

# Constructing algebraic theories

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# Outline

- ▶ Algebraic theory – definition and examples
- ▶ Combining algebraic theories: distributive laws
- ▶ Semigroups  $\xrightarrow{\text{generalise}}$  square tilings
- ▶ Combining algebraic theories: tensor product

# Algebraic theories

## Definition (algebraic theory)

An *algebraic theory* consists of a set of operation symbols  $\Sigma$  called the *signature*, a function  $\text{arity} : \Sigma \rightarrow \{0, 1, 2, \dots\}$ , and a set of *equations*  $E$ .

An *algebra* for the theory  $(\Sigma, E)$  is

- ▶ A set  $A$ ;
- ▶ for all  $\sigma \in \Sigma$ , a function  $A^{\text{arity } \sigma} \rightarrow A$ ;
- ▶ such that the equations in  $E$  are always satisfied.

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## Example

Commutative monoids:

$$\Sigma = \{ \cdot^{(2)}, 1^{(0)} \}, \quad (\text{i.e. } \text{arity}(\cdot) = 2, \text{arity}(1) = 0),$$

$$E = \{ a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad a \cdot b = b \cdot a, \quad a \cdot 1 = a \}.$$

# Algebraic theories

## Example

Abelian groups:

$$\Sigma = \{ +^{(2)}, -^{(1)}, 0^{(0)} \},$$

$$E = \left\{ \begin{array}{ll} a + (b + c) = (a + b) + c, & a + b = b + a, \\ a + (-a) = 0, & a + 0 = a \end{array} \right\}.$$

## Distributive laws

We can combine algebraic theories, e.g.

abelian groups  $\circ$  commutative monoids

= commutative rings with 1.

We take the operations and equations of abelian groups and commutative monoids, and add the *distributive law*

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Distributive laws:

- ▶ 'doing + first and then  $\cdot$ '  $\longrightarrow$  'doing  $\cdot$  first and then +'
- ▶ May or may not exist or be unique
- ▶ Very hard to find or to show that we have them all
- ▶ Applications to the structure of programming languages

# The theory of semigroups

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Elements of free semigroups are just non-empty lists. We can 'substitute': if  $W = xyz$ , then

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## Question

*Is there an algebraic theory that gives a two-dimensional generalisation of non-empty lists?*

## Labelled square tilings

Can we just use labelled 2d grids? No: we can't substitute.

$$U = \boxed{x}, \quad V = \boxed{y \mid z}$$

⇓

$U$	$V$
$U$	$U$

 $=$ 

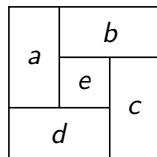
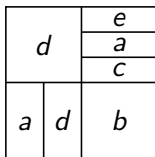
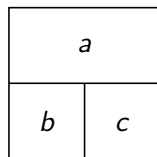
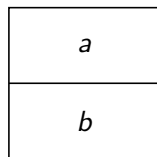
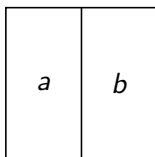
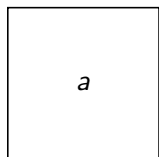
$x$	$y$	$z$
$x$	$x$	

Not a grid!

## Labelled square tilings

Let  $A$  be a set. We define a free algebra that has as elements all  $A$ -labelled tilings of the unit square by rectangles, up to...?

E.g. if  $A = \{a, b, c, d, e\}$ :



## Labelled square tilings

We can now substitute, by 'scaling the tilings to fit'. E.g.

$$U = \begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array}, \quad V = \begin{array}{|c|c|} \hline x & x \\ \hline x & y \\ \hline \end{array}, \quad W = \begin{array}{|c|c|} \hline y & z \\ \hline \end{array}$$

⇓

$$\begin{array}{|c|c|} \hline & U \\ \hline U & V \\ \hline & W \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & x & & \\ \hline & y & & \\ \hline x & x & x & \\ \hline & y & y & \\ \hline y & y & z & \\ \hline \end{array}$$

## Labelled square tilings

We can use these tilings to craft an algebraic theory. Our operations will be

$$a \cdot b = \begin{array}{|c|c|} \hline a & b \\ \hline \end{array}, \quad a * b = \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}.$$

E.g.

$$\begin{array}{|c|c|} \hline & b \\ \hline a & \\ \hline & c \\ \hline \end{array} = a \cdot (b * c).$$

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$$a \cdot b = \begin{array}{|c|c|} \hline a & b \\ \hline \end{array}, \quad a * b = \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}.$$

Our equations will represent geometric properties of the tilings:

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \text{ gives } a \cdot (b \cdot c) = (a \cdot b) \cdot c,$$

$$\begin{array}{|c|} \hline a \\ \hline b \\ \hline c \\ \hline \end{array} \text{ gives } a * (b * c) = (a * b) * c,$$

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \text{ gives } (a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d).$$

## Combining theories: tensor product

Our theory is as follows.

$$\Sigma = \{ \cdot^{(2)}, *^{(2)} \},$$

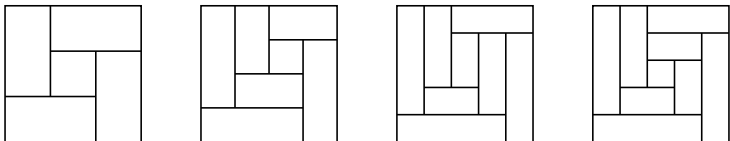
$$E = \left\{ \begin{array}{l} a \cdot (b \cdot c) = (a \cdot b) \cdot c \\ a * (b * c) = (a * b) * c \\ (a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d) \end{array} \right\}$$

This is an example of a (*commutative*) *tensor product* of algebraic theories.

Tensor products are vastly different from distributive laws. Tensor products always exist, and are always unique.

## Labelled square tilings

When considering all tilings, we have infinitely many 'irreducible' tilings that cannot be written only using  $\cdot$  and  $*$ . E.g.



We have an algebraic object, our tensor product of the theory of semigroups, sitting inside a geometric object, our square tilings.

We can also generalise to arbitrarily many dimensions!

## Acknowledgements

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Thank you for listening!