Monads and distributive laws

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Overview

- What is a monad?
- What is a distributive law?

Monads

Let **Sets** be the category of all sets.

A *monad* on **Sets** is a functor (think function) $T : \mathbf{Sets} \to \mathbf{Sets}$ with some extra structure.

Monads

Example (Free abelian group monad)

$$A: \quad \mathbf{Sets} \to \mathbf{Sets}$$
$$\{a, b, c\} \mapsto \{0,\ a,\ b,\ c,\ 2a,\ a-7b,\ -a+b+4c,\ \dots\}$$

An abelian group is just a set X and a function $A(X) \to X$ satisfying some properties.

Monads

Example (Free monoid monad)

$$M: \quad \mathbf{Sets} \to \mathbf{Sets}$$
$$\{a, b, c\} \mapsto \{1, a, b, c, aa, abba, bbbac, \dots\}$$

A monoid is just a set X and a function $M(X) \to X$ satisfying some properties.

Monads

What if we apply a monad twice?

For monoids:

$$MM(\{a, b\}) = \{(bbb)(a),\ (ab)(ba),\ (abb)(a),\ (ba)(1)(a),\ \dots\}$$

$$\downarrow$$

$$M(\{a, b\}) = \{\quad bbba \quad , \quad abba \ = \ abba \quad , \quad baa \quad , \dots\}$$

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It is part of the structure of all monads T that we can map $TT(X) \to T(X)$ to 'simplify terms of terms'.

Monads

Can we compose monads?

E.g. can we make $AM : \mathbf{Sets} \to \mathbf{Sets}$ into a monad?

Well, we'd need a way to map

$$AMAM(X) \to AM(X).$$

Distributive laws

To get a function

$$AMAM(X) \to AM(X),$$

we use something called a *distributive law*.

A distributive law here is a way to transform

$$\lambda : MA \to AM.$$

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Then we can do

$$AMAM(X) \xrightarrow{\lambda} AAMM(X) \xrightarrow{\text{simplify}} AM(X).$$

Distributive laws

Example

$$\lambda : \qquad MA \to AM$$
$$(a+b)(c+d) \mapsto ac + ad + bc + bd$$

Distributive laws

Example

$$\begin{aligned} \lambda : \quad & MA \to AM \\ & (a+b)(c+d) \mapsto ac + ad + bc + bd \end{aligned}$$

This distributive law lets us simplify elements of $AMAM(X)$ to elements of $AM(X)$:

$$\begin{aligned} & (aa+1)(b+1) + (b+1)(1+a) \\ & \quad \overset{\lambda}{\mapsto} ((aa)(b) + (aa)(1) + (1)(b) + (1)(1)) \\ & \qquad + ((b)(1) + (b)(a) + (1)(1) + (1)(a)) \\ & \quad \overset{\text{simplify}}{\mapsto} aab + aa + ba + a + 2b + 2. \end{aligned}$$

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This makes AM into the free ring monad!

Distributive laws

How many distributive laws $MA \to AM$ are there?

It turns out that there is only one, the λ from before giving multiplication distributing over addition (Zwart-Marsden, 2018).

My work has included looking for features of distributive laws that limit the forms they can take.

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Thank you!

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Hail the triumph of the corset
Hail the fair Philippa Fawcett
Victress in the fray
    Crown her queen of Hydrostatics
    And the other Mathematics
Wreathe her brow with bay.