. In fact, Ogata [48] shows that for $n \geq 3$, A^{n-1} satisfies N_1 and, building on this, he proves that A^{n-2+p} satisfies N_p for $p \geq 1$, see [49].

In order to get some more refined results, we have to study the regularity of a line bundle on a toric variety more carefully. Let X be a toric variety containing an open dense torus T. Let M be the character lattice associated to T, where we denote the character corresponding to $u \in M$ by χ^u. Then to any divisor D on X corresponds a polytope $P \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$H^0(X, \mathcal{O}_X(D)) \cong \bigoplus_{u \in P \cap M} K\chi^u \tag{2.1}$$

satisfying $P_{mD} = mP_D$ (cf. [18, Chapter 3] and Appendix C).

Recall that the Hilbert polynomial of a line bundle L is defined to be

$$h(m) = \chi(X, L^m) = \sum_{i=0}^{n} (-1)^i h^i(X, L^m).$$

Similarly, to a polytope P is associated a function

$$E(m) := |mP \cap M|.$$

In fact, $E(m)$ is a polynomial of degree $d = \dim P$, called the *Ehrhart polynomial* (cf. [13, Chapter IV.6] or [18, Chapter 5.3]).

Observe that the Ehrhart polynomial of P coincides with the Hilbert polynomial of $\mathcal{O}(D)$ if $\mathcal{O}(D)$ is globally generated. In fact, since the higher cohomology of a globally generated line bundle on a toric variety vanishes, we have

$$h(m) = \chi(X, \mathcal{O}(mD)) = h^0(X, \mathcal{O}(mD)) = |mP \cap M| = E(m).$$

The Ehrhart polynomial satisfies *Ehrhart reciprocity*:

$$|\operatorname{int}(mP) \cap M| = (-1)^{\dim P} E(-m), \tag{2.2}$$

where $\operatorname{int}(P)$ denotes the interior of P (cf. [13, Chapter IV.6]). Let r be the smallest number such that rP does not contain any interior lattice points. By (2.2), $E(-m) = 0$ if and only if $\operatorname{int}(mP) \cap M = \emptyset$. Hence the integer roots of the Ehrhart polynomial are at $\{-1, \ldots, -r\}$. In particular, r coincides with the number of distinct integer roots of the Hilbert polynomial of $\mathcal{O}(D)$.

Proposition IV.5. *Let L be an ample line bundle on a toric variety X of dimension n, and let r denote the number of distinct integer roots of the Hilbert polynomial of L. Then L^{n-r} is $\mathcal{O}_X$-regular with respect to L.*

The main argument in the proof follows Batyrev and Borisov [2, Theorem 2.5].

Proof. We have to show that

$$H^i\left(X, L^{n-r-i}\right) = 0 \text{ for all } i > 0.$$

If $n - r - i \geq 0$, this follows from the vanishing of higher cohomology of globally generated line bundles on toric varieties (cf. [18, 3.5]). If $n - r - i < 0$, it follows from the toric Kodaira vanishing theorem C.13 (ii) that $H^i\left(X, L^{n-r-i}\right) = 0$ for $i \neq n$. When $i = n$, we have

$$(-1)^n h^n\left(X, L^{-r}\right) = \chi\left(X, L^{-r}\right) = h\left(-r\right) = 0. \qquad \square$$

Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is the only line bundle on a toric variety of dimension n with $r = n$, Lemma III.2 implies the following criterion for N_p.

Theorem IV.6. *Le $L \ncong \mathcal{O}_{\mathbb{P}^n}(1)$ be an ample line bundle on a projective toric variety X of dimension n and let r be the number of distinct integer roots of the Hilbert polynomial of L. Then*

$$L^{n-r-1+p} \textit{ satisfies } N_p \textit{ for all } p \geq 1.$$

Remark IV.7. By Proposition C.16 we can use the same argument as in the proof of Proposition IV.5 to see that for a globally generated line bundle L on a toric variety X, $L^{\deg(h)-r}$ is $\mathcal{O}_X$-regular. In particular, $L^{\deg(h)-r-1+p}$ satisfies N_p.

4.3 Normal polytopes

Normal generation for a globally generated line bundle $\mathcal{O}_X(D)$ on a toric variety corresponds to the following combinatorial property of the polytope corresponding to D.

Definition IV.8. A lattice polytope P is called *normal* if for all $m \in \mathbb{N}$ the map

$$\underbrace{(P \cap M) + \cdots + (P \cap M)}_{m} \to mP \cap M \tag{3.3}$$

is surjective.

In other words, every lattice point in mP can be written as a sum of m lattice points in P. To see that this is equivalent to the normality of $\mathcal{O}_X(D)$, observe that by (C.1) the surjectivity of (3.3) is equivalent to the surjectivity of the corresponding map

$$\underbrace{H^0(X, \mathcal{O}_X(D)) \otimes \cdots \otimes H^0(X, \mathcal{O}_X(D))}_{m} \to H^0(X, \mathcal{O}_X(mD))$$

for all m. This in turn is equivalent to the surjectivity of the map $Sym^\bullet H^0(X, D) \to \bigoplus_m H^0(X, mD)$ and so $\mathcal{O}_X(D)$ is normally generated (satisfies N_0) if and only if the corresponding polytope P is normal.

Example IV.9. Let $P = \mathrm{conv}\langle(0,0,0),(1,0,0),(0,1,0),(1,1,2)\rangle$. One can check that the vertices are the only lattice points in P. Then $(1,1,1) \in 2P$ cannot be written as a sum of two lattice points in P, in particular, P is not normal. But it follows from Proposition IV.12 that $2P$ is normal.

The following gives a criterion for a lattice polytope to be normal.

Proposition IV.10. *Let P be a lattice polytope of dimension n and let r be the largest integer such that rP does not contain any lattice points in its interior. Then $(n-r)P$ is normal.*

Proof. We claim that the map

$$mP \cap M + P \cap M \to (m+1)\, P \cap M \tag{3.4}$$

is surjective for all $m \geq n - r$. This implies that for $m \geq n - r$, every lattice point in mP can be written as a sum of a lattice point in $(n-r)\,P$ and $m - n + r$ lattice points in P. In particular, $(n-r)\,P$ is normal.

Let $v \in (m+1)P \cap M$. Then we can express v as a positive combination of $n+1$ vertices $v = \sum_{i=0}^{n} a_i v_i$ with $\sum_{i=0}^{n} a_i = m + 1$. It suffices to show that $a_j \geq 1$ for some j. Suppose $a_i < 1$ for all i. Then $v' = \sum_{i=0}^{n} (1 - a_i)\, v_i \in \operatorname{int}(n-m)\, P \cap M$, a contradiction since $n - m \leq r$. □

Remark IV.11. This also follows from Theorem IV.6, since such a polytope P gives rise to an ample divisor D on a projective toric variety X_P of dimension n such that r is the number of distinct integer roots of the Hilbert polynomial of $\mathcal{O}_X(D)$.

Proposition IV.12 ([14],[41],[5]). *Let P be a lattice polytope of dimension n. Then $(n-1)\,P$ is normal.*

Proof. Observe that any lattice polytope is a union of polytopes that do not contain any lattice points in their interior. In fact, if P contains a lattice point u in its interior, replace it with the union of the convex hulls $\operatorname{conv}\langle F, u\rangle$ for all facets F of P, i.e., with its stellar subdivision in direction of the lattice point (see, for example, [13, III 2.]). Since a polytope satisfies (3.4) if it is a union of polytopes satisfying (3.4), the claim follows from Proposition IV.10 for polytopes not containing any lattice points in their interior. □

The following well known example shows that this result is sharp (cf. [14]).

Example IV.13. Let $P = \operatorname{conv}\langle 0, e_1, \ldots, e_{n-1}, e_1 + \cdots + e_{n-1} + (n-1)\, e_n \rangle \subset \mathbb{R}^n$. Then the vertices are the only lattice points in P. Moreover, $(1, \ldots, 1) \in (n-1)\, P$ cannot be expressed as a sum of lattice points in $(n-2)\, P$.

Corollary IV.14. *Let A be an ample line bundle on a toric variety X of dimension n. Then A^{n-1} is normally generated.*

There are some intriguing open questions related ample line bundles on smooth toric varieties (see also Section C.3).

Question IV.15.

(i) Let A be an ample line bundle on a smooth toric variety X. Is A normally generated? Equivalently, is a smooth polytope normal?

(ii) (Sturmfels [66, Conjecture 13.19]) Let A be an normally generated line bundle on a smooth toric variety. Does A satisfy N_1?

The following example due to Bruns and Gubeladze shows that there are very ample line bundles on singular toric varieties that are not normally generated.

Example IV.16 (Bruns and Gubeladze [4, Example 5.1]). Let

$$\begin{aligned}
v_1 &= (1,1,1,0,0,0) & v_6 &= (0,1,1,0,0,1) \\
v_2 &= (1,1,0,1,0,0) & v_7 &= (0,1,0,1,1,0) \\
v_3 &= (1,0,1,0,1,0) & v_8 &= (0,1,0,0,1,1) \\
v_4 &= (1,0,0,1,0,1) & v_9 &= (0,0,1,1,1,0) \\
v_5 &= (1,0,0,0,1,1) & v_{10} &= (0,0,1,1,0,1),
\end{aligned}$$

and let $P = \langle v_1, \ldots, v_{10} \rangle$. Then

$$(1,1,1,1,1,1) = \sum_{i=1}^{10} \frac{1}{5} v_i \in 2P \cap M$$

cannot be written as a sum of two lattice points in P and so P is not normal. On the other hand, P gives rise to a very ample line bundle A on X_P.

A related question is whether for two line bundles L and M on a projective variety X the natural map

$$H^0(X,L)\otimes H^0(X,M)\to H^0(X,L\otimes M) \tag{3.5}$$

is surjective. Fakhruddin [15] shows that when L is an ample line bundle on a smooth projective toric surface X, and M is globally generated, the map (3.5) is surjective.

4.4 Rational surfaces and applications to toric surfaces

For convenience, we will use divisorial (additive) notation in this section.

In this section we study Mukai's conjecture for toric surfaces. In particular, when X is a smooth toric surface, we give a sharp combinatorial criterion for an adjoint divisor to satisfy N_p.

For ample line bundles on rational surfaces, Gallego and Purnaprajna proved the following criterion for N_p.

Theorem IV.17. *[23, Theorem 1.3.] Let X be a smooth rational surface, let L be a globally generated line bundle on X and let K_X denote the canonical divisor. If $-K_X\cdot c_1(L)\geq p+3$, then L satisfies N_p. Moreover, if L is ample and $-K_X$ is effective, the converse holds.*

They also prove a strong form of Mukai's conjecture [23, Theorem 1.23] for anticanonical surfaces ($-K_X$ effective) as well as a Reider-type theorem for rational surfaces [23, Theorem 1.24], which implies Mukai's conjectures in this case.

In fact, combining Theorem IV.17 with Noether's formula

$$\chi(X,\mathcal{O}_X)=\frac{1}{12}\left(K_X^2+\chi_{\mathrm{top}}(X)\right), \tag{4.6}$$

we obtain the following corollary.

Corollary IV.18. *Let A be a divisor on a smooth rational surface X such that $A+K_X$ is base point free and*

$$-K_X \cdot A \geq p + 15 - \chi_{\text{top}}(X).$$

Then $K_X + A$ satisfies N_p.

Proof. Since X is a rational surface, $\chi(X, \mathcal{O}_X) = 1$. Then

$$\begin{aligned}(A+K_X)\cdot(-K_X) &= A\cdot(-K_X) - K_X^2 \\ &= A\cdot(-K_X) - 12 + \chi_{\text{top}}(X) \\ &\geq p+3\end{aligned}$$

if and only if

$$A\cdot(-K_X) \geq p + 15 - \chi_{\text{top}}(X). \qquad \square$$

For an ample line bundle A on a toric surface, we have a nice combinatorial interpretation of $-K_X \cdot A$.

Lemma IV.19. *Let A be an ample divisor on a toric surface X and let K_X be the canonical divisor of X. Then*

$$(-K_X)\cdot A = |\partial P \cap M|.$$

Proof. Let $K_X \sim -\sum D_\rho$ (see Section C.5) and let E_ρ be the edge of P corresponding to D_ρ (cf. Section C.2). Then by Proposition C.9,

$$A\cdot(-K_X) = \sum A\cdot D_\rho = \sum\left(|E_\rho \cap M| - 1\right) = |\partial P \cap M|. \qquad \square$$

The combinatorial interpretation of the above results for ample divisors on toric surfaces follows from a more general formula for N_p for a divisor D in terms of the volume of the polytope P corresponding to D and the volumes of the facets of P due to Schenck [65].

Theorem IV.20. *[65] Let A be an ample divisor on a toric surface X corresponding to a lattice polygon P. Then A satisfies N_p if and only if*

$$|\partial P \cap M| \geq p + 3.$$

The case $p = 1$ is due to Koelman [35, 34].

In fact, the proof of Theorem IV.17 goes through for singular toric varieties, since toric surfaces are normal and since an ample line bundle on a toric surface is normally generated and therefore induces an embedding that is arithmetically Cohen-Macaulay.

Example IV.21. If $X = \mathbb{P}^2$, then the lattice polygon corresponding to the hyperplane divisor H is the standard simplex $\Delta = \operatorname{conv}\langle (0,0), (1,0), (0,1)\rangle$. It is easy to see that

$$|\partial (d\Delta) \cap M| = \begin{cases} 5 & \text{if } d = 2 \\ 3d - 3 & \text{if } d \geq 3, \end{cases}$$

and so we obtain the well known fact that $\mathcal{O}_{\mathbb{P}^2}(d)$ satisfies N_5 when $d = 2$ and N_{3d-3} when $d \geq 3$, and that these bounds are sharp.

Example IV.22. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $A = \mathcal{O}_X(d_1, d_2)$. Then A corresponds to a rectangle $P = \operatorname{conv}\langle (0,0), (d_1, 0), (0, d_2), (d_1, d_2)\rangle$. Since $|\partial P \cap M| = 2d_1, +2d_2$, we see that A satisfies N_P if and only if $p \leq 2d_1 + 2d_2 - 3$.

The topological Euler characteristic of a toric variety $X = X(\Sigma)$ coincides with the number of maximal cones of Σ, or the number of vertices of any polytope corresponding to an ample divisor on X (cf. [18, Section 3.2]). So Corollary IV.18 and Lemma IV.19 combine to yield the following Corollary.

Corollary IV.23. *Let A be an ample divisor on a smooth toric surface X corresponding to a lattice polygon P. Then $A+K_X$ satisfies N_p if and only if*

$$|\partial P \cap M| + |\{\textit{vertices of } P\}| \geq p+15.$$

Theorem IV.24. *Let $X = X(\Sigma)$ be a toric surface and let $D = -D_1 - \cdots - D_r$ for distinct torus invariant prime divisors D_i. Assume that $X \ncong \mathbb{P}^2$. Let A be an ample divisor on X such that*

$$A \cdot C \geq 3+p$$

for every torus invariant curve C. Then $A+D$ satisfies N_p.

Proof. Let P be the polygon corresponding to $A+D$ and let $d = |\Sigma(1)|$ be the number of rays ρ in Σ. Then $d \geq 3$. By Fujinos' Theorem IV.1 (i), $\frac{2}{3+p}A + D$ is nef. Therefore

$$\begin{aligned} |\partial P \cap M| &= \sum (A+D) \cdot D_\rho \\ &= \sum \left(\frac{p+1}{p+3} A + \frac{2}{p+3} A + D \right) \cdot D_\rho \\ &\geq \sum \frac{p+1}{p+3} A \cdot D_\rho \\ &\geq \sum (p+1) \geq d(p+1) \geq 3(p+1) \geq p+3. \end{aligned}$$

Now the result follows from Theorem IV.20. □

4.5 Segre-Veronese embeddings

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, and let $L = \mathcal{O}_X(d_1, \ldots, d_r)$. Even for these line bundles, N_p is not fully understood.

When $r = 1$, the resulting embedding is the Veronese embedding. The rational normal curve satisfies N_p for all p (cf. [11, Corollary 6.2]). Moreover, Green proves that $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies N_d [27]. On the other hand, when $n = 2$ and $d \geq 3$, then $\mathcal{O}_{\mathbb{P}^2}(d)$ satisfies N_{3d-3} (cf. Example IV.21) and for arbitrary n, Józefiak, Pragacz and Weyman [31] show that $\mathcal{O}_{\mathbb{P}^n}(2)$ satisfies N_5. Ottaviani and Paoletti [50] conjecture that for $n \geq 3$ and $d \geq 3$, $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies N_{3d-3} and they prove that these bounds would be sharp. Rubei [61] shows that $\mathcal{O}_{\mathbb{P}^n}(3)$ satisfies N_4.

For $X = \mathbb{P}^1 \times \mathbb{P}^1$, we have that $\mathcal{O}_X(d_1, d_2)$ satisfies N_p if and only if $p \leq 2d_1 + 2d_2 - 3$ (or the resolution is linear) by Example IV.22. For general r, the Segre embedding $\mathcal{O}_X(1, \ldots, 1)$ satisfies N_3, see Lascoux [37] and Pragacz and Weyman [57] for $r \leq 2$ and Rubei [62] for $r \geq 3$.

The following generalization of Green's result first appeared in a preprint by H. Schenck and G. Smith.

Theorem IV.25. *[29, Corollary 1.5.] Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and let $L = \mathcal{O}_X(d_1, \ldots, d_r)$. Then L satisfies $N_{\min\{d_i\}}$.*

Proof. Let $p_i : X \to \mathbb{P}^{n_i}$ be the projection onto the ith factor. Observe that $B_i := p_i^* \mathcal{O}_{\mathbb{P}^{n_i}}(1)$ generate the semigroup of nef divisors on X. Let $d := \min\{d_i - 1\}$. Since $\mathcal{O}_{\mathbb{P}^n}$ is $\mathcal{O}_{\mathbb{P}^n}$-regular, it follows from Lemma B.6 that $\mathcal{O}_X$ is $\mathcal{O}_X$-regular (see also Maclagan and Smith [42, Proposition 6.10]). Thus, by Theorem B.3.1, $\mathcal{O}_X(d_1 - d, \ldots, d_r - d)$ is $\mathcal{O}_X$-regular and $\mathcal{O}_X(d_1 - d, \ldots, d_j - d - 1, \ldots, d_r - d)$ is nef for all j. Hence Lemma III.8 applies for $B_1, \ldots, B_r$ with $w_1 = (d_1 - d, \ldots, d_\ell - d)$ and $w_k = (1, \ldots, 1)$ for $k \geq 2$. $\square$

4.6 Polytopal semigroup rings and the Koszul property

In this section we assume that K is an algebraically closed field of arbitrary characteristic. The homogeneous coordinate rings of projective toric varieties correspond to *polytopal semigroup rings*. These rings were studied by Hochster [30], who proved that a normal polytopal semigroup ring is Cohen-Macaulay, and by Bruns, Gubeladze and Trung in [5, 6]. In this section we give criteria for polytopal semigroup rings to be Koszul.

Let P be a lattice polytope in $\mathbb{R}^n$ and let K be a field. Let $P \times 1 \in \mathbb{R}^n \times \mathbb{R}$ be the image of P under the map $\iota : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R} : x \mapsto (x, 1)$. Let S_P be the semigroup generated by the lattice points in $\iota(P)$ and let C_P be the cone over $\iota(P)$ with apex the origin. Then the associated algebra $K[S_P]$ is a graded K-algebra and an algebra arising in this way is called a polytopal semigroup algebra.

A semigroup S is called *saturated* if whenever $mu \in S$ for some positive integer m, then $u \in S$. The following proposition is a nice illustration of the interaction of algebraic geometry, combinatorics and commutative algebra (cf. [30]).

Proposition IV.26. *Let P be a lattice polytope corresponding to an ample divisor A on the projective toric variety X_P, let S_P be the associated semigroup, and let $K[S_P]$ be the associated semigroup algebra. The following are equivalent:*

(i) $\mathcal{O}_X(A)$ *is normally generated,*

(ii) P *is normal,*

(iii) $S_P = C_P \cap \mathbb{Z}^{n+1}$,

(iv) $K[S_P]$ *is integrally closed,*

(v) S_P *is saturated.*

Proof. The equivalence of (i) and (ii) is discussed after Definition IV.8. It follows from the definitions that (ii) implies (iii). For the implication $(iii) \Rightarrow (iv)$, we refer to [18, Section 2.1]. To see that $(iv) \Rightarrow (v)$, let $u \in \mathbb{Z}^{n+1}$ such that $mu \in S_P$ for some $m \in \mathbb{N}$. Then χ^u is integral over $K[S_P]$, since $\chi^{mu} \in K[S_P]$, and so

$\chi^u \in K[S_P]$. Hence $u \in S_P$. To see that (v) implies (ii), observe that for a lattice point $x \in mP \cap \mathbb{Z}^n$, there exists $c \in \mathbb{N}$ such that $c(x, m) \in S_P$. In fact, if $v_1, \ldots, v_s$ are the vertices of P, then $x = \sum a_i v_i$ with $a_i \in \mathbb{Q}_{\geq 0}$, so we can choose $c \in \mathbb{N}$ with $ca_i \in \mathbb{N}$ for all i. Hence $(x, m) \in S_P$. In particular, (x, m) can be written as a sum of lattice points of the form $(p, 1) \in \mathbb{Z}^{n+1}$ with $p \in P \cap \mathbb{Z}^n$, and therefore x can be written as a sum of m lattice points in $P \cap \mathbb{Z}^n$. □

We obtain the following criterion for $R(L)$ to be Koszul.

Theorem IV.27. *Let L be a globally generated line bundle on a projective toric variety X. Let r be the number of distinct integer roots of the Hilbert polynomial of L and let d be the degree of the Hilbert polynomial of L. Then for $m \geq d - r$, $R(L^m)$ is Koszul.*

Proof. Without loss of generality we may assume that L is ample. In fact, if P is a polytope corresponding to L giving rise to an ample line bundle A on the projective toric variety X_P, then $R(A) \cong R(L)$ by Proposition C.15. Then the claim follows from Proposition IV.5 and Corollary III.10. □

Expressing this in the language of polytopal semigroup rings,

we obtain the following

Corollary IV.28. *Let P be a polytope of dimension n and let r be the largest integer such that rP does not contain any interior lattice points. Then for $d \geq n - r$, the polytopal semigroup ring $K[S_{dP}]$ is Koszul.*

Bruns, Gubeladze and Trung [5, Theorem 1.3.3.] prove this for $d \geq n$.

Similarly we get the following variant of Theorem II.6 for toric varieties.

Proposition IV.29. *Let X be a Gorenstein projective toric variety of dimension n not isomorphic to $\mathbb{P}^n$ and let A be an ample line bundle on X. Let $B_1, \ldots, B_r$ be the* *generators of the nef cone of* X *and suppose that* $A \otimes B^{-1}$ *is nef for all* $1 \leq i \leq r$. *Let* N *be a nef line bundle and let* $L = K_X \otimes A^d \otimes N$. *Then* $R(L)$ *is Koszul for* $d \geq n+1$.

Proof. Arguing as in the proof of Theorem IV.2, we see that L is $\mathcal{O}_X$-regular and so the statement follows from Corollary III.10. $\square$

APPENDICES

APPENDIX A

Introduction to the Property N_p

A.1 The definition of N_p.

Let L be a globally generated line bundle on a projective variety X. The bundle L induces a morphism

$$\phi_L : X \to \mathbb{P}\left(H^0(X, L)^*\right).$$

Let $R = \oplus_{m \geq 0} H^0(X, L^m)$ and let $S := Sym^\bullet H^0(X, L) \cong K[X_0, \ldots, X_N]$, where $N + 1 = \dim H^0(X, L)$. Then R is a graded S-module. Assume that R is finitely generated (e.g. assume X is normal, cf. [40, Theorem 2.1.30]).

Let

$$\cdots \to E_i \to \cdots \to E_0 \to R \to 0$$

be a minimal free graded resolution of R as an S-module. The modules $E_i \cong \oplus S(-a_{ij})$ encode the information about the syzygies, and so we want to study the algebraic properties of these modules. For example, if $E_0 \cong S$, i.e., if S surjects onto R, then R is generated in degree 1. This implies that R is integrally closed. Moreover, if L is ample, this condition implies that L is very ample (see Mumford [46, Section 1]).

Similarly, if S surjects onto R, then $E_1 \cong \oplus S(-a_{1j})$ is the free module on a set of minimal homogeneous generators of the ideal I of $\phi_L(X)$, in particular the a_{1j} are the degrees of these generators. Similarly, the a_{2j} give the degrees of the minimal first syzygies of the ideal I.

Definition A.1. L satisfies *property* N_0 if $E_0 \cong S$ and L satisfies *property* N_p if

$$E_0 \cong S \text{ and } E_i \cong \bigoplus S(-i-1) \text{ for all } 1 \leq i \leq p.$$

Hence $\phi_L(X)$ is projectively normal if and only if L satisfies N_0 and $\phi_L(X)$ is normal. Moreover, L satisfies N_1 if and only if the homogeneous ideal I of $\phi_L(X)$ is generated by quadrics and N_2 implies that the syzygies among the minimal generators of I are linear.

The concepts of normal generation and normal presentation introduced by Mumford [46] correspond to N_0 and N_1 respectively.

Example A.2 (The twisted cubic). Let $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(3)$, i.e., $X \subset \mathbb{P}^3$ is the twisted cubic. Let $X_0, \ldots, X_3$ denote the coordinates of $\mathbb{P}^3$. The homogeneous ideal of $X \subset \mathbb{P}^3$ is given by (F_1, F_2, F_3), where $F_1 = X_0X_3 - X_1X_2$, $F_2 = X_0X_2 - X_1^2$ and $F_3 = X_1X_3 - X_2^2$. F_1, F_2 and F_3 satisfy the relations $X_1F_1 + X_2F_2 - X_0F_3 = 0$ and $X_2F_1 + X_3F_2 - X_1F_3 = 0$. These generate the module of relations and so the first syzygy module is a free module of rank 2. Since $(X_1, X_2, -X_0)$ and $(X_2, X_3, -X_1)$ satisfy no relations, the higher syzygy modules are zero.

Observe that the relations satisfied by the equations for the twisted cubic are linear. In fact, $\mathcal{O}_{\mathbb{P}^1}(d)$ satisfies N_p for all p, so all the syzygies of the resolution of any rational normal curve are linear.

In general, Green [27] has shown that L^m satisfies N_p for m sufficiently large. But, much as with the question which power of an ample line bundle is very ample, sufficient criteria for m are often not known.

A.2 Criteria for the property N_p

This section is a selection of statements from [24] and [39] slightly adapted for our purposes.

A main tool we use for studying N_p is the vector bundle M_L associated to L which is defined by the short exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0. \tag{A.1}$$

Theorem A.3 (Cohomological criterion for N_p, compare [10, Lemma 1.6]). *Let L be a globally generated line bundle on a projective variety X. Then L satisfies N_p if $H^1\left(X, \wedge^k M_L \otimes L^\ell\right) = 0$ for $1 \leq k \leq p+1$ and $\ell \geq 1$. If also $H^1\left(X, L^\ell\right) = 0$ for $\ell \geq 0$, the converse holds.*

When the characterstic of K is zero, then $\wedge^k M_L$ is a direct summand of $M_L^{\otimes k}$ and we get the following Corollary:

Corollary A.4. *Suppose K has characteristic zero. Then L satisfies N_p if $H^1\left(X, M_L^{\otimes k} \otimes L^\ell\right) = 0$ for $1 \leq k \leq p+1$ and $\ell \geq 1$.*

Another important tool that we also will use in the proof of Theorem A.3, is Koszul cohomology.

Definition A.5. Let $\mathcal{K}_{k,\ell}(X, L)$ denote the middle cohomology of

$$\begin{aligned} \wedge^{k+1} H^0(X, L) \otimes H^0\left(X, L^{\ell-1}\right) &\to \wedge^k H^0(X, L) \otimes H^0\left(X, L^\ell\right) \\ &\to \wedge^{k-1} H^0(X, L) \otimes H^0\left(X, L^{\ell+1}\right). \end{aligned} \tag{A.2}$$

The following Lemma shows that the Koszul cohomology group is simply a way to compute $\operatorname{Tor}^S(R, K)$, where K is the residue field of S at the irrelevant ideal $(X_0, \ldots, X_N)$.

Lemma A.6. $\mathcal{K}_{k,\ell}(X,L) = \operatorname{Tor}_k^S(R,K)_{k+\ell}$.

Proof. Tensoring the Koszul resolution

$$\text{(A.3)} \qquad \cdots \to \wedge^k H^0(X,L) \otimes S(-k) \to \cdots \to H^0(X,L) \otimes S \to S \to K \to 0$$

with R, we see that the degree $(\ell + k)$ part of the k'th module is $\wedge^k H^0(X,L) \otimes H^0\left(X, L^\ell\right)$ and the claim follows. ◻

Now we can interpret N_p in terms of Koszul cohomology.

Proposition A.7. *L satisfies N_p if and only if $\mathcal{K}_{k,\ell}(X,L) = 0$ for $0 \le k \le p$ and $\ell \ge 2$.*

Proof. Let $E_\bullet \to R$ be a minimal free graded resolution of R as an S-module. Then L satisfies N_p if and only if for $0 \le k \le p$, E_k has no minimal generator of degree larger than $k+1$. Since the resolution $E_\bullet \to R$ is minimal, all maps become zero after tensoring with K. So, computing $\operatorname{Tor}_\bullet^S(R,K)$ using the resolution $E_\bullet \to R$, we see that $\operatorname{Tor}_k^S(R,K) \cong E_k \otimes K$. In particular,

$$\dim \operatorname{Tor}_k^S(R,K)_m = \text{ number of generators of } E_k \text{ of degree } m.$$

Hence L satisfies N_p if and only if for $0 \le k \le p$, $\operatorname{Tor}_k^S(R,K)_m = 0$ for $m \ge k+2$. Now the result follows from Lemma A.6. ◻

Koszul cohomology is related to the vector bundle M_L in the following way.

Lemma A.8. *We have an isomorphism*

$$\mathcal{K}_{k,\ell}(X,L) \cong \operatorname{Ker}\left(H^1\left(X, \wedge^{k+1} M_L \otimes L^{\ell-1}\right) \to \wedge^{k+1} H^0(X,L) \otimes H^1\left(X, L^{\ell-1}\right)\right).$$

Proof. Consider the following diagram (see [39, Diagram (1.3.5)]).

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& \wedge^{k+1} M_L \otimes L^{\ell-1} \arrow[d] & & & \\
& \wedge^{k+1} H^0(L) \otimes L^{\ell-1} \arrow[d] \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \wedge^{k} M_L \otimes L^{\ell} \arrow[r] \arrow[d] & \wedge^{k} H^0(L) \otimes L^{\ell} \arrow[r] \arrow[dr] & \wedge^{k-1} M_L \otimes L^{\ell+1} \arrow[r] \arrow[d] & 0 \\
& 0 & & \wedge^{k-1} H^0(L) \otimes L^{\ell+1} &
\end{tikzcd}$$

The vertical and horizontal sequences are induced by (A.1). Taking global sections, we obtain the following diagram.

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& H^0\left(\wedge^{k+1} M_L \otimes L^{\ell-1}\right) \arrow[d] & & & \\
& \wedge^{k+1} H^0(L) \otimes H^0\left(L^{\ell-1}\right) \arrow[d] \arrow[dr, "\alpha"] & & 0 \arrow[d] & \\
0 \arrow[r] & H^0\left(\wedge^{k} M_L \otimes L^{\ell}\right) \arrow[r, "\lambda"] \arrow[d, "\psi"'] & \wedge^{k} H^0(L) \otimes H^0\left(L^{\ell}\right) \arrow[r, "\mu"] \arrow[dr, "\beta"] & H^0\left(\wedge^{k-1} M_L \otimes L^{\ell+1}\right) \arrow[d] & \\
& H^1\left(\wedge^{k+1} M_L \otimes L^{\ell-1}\right) \arrow[d, "\phi"'] & & \wedge^{k-1} H^0(L) \otimes H^0\left(L^{\ell+1}\right) & \\
& \wedge^{k+1} H^0(L) \otimes H^1\left(L^{\ell-1}\right) & & &
\end{tikzcd}$$

Then $\mathcal{K}_{k,\ell}(X,L) = \operatorname{Ker}\beta / \operatorname{Im}\alpha$ is the cohomology of the diagonal sequence. Observe that $\operatorname{Ker}\beta = \operatorname{Ker}\mu = \operatorname{Im}\lambda$, so in particular λ maps $H^0\left(\wedge^k M_L \otimes L^\ell\right)$ isomorphically onto $\operatorname{Ker}\beta$. Let

$$\Phi : \mathcal{K}_{k,\ell}(X,L) \to \operatorname{Ker}\phi, \;\; x \mapsto \psi\lambda^{-1}(x).$$

Then Φ is well defined and an isomorphism. $\square$

Proof of Theorem A.3. If $H^1\left(\wedge^k M_L \otimes L^\ell\right) = 0$ for $1 \le k \le p+1$ and $\ell \ge 1$, then by Lemma A.8, $\mathcal{K}_{k,\ell}(X, L) = 0$ for $0 \le k \le p$ and $\ell \ge 2$, and L satisfies N_p by Proposition A.7. Conversely, Lemma A.8 shows that the vanishing of $\mathcal{K}_{k,\ell}(X, L)$ implies the vanishing of $H^1\left(\wedge^{k+1} M_L \otimes L^{\ell-1}\right)$ provided that $H^1\left(L^\ell\right) = 0$. □

A.3 Koszul rings

When L is a globally generated line bundle and $R(L) = \bigoplus_{m\ge 0} H^0(X, L^m)$ is the section ring associated to L the question whether $R(L)$ is Koszul is closely related to whether L satisfies N_1. We follow closely the exposition in Pareschi [51].

Definition A.9. Let $R = K \oplus \bigoplus_{m\ge 1} R_m$ be a graded K-algebra. R is *Koszul* if and only if $\mathrm{Tor}_h^R(K, K)_\ell = 0$ for all $\ell \ne h$.

If

$$\cdots E_2 \to E_1 \to R \to K \to 0 \tag{A.4}$$

is the free minimal resolution of K as an R-module, then R is Koszul if and only if $E_i \cong \bigoplus_j R(-a_{ij})$ with $a_{ij} = i$ for all j. To see this, observe that all maps in (A.4) become zero after tensoring with K. The following well known fact explains the relationship between the Koszul property of the section ring $R(L)$ and N_1.

Proposition A.10 ([3, Lemma 1.5.11]). *Let L be a globally generated line bundle. If $R(L)$ is Koszul, then L satisfies N_1.*

The converse does not hold: Sturmfels [67, Theorem 3.1.] gives an example of a smooth projectively normal curve of genus 7 in $\mathbb{P}^5$ whose coordinate ring is presented by quadrics that is not Koszul.

There is a cohomological criterion similar to Theorem A.3 for the section ring of a globally generated line bundle to be Koszul. First, we need to introduce some notation.

Let E be a vector bundle that is generated by its global sections. Just like in (A.1) we associate to E a vector bundle M_E via the short exact sequence

$$0 \to M_E \to H^0(X, E) \otimes \mathcal{O}_X \to E \to 0. \tag{A.5}$$

Given a line bundle L, let $M^{(0,L)} = L$ and let $M^{(1,L)} = M_L \otimes L$. If $M^{(1,L)}$ is globally generated, we let $M^{(2,L)} = M_{M^{(1,L)}} \otimes L$. Continuing inductively, if $M^{(h-1,L)}$ is globally generated, we let $M^{(h,L)} = M_{M^{(h-1,L)}} \otimes L$.

Lemma A.11 (Lazarsfeld, see [51, Lemma 1]). *Let X be a projective variety, let L be a globally generated line bundle on X and let $R(L)$ be the section ring associated to L. Assume that the vector bundles $M^{(h,L)}$ are defined for all $h \geq 0$. If $H^1\left(M^{(h,L)} \otimes L^\ell\right) = 0$ for all $\ell \geq 0$ then $R(L)$ is Koszul. Moreover, if $H^1\left(L^\ell\right) = 0$ for all $\ell \geq 1$, the converse also holds.*

Observe that the proof goes through for a singular variety over an arbitrary field.

APPENDIX B

Regularity

In this section we review multigraded regularity as introduced by Maclagan and Smith [42]. Let $B_1, \ldots, B_r$ be globally generated line bundles on a projective variety X. For $u := (u_1, \ldots, u_r) \in \mathbb{Z}^r$, let $B^u := B_1^{u_1} \otimes \cdots \otimes B_r^{u_r}$. Observe that if $e_1, \ldots, e_r$ is the standard basis for $\mathbb{Z}^r$, then $B^{e_j} = B_j$.

Definition B.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let L be a line bundle on X. Define $\mathcal{F}$ to be L-regular (with respect to $B_1, \ldots, B_r$) if $H^i(X, \mathcal{F} \otimes L \otimes B^{-u}) = 0$ for all $i > 0$ and all $u \in \mathbb{N}^r$ satisfying $|u| := u_1 + \cdots + u_r = i$.

Note that when $r = 1$, this is the usual definition of Castelnuovo-Mumford regularity—see regularity.

Remark B.2. Observe that if $\mathcal{F}$ is L-regular, so is $\mathcal{F} \oplus \cdots \oplus \mathcal{F}$ as cohomology commutes with direct sums.

Mumford's theorem [40, Theorem 1.8.5.] generalizes to the multigraded case in the following way.

Theorem B.3 ([29], Theorem 2.1). *Let $\mathcal{F}$ be L-regular. Then for all $u \in \mathbb{N}^r$*

1. *$\mathcal{F}$ is $(L \otimes B^u)$-regular;*

2. *the natural map*

$$H^0(X, \mathcal{F} \otimes L \otimes B^u) \otimes H^0(X, B^v) \longrightarrow H^0\left(X, \mathcal{F} \otimes L \otimes B^{u+v}\right)$$

is surjective for all $v \in \mathbb{N}^r$;

3. *$\mathcal{F} \otimes L \otimes B^u$ is generated by its global sections, provided there exists $w \in \mathbb{N}^r$ such that B^w is ample.*

When X is a toric variety, this follows from results in Maclagan and Smith [42]. Our proof imitates Mumford [46, Theorem 2] and Kleiman [33, Proposition II.1.1].

Proof. By replacing $\mathcal{F}$ with $\mathcal{F} \otimes L$, we may assume that the coherent sheaf $\mathcal{F}$ is $\mathcal{O}_X$-regular. We proceed by Noetherian induction on $\operatorname{Supp}(\mathcal{F})$. When $\operatorname{Supp}(\mathcal{F}) = \emptyset$, the claim is trivial. As each B_j is base point-free, we may choose a section $s_j \in H^0(X, B_j)$ such that the induced map $\mathcal{F} \otimes B^{-e_j} \to \mathcal{F}$ is injective (see Mumford [46, page 43]). Let $\mathcal{G}_j$ be the cokernel defined by

$$0 \to \mathcal{F} \otimes B^{-e_j} \to \mathcal{F} \to \mathcal{G}_j \to 0.$$

Then $\operatorname{Supp}(\mathcal{G}_j) \subsetneq \operatorname{Supp}(\mathcal{F})$. Twisting by B^{-u} and taking the long exact sequence of cohomology, we get

$$\begin{aligned} \text{(B.1)} \quad \cdots \to H^i\left(X, \mathcal{F} \otimes B^{-u-e_j}\right) &\to H^i\left(X, \mathcal{F} \otimes B^{-u}\right) \\ &\to H^i\left(X, \mathcal{G}_j \otimes B^{-u}\right) \to H^{i+1}\left(X, \mathcal{F} \otimes B^{-u-e_j}\right) \to \cdots . \end{aligned}$$

Letting $|u| = i$, we see that $\mathcal{G}_j$ is $\mathcal{O}_X$-regular.

For (1) observe that it suffices to show that $\mathcal{F}$ is B_j-regular for all j. The induction hypothesis implies that $\mathcal{G}_j$ is B_j-regular. Setting $u = -e_j + u'$ with $|u'| = i$ in (B.1), the claim follows.

For (2), consider the commutative diagram:

$$\begin{tikzcd}
 & & H^0(X,\mathcal{F}) \otimes H^0(X,B_j) \arrow[r] \arrow[d] & H^0(X,\mathcal{G}_j) \otimes H^0(X,B_j) \arrow[d] \\
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[ur] \arrow[r] & H^0(X,\mathcal{F} \otimes B_j) \arrow[r] & H^0(X,\mathcal{G}_j \otimes B_j) .
\end{tikzcd}$$

Since $\mathcal{F}$ is $\mathcal{O}_X$-regular, the map in the top row is surjective. The induction hypothesis guarantees that the map in the right column is surjective. Thus, the Snake Lemma implies that the map in the middle column is also surjective. Therefore, (2) follows from the associativity of the tensor product and (1).

Lastly, consider the commutative diagram:

$$\begin{tikzcd} H^0\left(X,\mathcal{F}\otimes B^u\right)\otimes H^0\left(X,B^v\right)\otimes\mathcal{O}_X \arrow[r] \arrow[d] & H^0\left(X,\mathcal{F}\otimes B^{u+v}\right)\otimes\mathcal{O}_X \arrow[d, "\beta_{u+v}"'] \\ H^0\left(X,\mathcal{F}\otimes B^u\right)\otimes B^v \arrow[r, "\beta_u\otimes\mathrm{id}"] & \mathcal{F}\otimes B^{u+v} \end{tikzcd}$$

Applying (2), we see that the map in the top row is surjective. By assumption, there is $w\in\mathbb{N}^r$ such that B^{mw} is ample. If $v:=kw$, then Serre's vanishing Theorem ([40, Theorem 1.2.6]) implies that β_{u+v} is surjective for k sufficiently large. Hence, β_u is also surjective which proves (3). $\square$

Observe that Part 2. of this theorem implies in particular the following vanishing for regular sheaves.

Lemma B.4. *Let $\mathcal{F}$ be a coherent sheaf on a projective variety X. Suppose that $\mathcal{F}$ is $\mathcal{O}_X$-regular. Then for all $u,u'\in\mathbb{N}^r$ with $|u|=i$,*

$$H^i\left(X,\mathcal{F}\otimes B^{-u+u'}\right)=0.$$

The following lemma gives an example of how regularity behaves in short exact sequences. It plays a key part in the proof of Theorem III.8.

Lemma B.5. *Let $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ be a short exact sequence of coherent $\mathcal{O}_X$-modules. If $\mathcal{F}$ is L-regular, $\mathcal{F}''$ is $(L\otimes B^{-e_j})$-regular for all $1\le j\le r$ and $H^0\left(X,\mathcal{F}\otimes L\otimes B^{-e_j}\right)\to H^0\left(X,\mathcal{F}''\otimes L\otimes B^{-e_j}\right)$ is surjective for all $1\le j\le r$, then $\mathcal{F}'$ is also L-regular.*

Sketch of Proof. This is similar to the proof of Theorem B.3.1: tensor the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

with $L \otimes B^u$ and analyze the associated long exact sequence. $\square$

Multigraded regularity also behaves well with respect to products.

Lemma B.6. *For $1 \leq i \leq r$, let X_i be a projective algebraic variety, B_i a globally generated line bundle on X_i and let $\mathcal{F}_i$ be a coherent sheaf on X_i. Let*

$$X = X_1 \times \cdots \times X_r$$

be the product, and let $p_i : X \to X_i$ be the projection onto the i'th factor. Suppose $\mathcal{F}_i$ is L_i-regular with respect to B_i for all $1 \leq i \leq r$. Then $p_1^*\mathcal{F}_1 \otimes \cdots \otimes p_r^*\mathcal{F}_r$ is $p_1^*L_1 \otimes \cdots \otimes p_r^*L_r$-regular with respect to $p_1^*B_1, \ldots, p_r^*B_r$.

Proof. For simplicity of notation, we will omit the p_i^*. Replacing $\mathcal{F}_i$ by $\mathcal{F}_i \otimes L_i$, we may assume that $\mathcal{F}_i$ is $\mathcal{O}_{X_i}$-regular. We proceed by induction on r. Let $X' = X_1 \times \cdots \times X_{r-1}$ and let $\mathcal{F}' = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_{r-1}$ on X'. By the induction hypothesis, $\mathcal{F}'$ is $\mathcal{O}_{X'}$-regular with respect to $B_1, \ldots, B_{r-1}$. For $u = (u_1, \ldots, u_r) \in \mathbb{N}^r$ with $|u| = i$, let $u' = (u_1, \ldots, u_{r-1}, 0) \in \mathbb{N}^r$. Then, considering $B^{u'}$ as a sheaf on X', we have

$$\begin{aligned} H^i\left(X' \times X_r, \mathcal{F}' \otimes \mathcal{F}_r \otimes B^{-u}\right) &= H^i\left(X' \times X_r, \mathcal{F}' \otimes B^{-u'} \otimes \mathcal{F}_r \otimes B^{-u_r}\right) \\ &= \bigoplus_{k'+k=i} H^{k'}\left(X', \mathcal{F}' \otimes B^{-u'}\right) \otimes H^k\left(X_r, \mathcal{F}_r \otimes B_r^{u_r}\right), \end{aligned}$$

where the last equality follows from the the Künneth formula [64]. Let $k' + k = i$. If $|u'| \leq k'$. Then by Lemma B.4, $H^{k'}\left(X', \mathcal{F} \otimes B^{u'}\right) = 0$. If $|u'| > k'$, then $u_r < k$ and again by Lemma B.4, $H^k\left(X_r, \mathcal{F} \otimes B_r^{-u_r}\right) = 0$. In particular, $H^i\left(X' \times X_r, \mathcal{F}' \otimes \mathcal{F}_r \otimes B^{-u}\right) = 0$. $\square$

APPENDIX C

Toric varieties and vanishing theorems

C.1 Introduction and Notation

In this appendix we present the results on line bundles on toric varieties that we use in the course of the thesis. It is by no means supposed to be an introduction to the theory of toric varieties. The main reference is Fulton's book "Introduction to Toric varieties" [18] and the definitions as well as the proofs of the stated theorems can be found there unless otherwise noted.

Let K be an algebraically closed field. A *toric variety* is a normal algebraic variety containing an open dense torus $T \cong (K^*)^n$ together with an action of T on X that naturally extends the torus action of T on itself. Let $N \cong \mathbb{Z}^n$ be a lattice and let $M := \operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice to N. An element $u = (u_1, \ldots, u_n) \in M$ gives rise to a character of the torus T

$$\chi^u := x_1^{u_1} \cdots x_n^{u_n} \in k[T] = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

which can be identified with the corresponding regular function in $k[T]$.

Let $\sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$ be a rational polyhedral cone of dimension n and let $\sigma^\vee = \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$ be the dual cone. Let $S_\sigma = \sigma^\vee \cap M$, a finitely generated semigroup. Then $K[S_\sigma] = \{\chi^u \mid u \in S_\sigma\}$ is a finitely generated

K-algebra giving rise to an affine algebraic variety $U_\sigma = \operatorname{Spec}(K[S_\sigma])$ of dimension n. When $\Sigma \subset N_{\mathbb{R}}$ is a rational polyhedral fan, then the open sets U_σ for $\sigma \in \Sigma$ glue together to a toric variety $X(\Sigma)$. In fact, all toric varieties arise in this way.

Given two toric varieties X, X' corresponding to fans $\Sigma \subset N_{\mathbb{R}}$ and $\Sigma' \subset N'_{\mathbb{R}}$, a lattice homomorphism $\phi : N \to N'$ induces a rational map $f : X \to X'$. Moreover, if for every cone $\sigma \in \Sigma$ there exists a cone $\sigma' \in \Sigma'$ such that $\phi(\sigma) \subset \sigma'$, then f is a morphism.

C.2 Divisors on Toric varieties

There is an order reversing correspondence between the cones $\sigma \subset \Sigma$ and the irreducible torus invariant subvarieties of $X(\Sigma)$. In particular, if $\Sigma(1)$ denotes the set of rays of Σ, then to each ray $\rho \in \Sigma(1)$ is associated a torus invariant prime divisor D_ρ. In fact, the torus invariant prime divisors generate the class group $\operatorname{Cl}(X)$, so for any divisor D we have that $D \sim \sum a_\rho D_\rho$, where $\sim$ denotes linear equivalence.

Let v_ρ be the primitive lattice vector generating the ray ρ. A divisor $\sum a_\rho D_\rho$ is Cartier if and only if there exists a real valued function $\psi_D : |\Sigma| \to \mathbb{R}$ on the support $|\Sigma|$ of Σ that is linear on each cone $\sigma \subset \Sigma$ with $\psi_D(v_\rho) = -a_\rho$. Let $u(\sigma) = \psi_D|_\sigma \in M$. The function ψ_D gives convenient criteria for global generation and ampleness of a divisor. Recall that a line bundle L on an algebraic variety X is *globally generated* if for every point $x \in X$ there exists a global section of L not vanishing at x. Such a line bundle induces a morphism $\phi_L : X \to \mathbb{P}(H^0(X, L))$. The line bundle L is called *ample* if some multiple $L^m := \underbrace{L \otimes \cdots \otimes L}_{m}$ is *very ample*, i.e., it induces an embedding. A Cartier divisor D is called ample (globally generated) if $\mathcal{O}_X(D)$ is ample (globally generated), and D is called $\mathbb{Q}$-ample ($\mathbb{Q}$-globally generated) if mD is Cartier and ample (globally generated) for some $m \in \mathbb{N}$. Then

$$\begin{array}{lcl} D \text{ is globally generated} & \Leftrightarrow & \psi_D \text{ is convex} \\ D \text{ is ample} & \Leftrightarrow & \psi_D \text{ is strictly convex with respect} \\ & & \text{to the maximal cones of } \Sigma. \end{array}$$

In particular, every ample line bundle on a toric variety is globally generated.

To the divisor $D = \sum a_\rho D_\rho$ is associated a polytope

$$P_D := \{u \in M_{\mathbb{R}} \mid \langle u, v_\rho \rangle \geq -a_\rho\}$$

with the property that

$$H^0\left(X, \mathcal{O}\left(D\right)\right) \cong \bigoplus_{u \in P_D \cap M} K\chi^u. \tag{C.1}$$

Moreover, a Cartier divisor D is globally generated if and only if P_D is a lattice polytope.

Conversely, given a lattice polytope $P \subset M_{\mathbb{R}}$, we can construct a (possibly degenerate) fan Σ_P in $N_{\mathbb{R}}$ whose cones are in order reversing correspondence with the faces of P as follows. To P is associated a piecewise linear convex function

$$\Xi_P : N_{\mathbb{R}} \to \mathbb{R}, v \mapsto \min_{u \in P}\{\langle u, v \rangle\}.$$

For a face Q of P, we let

$$\sigma_Q := \{v \in N_{\mathbb{R}} \mid \langle u, v \rangle = \Xi_P\left(v\right) \text{ for all } u \in Q\}.$$

The maximal cones of Σ_P correspond to the maximal domains of linearity of Ξ_P. In particular, a lattice polytope P of dimension n gives rise to an ample line bundle A_P on the toric variety of dimension n, $X\left(\Sigma_P\right)$. Similarly, a divisor D on a toric variety $X\left(\Sigma\right)$ is $\mathbb{Q}$-ample if and only if $\Sigma_{P_D} = \Sigma$ and then there is an order preserving correspondence between the faces of the polytope and the torus invariant subvarieties of X. Moreover, D is $\mathbb{Q}$-globally generated if and only if Σ refines the (possibly degenerate) fan Σ_{P_D}, i.e., for every cone $\sigma \in \Sigma$ there is a cone $\sigma' \in \Sigma_{P_D}$ such that $\sigma \subset \sigma'$.

The following lemma gives a criterion for very ampleness of a divisor.

Lemma C.1. *Let D be a Cartier divisor on a toric variety X. Then D is very ample if and only if ψ_D is strictly convex and for all maximal cones $\sigma \in \Sigma$, the semigroup S_σ is generated by $\{u - u(\sigma) \in P_D \cap M\}$.*

C.3 Smooth toric varieties

The following gives a criterion for a toric variety to be smooth.

Proposition C.2. *Let $X = X(\Sigma)$ be a toric variety. Then X is smooth if and only if each cone $\sigma \in \Sigma$ can be generated by part of a basis for the lattice N.*

On a smooth toric variety, the question of which power of an ample line bundle is very ample has a very simple answer.

Proposition C.3. *Let A be an ample line bundle on a smooth toric variety X. Then A is very ample.*

Definition C.4. Let P be a polytope of dimension n. P is called *simple* if every vertex of P is contained in exactly n facets (codimension 1 faces).

Definition C.5. Let P be a polytope corresponding to an ample line bundle A on a toric variety X. P is called *smooth* if X is smooth.

Then Proposition C.2 implies that a smooth polytope is simple and that for any vertex v of P, the set of vectors obtained by taking the difference between the first lattice vector on an edge going out from v and v forms a basis of M.

C.4 Numerical criteria for toric divisors

A good way to measure the positivity of a line bundle is by intersecting the divisor with certain subvarieties.

Definition C.6. Let D be a Cartier divisor on a complete variety X. Then D is called *nef* if and only if

$$(D \cdot C) \geq 0$$

for all irreducible curves $C \subset X$.

The nef divisors form a cone, the *nef cone* $\mathrm{Nef}\,(X)$ whose interior is the ample cone (see [40, Theorem 1.4.23]).

When X is toric, the criterion for nefness as well the for ampleness (for example, [40, Theorem 1.2.23]) take a particularly simple form.

Theorem C.7. *[47, Theorem 3.1 and 3.2] Let D be a Cartier divisor on a complete toric variety X. Then*

(i) D is nef if and only if for every torus invariant curve C, $(D \cdot C) \geq 0$, and

(ii) D is ample if and only if for every torus invariant curve C, $(D \cdot C) > 0$.

In particular, we see that the nef cone of a toric variety is a polyhedral cone as there are only finitely many irreducible torus invariant curves on a toric variety.

In general, every divisor that is globally generated is nef. When X is toric, the converse holds.

Proposition C.8. *[38, Proposition 1.5] Let D be a Cartier divisor on a complete toric variety X. Then D is nef if and only if D is globally generated.*

The following criterion due to Laterveer gives a nice combinatorial description of the intersection of an ample line bundle with a torus invariant curve.

Proposition C.9. *[38, 1.4] Let A be an ample line bundle on a projective variety X corresponding to a polytope P. For a torus invariant curve C, let E be the corresponding edge of P. Then $A \cdot C$ equals the lattice length of E, i.e.,* $(\#\{E \cap M\} - 1)$.

C.5 Vanishing theorems on toric varieties

We start with a vanishing theorem for globally generated line bundles.

Theorem C.10. *Let L be a globally generated line bundle on a toric variety X, then $H^i(X, L) = 0$ for all $i > 0$.*

A toric variety is Cohen-Macaulay, so it has a *dualizing sheaf* ω_X and Serre duality holds. In fact, the dualizing sheaf is isomorphic to $\mathcal{O}_X(-\sum D_\rho)$, where we sum over all torus invariant prime divisors. In particular, if X is Gorenstein, then $-\sum D_\rho$ is a representative of the canonical divisor K_X.

Theorem C.11 (Serre duality). *Let X be a complete toric variety and let $\omega_X = \mathcal{O}_X(-\sum D_\rho)$. Then for any line bundle L on X we have*

$$H^{n-i}\left(X, L^{-1} \otimes \omega_X\right) \cong H^i(X, L)^*.$$

We will also make use of the Kodaira vanishing theorem.

Theorem C.12 (Kodaira vanishing). *Let X be a smooth irreducible complex projective variety of dimension n, and let L be an ample line bundle on X. Then*

$$H^i(X, L \otimes K_X) = 0 \text{ for } i > 0.$$

On toric varieties, a more general form of Kodaira vanishing holds.

Theorem C.13 (Kodaira vanishing for toric varieties [47, Corollary 2.5]). *Let X be a projective toric variety of dimension n and let A be an ample divisor bundle on X. Let $D_1, \ldots, D_r$ be distinct torus invariant prime divisors. Then* *(i)* $H^i(X, \mathcal{O}_X(A - D_1 - \cdots - D_r)) = 0$ *for all* $i \geq 1$, *and*

(ii) $H^i(X, \mathcal{O}_X(-A)) = 0$ *for all* $i < n$.

Here C.13 (ii) follows from C.13 with $K_X = \sum D_\rho$ (i) and Serre duality.

Proposition C.14. *Let* N *be a lattice, let* Σ' *be a fan that refines a possibly degenerate fan* Σ *in* $N_{\mathbb{R}}$, *and let* $f : X(\Sigma') \to X(\Sigma)$ *be the map induced by the identity on* N. *Then*

(i) $f_*\mathcal{O}_{X(\Sigma')} = \mathcal{O}_{X(\Sigma)}$, *and*

(ii) $R^i f_* \mathcal{O}_{X(\Sigma')} = 0$ *for* $i \geq 1$.

Observe that the proof of the last Proposition in [18, Section 3.5] also goes through in the degenerate case.

Proposition C.15. *Let* N *be a lattice and let* Σ *be a complete fan in* $N_{\mathbb{R}}$ *giving rise to a complete toric variety* $X = X(\Sigma)$. *Let* D *be a nef divisor on* X *corresponding to a lattice polytope* P. *Let* (A, X_P) *be the ample line bundle on the toric variety* X_P *associated to* P *and let* $f\colon X \to X_P$ *be the map induced by the identity on* N. *Then for any Cartier divisor* D' *on* X_P,

$$H^i(X, f^*D') = H^i(X_P, D')$$

for all $i \geq 0$.

Proof. Since the fan Σ defining X is a refinement of the (possibly degenerate) fan Σ_P defining X_P, it follows from the projection formula and Proposition C.14 that $f_*f^*D' = D'$ and $R^i f_*(\mathcal{O}_X(f^*D')) = 0$ for all $i \geq 1$ implying the claim. □

Often we can use Proposition C.15 to generalize results for ample line bundles to globally generated line bundles. For example, combining it with Theorem C.12 (ii), we obtain the following.

Proposition C.16. *Let D be a globally generated divisor on a projective toric variety X and let P_D be the polytope associated to D. Then*

$$H^i\left(X, \mathcal{O}_X\left(-D\right)\right) = 0 \text{ for } i < \dim P_D.$$

APPENDIX D

Arbitrary fields

Since the results on toric varieties in Chapter IV and Appendix C only depend on combinatorial data, they hold for varieties over any field.

The cohomological criteria A.3 and A.11 discussed in Appendix A hold for any field. Most applications require Corollary A.4 though, which uses the assumption that K has characteristic zero. We can set up the theory of regularity as discussed in Appendix B over any field, but our proof of Mumford's theorem B.3 uses prime avoidance, hence we need to assume that K is infinite.

The results in Sections 3.1 and 3.2 require that K has characteristic zero, since they rely on Corollary A.4 and some of them use the Kodaira vanishing theorem. In Section 3.3 we need to assume that K is infinite, since Lemma III.9 uses Mumford's theorem B.3.

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ABSTRACT

Syzygies of toric varieties

by

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Studying the equations defining the embedding of a projective variety and the higher relations (syzygies) between them is a classical problem in algebraic geometry. We give criteria for ample line bundles on toric varieties to give rise to a projectively normal embedding whose ideal is generated by quadratic equations and whose first q syzygies are linear. We illustrate the interactions with the combinatorics of lattice polytopes, and we study the related question of when the homogeneous coordinate ring of the embedding is Koszul. We obtain these results by exploiting the connection between the regularity of an ample line bundle L on a projective variety and the syzygies of embeddings induced by powers of L. Much of this has also appeared in a preprint with H. Schenck and G. Smith.