Algebraic theories all of whose algebras are free

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Declaration

I hereby declare that I made a significant contribution to the work described in this thesis, that I wrote this thesis myself, and that this thesis has not been submitted for any other degree or professional qualification. At time of writing, no part of this thesis has been submitted for publication elsewhere.

Maia Woolf
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Lay summary

There are lots of different types of houses, and there are lots of different types of chairs. But it's possible to give a description of what it means for something to be a house or a chair. It's probably even possible to give a criterion that can be applied to any object, to check whether it's a house: maybe we'll say a house is 'a building with a kitchen and at least one bedroom'.

There are lots of different mathematical objects that mathematicians like studying. They often come in the form 'some things, and some ways to operate on those things'. For example, you might look at all whole numbers, and look at adding them: this is a way to take two whole numbers (2 and 3, say), and turn them into one ($2+3=5$). We could, however, look at other collections of things that can be combined: for example, we could look at the collection of all colours (red, blue, etc.), and combine them by mixing them (red + blue = purple). This is actually quite similar to the numbers example: for instance, $2+3=3+2=5$, and similarly red + blue = blue + red = purple.

So maybe, like we did with houses, we could come up with a criterion that fits both the numbers and the colours examples. Maybe 'a collection of things, and a way to add them together such that, for all things a and b in that collection, $a+b=b+a$'. This is an example of an *algebraic theory*!

In this thesis, I look at a particular property that some algebraic theories have (I call these algebraic theories *totally free*). In the 1970s, Steven Givant described exactly which algebraic theories have this property, and in the 2010s, Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei gave another way to prove this fact. The difference between the two approaches is one of mathematical language: Givant used language and tools from an area of mathematics called logic, and Kearnes, Kiss, and Szendrei used language and tools from an area of mathematics called classical universal algebra. Tom Leinster and I have been working to understand this result using the language and tools of an area of mathematics called category theory.

Using different language to talk about mathematics can give us different insights. Just like the fact that the Danish and Nynorsk word *hygge* can be translated into English, just not particularly elegantly, sometimes we benefit from working in a particular mathematical language. And that's this thesis: trying to translate an interesting fact about algebraic theories into the language of category theory. We don't have a full translation, but by trying to translate what we can, we've found some interesting new points of view.

Abstract

Which algebraic theories have the property that all their algebras are free? In the 1970s, Steven Givant answered this question – other than degenerate cases, the only ones are the theories of sets, pointed sets, and the theories of affine spaces and vector spaces over division rings. Givant's proof is in the language of logic. In the 2010s, Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei gave another proof, this time in the language of classical universal algebra.

But algebraic theories can be equivalently thought of as finitary monads, hence we can ask for an understanding of why this result holds in the language of category theory. We do not have a complete answer, but in this thesis I will present some progress we have made. I will exhibit some of the properties of these special classes of algebraic theories, which are made clearer by looking at them through the lens of category theory: I will show that all such theories are in a certain way minimal, and all have a notion of dimension for their algebras. Where possible, I will not rely on the classification result of Givant/Kearnes, Kiss, and Szendrei; this approach is, among other things, a good first step to a wider category theoretic understanding of the result.

With my sincere thanks to all the people who made this thesis possible, and provided laughter and support throughout the year. They include all who go too-often uncredited. They include my friends, and my family. And they of course include Tom.

Contents

Chapter 1

Introduction

1.1 Background

Algebraic theories are of interest to, among others, mathematicians and computer scientists working in, among others, the areas of category theory, higher algebra, logic, and programming languages. From their status as a unifying generalisation of various theories of familiar algebraic objects to their uses in modelling computational effects, there is much to study about these rich and subtle mathematical objects. Each algebraic theory has some free algebras; in this thesis, I describe work done with Tom Leinster in investigating those algebraic theories for which *all* their algebras are free. I will refer to these theories as *totally free.*

In the 1970s, Steven Givant in [Giv78] proved, in the language of logic, that a nondegenerate algebraic theory with only free algebras must be isomorphic to one of the following theories: sets, pointed sets, affine spaces over a division ring, or vector spaces over a division ring. A discussion about this question arose independently on *MathOverflow* in 2014 [C$^+$14], and led to an alternative proof (in fact of a slightly stronger fact) by Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei [KKS18]. Further discussion online occurred in 2023 on the *n-Category Café* blog [B$^+$23], of a more category theoretic flavour.

Kearnes, Kiss, and Szendrei's proof is in the language of classical universal algebra. This is a powerful approach, but tends to be quite syntactic. The driving idea behind this thesis is that we might gain some different and interesting understanding of the result by instead approaching it with the more abstract language of category theory. Whilst we do not have a complete category theoretic proof of the result, looking at the result in this way highlights some particularly interesting properties of the theories in question, and does so in a way that is not obvious when approaching it via logic or classical universal algebra.

1.2 Overview of this thesis

After this overview, I will end Chapter 1 with a brief overview of the notational choices I will use in this thesis. Chapter 2 will be a recapitulation of the relevant foundations of monads, the category of sets, and algebraic theories (via both the classical universal algebraic and the category theoretic approaches), which will be needed in the rest of the thesis. Whilst some are common knowledge, I will cover some less-well-known facts also, such as the fact that monads on **Set** preserve monics.

In Chapter 3, I will state the classification theorem, and will give some explanation of the shape of Kearnes, Kiss, and Szendrei's result. The subsequent three chapters will be each devoted to a different property of the algebraic theories with all their algebras free. Chapter 4 will be the fact that each such theory is either *affine* or *pointed*, and that these properties interact with each other via an adjunction between the category of affine monads and the category of pointed monads. Chapter 5 will be the fact that each of the theories is either degenerate, or minimal in a certain important way. And Chapter 6 will be the fact that each of the theories is either degenerate or has a notion of dimension, generalising the familiar notion of dimension for vector spaces.

Penultimately, Appendix A will be an overview of the often-conflicting different terminologies used by classical universal algebraists and by category theorists. Finally, a bibliography.

The key points of the thesis are as follows. A reader looking for the highlights might be satisfied with the following:

- Sufficient prerequisites, including a definition of algebraic theory, understanding what is meant by algebras, homomorphisms, and quotients; the category of algebras for a theory. Knowing what *degenerate theory* and *trivial algebra* mean in this thesis.
- The statement of the classification theorem.
- Understanding what is meant by theories being *affine* or *pointed* in this thesis. The adjunction between affine monads and pointed monads.
- Understanding what is meant by *minimality* in this thesis; the fact that all nondegenerate totally free theories are minimal.
- Understanding what is meant by *dimension* in this thesis; the fact that all nondegenerate totally free theories have dimension.

1.3 Notation and convention

I will write $0, 1, 2, \dots$ to mean sets with $0, 1, 2, \dots$ elements respectively, and I will at times, when unproblematic, conflate sets and their cardinalities. For sets X and Y, I will write $X \leq Y$ to mean that there exists an injection $X \hookrightarrow Y$, $X < Y$ to mean that $X \leq Y$ and $X \ncong Y$, and similarly for $\geq$ and $>$. I will write $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$ to mean the ordinal categories with $0, 1, 2, \dots$ objects respectively (e.g. $\mathbf{1}$ is the terminal category, and $\mathbf{2}$ is the *walking arrow*). I will write 1_X for the identity morphism on an object X. I will mostly write natural transformations with a single arrow, e.g. $\alpha : F \to G$, since often they will form the morphisms in a regular 1-category of endofunctors. I will write horizontal composition of natural transformations with the symbol $*$. I will write the set of natural numbers as $\mathbb{N} = \{0, 1, 2, \dots\}$.

I will write the coprojections of a binary coproduct $X + Y$ as $\mathrm{copr}_1 : X \to X + Y$ and $\mathrm{copr}_2 : Y \to X + Y$. Given a coproduct $X + Y$ and a cocone as in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{copr}_1} & X+Y & \xleftarrow{\mathrm{copr}_2} & Y \\ & {\scriptstyle f}\searrow & \downarrow & \swarrow{\scriptstyle g} & \\ & & Z, & & \end{array}$$

I will write $\binom{f}{g}$ to denote the induced dashed arrow. Given maps between the comultiplicands of two coproducts as in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{copr}_1} & X+Y & \xleftarrow{\mathrm{copr}_2} & Y \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[\mathrm{copr}_1]{} & X'+Y' & \xleftarrow[\mathrm{copr}_2]{} & Y', \end{array}$$

I will write $f+g$ to denote the induced dashed arrow: this notation complements well the functoriality of coproducts (when enough exist).

Throughout this thesis, I will be careful to distinguish between algebras, which I will write in bold, e.g. $\mathbf{A}$; and their underlying sets, which I will write in italic, e.g. A. These are distinct mathematical objects, and live in different categories. The same could (and perhaps should) also be said about algebra homomorphisms, but I will just write these in italic. With monads I will also not be so careful, so I will conflate a monad with its endofunctor: whilst monads *are* algebras, in this case, the distinction is less important here.

By *ring*, I will mean a not-necessarily-commutative ring with a multiplicative unit.

Chapter 2

Prerequisites

Here I will assume a basic knowledge of category theory, e.g. the content of Tom Leinster's book [Lei16] (but no more). An alternative textbook is Emily Riehl's, [Rie16], which covers significant portions of the following.

2.1 Monads

This section will be an overview of some basics of the theory of monads. Those with monadic experience may wish to skip to Section 2.2.

Definition 2.1.1. A **monad** on a category $\mathcal{C}$ is

- an endofunctor $T : \mathcal{C} \to \mathcal{C}$,
- a natural transformation $\eta : 1_{\mathcal{C}} \to T$, called the **unit**, and
- a natural transformation $\mu : TT \to T$, called the **multiplication**,

such that the diagrams

$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

commute.

That is, a monad on $\mathcal{C}$ is just a $\otimes$-monoid in the monoidal category $\mathbf{End}(\mathcal{C}) := [\mathcal{C}, \mathcal{C}]$ of endofunctors on $\mathcal{C}$, where the tensor product $\otimes$ is given by composition of endofunctors. We often refer to the monad (T, η, μ) just by its endofunctor T. Where unspecified, I will write η, μ for the unit and multiplication of a monad in question, and I will write η^T, μ^T, etc., when there is more than one monad in play.

Definition 2.1.2. Given monads (T, η^T, μ^T) and (S, η^S, μ^S) on a category $\mathcal{C}$, a **monad map** is a natural transformation $\alpha : T \to S$ that respects the units and multiplications of T and S. That is, the diagrams

$$\begin{array}{ccc} & 1_{\mathcal{C}} & \\ {\scriptstyle \eta^T}\swarrow & & \searrow{\scriptstyle \eta^S} \\ T & \xrightarrow[\alpha]{} & S \end{array} \qquad\qquad \begin{array}{ccc} T^2 & \xrightarrow{\alpha * \alpha} & S^2 \\ {\scriptstyle \mu^T}\downarrow & & \downarrow{\scriptstyle \mu^S} \\ T & \xrightarrow[\alpha]{} & S \end{array}$$

commute.

Thinking of monads as monoids, a monad map is just a monoid homomorphism. The composite (as natural transformations) of two monad maps is again a monad map. Given a monad T, the identity natural transformation 1_T is a monad map. Hence:

Definition 2.1.3. Given a category $\mathcal{C}$, the monads on $\mathcal{C}$, along with the monad maps between them, form a category $\mathbf{Mnd}(\mathcal{C})$.

And as one might expect:

Definition 2.1.4. A monad map is called an **isomorphism of monads** when it is a natural isomorphism. Given monads T and S and an isomorphism $T \to S$, we say that T and S are **isomorphic**.

There are more general (e.g. 2-categorical) notions of monad and monad map, but the above will be sufficient for us here. For a more general formulation, I highly recommend Ross Street's classic 1972 paper *The formal theory of monads* [Str72].

Just as monoids in the category of sets can act on sets, monads can act on objects in their base category.

Definition 2.1.5. Let T be a monad on a category $\mathcal{C}$. A T**-algebra** is an object $A \in \mathcal{C}$ along with a map $a : TA \to A$ in $\mathcal{C}$, such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} T^2A & \xrightarrow{\mu_A} & TA \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array}$$

commute. We might call a a T**-algebra structure (on** A**)**.

Definition 2.1.6. Let (A, a) and (B, b) be algebras for a monad T on a category $\mathcal{C}$. A T**-algebra homomorphism**, or T**-algebra map**, is a map $h : A \to B$ in $\mathcal{C}$ that respects the T-algebra structures on A and B. That is, the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Th} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[h]{} & B \end{array}$$

commutes.

The composition (as maps in $\mathcal{C}$) of two T-algebra homomorphisms is again a T-algebra homomorphism. The identity homomorphism 1_A on the object of an algebra (A, a) is a T-algebra homomorphism. Hence:

Definition 2.1.7. Given a monad T on a category $\mathcal{C}$, the T-algebras and T-algebra homomorphisms form a category T**-Alg** (also written $\mathcal{C}^T$), the **category of algebras for** T (also called the **Eilenberg–Moore category of** T).

The category of algebras for a monad on $\mathcal{C}$ lies above $\mathcal{C}$ via an adjunction. We have a forgetful functor $U : T\textbf{-Alg} \to \mathcal{C}$ given by $(A, a) \mapsto A$ and $h \mapsto h$, which is faithful, and in fact always has a left adjoint, the *free functor* which we call F. As with η and μ, I will write U^T and F^T when we are working with multiple monads. This left adjoint F sends an object A in $\mathcal{C}$ to the **free** T**-algebra on** A, given by (TA, μ_A).

Lemma 2.1.8. *The following is a T-algebra.*

$$\begin{array}{c} T^2A \\ \downarrow{\scriptstyle \mu_A} \\ TA \end{array}$$

Proof. Consider the diagrams

$$\begin{array}{ccc} TA & \xrightarrow{\eta_{TA}} & T^2A \\ & {\scriptstyle 1_{TA}}\searrow & \downarrow{\scriptstyle \mu_A} \\ & & TA \end{array} \qquad \begin{array}{ccc} T^3A & \xrightarrow{\mu_{TA}} & T^2A \\ {\scriptstyle T\mu_A}\downarrow & & \downarrow{\scriptstyle \mu_A} \\ T^2A & \xrightarrow[\mu_A]{} & TA, \end{array}$$

which are what we need to commute for (TA, μ_A) to be a T-algebra. These are precisely two of the monad axioms (at A). ☐

Lemma 2.1.9. *Let T be a monad on a category $\mathcal{C}$. Then the following defines a functor, where A is an object and $f : A \to B$ is a morphism in $\mathcal{C}$.*

$$\begin{aligned} F^T &: \mathcal{C} \to T\textbf{-Alg}, \\ &A \mapsto (TA, \mu_A), \\ &f \mapsto Tf. \end{aligned}$$

Proof. Let us first check that Tf actually is a T-algebra homomorphism. We require the diagram

$$\begin{array}{ccc} T^2A & \xrightarrow{T^2f} & T^2B \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ TA & \xrightarrow[Tf]{} & TB \end{array}$$

to commute, which it does: this is just naturality of μ. And then F's respect for identities and associativity is inherited from T, and we are done. ☐

Importantly: note that $T = UF$.

Lemma 2.1.10. *Let T be a monad on a category $\mathcal{C}$, and let F and U be the free and forgetful functors to and from T*-**Alg** *respectively. Then $F \dashv U$.*

Proof. I will show that we have a unit and a counit, and that they satisfy the required triangle identities. Our unit will be T's unit, η. For the counit, first note that $FU(A,a) = (TA, \mu_A)$. We then take the counit

$$\begin{array}{ccc} T^2A & \xrightarrow{T\varepsilon_A} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[\varepsilon_A]{} & A \end{array}$$

to be given by $\varepsilon_{(A,a)} = a$ considered as a T-algebra homomorphism: by definition of T-algebra, the above diagram commutes and thus a is indeed a homomorphism $(TA, \mu_A) \to (A, a)$.

Now for the triangle identities. The identity

$$\begin{array}{ccc} U(A,a) & \xrightarrow{\eta_{U(A,a)}} & UFU(A,a) \\ & {\scriptstyle 1_{U(A,a)}}\searrow & \downarrow{\scriptstyle U\varepsilon_{(A,a)}} \\ & & U(A,a) \end{array}$$

is precisely the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array}$$

from the definition of T-algebra, so commutes. The other triangle is the diagram

$$\begin{array}{ccc} FA & \xrightarrow{F\eta_A} & FUFA \\ & {\scriptstyle 1_{FA}}\searrow & \downarrow{\scriptstyle \varepsilon_{FA}} \\ & & FA. \end{array}$$

Because U is faithful, for the above to commute it suffices for

$$\begin{array}{ccc} UFA & \xrightarrow{UF\eta_A} & UFUFA \\ & {\scriptstyle U1_{FA}}\searrow & \downarrow{\scriptstyle U\varepsilon_{FA}} \\ & & UFA \end{array}$$

to commute. And this is just

$$\begin{array}{ccc} TA & \xrightarrow{T\eta_A} & TTA \\ & {\scriptstyle 1_{TA}}\searrow & \downarrow{\scriptstyle \mu_A} \\ & & TA, \end{array}$$

which is one of the diagrams in the definition of monad. So $F \dashv U$. □

As well as its category of algebras, each monad has a *Kleisli category*, which is the full subcategory of free algebras of the monad. It comes with an inclusion functor into the category of algebras. For more information, see for example Section 5.2 of [Rie16]. A monad is totally free if and only if this inclusion functor is an equivalence; I do not make use of this fact in this thesis.

2.2 The category of sets

Throughout this thesis, our monads will be on **Set**, the category of sets and functions. For the avoidance of doubt: throughout this thesis, I will take **Set** to have the axiom of choice; that is, every surjection in **Set** has a section. Without the axiom of choice, we cannot even show that every vector space is free. Below are some useful lemmas about **Set**.

Firstly, I want to highlight a particular feature of the category of sets. This is the sort of thing that is in most introductory category theory textbooks; I highlight it here because it will be an important computational tool in this thesis. We can 'calculate with elements' in the category of sets (we say that **Set** is *well-pointed*). Here are some examples.

- A monomorphism in **Set** is an injection, and so to check if a morphism $f : X \to Y$ of sets is monic, we need only check that for all $x, x' \in X$, $f(x) = f'(x)$ implies $x = x'$. Similarly, an epimorphism in **Set** is a surjection.

- When constructing limits or colimits in **Set**, we have explicit formulae for these: for example the equaliser of the diagram

  $$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

  is (isomorphic to) the set $\{x \in X : f(x) = g(x)\}$ (along with the appropriate inclusion map).

- In the diagram

  $$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow f_1 \\ X_2 & \xrightarrow[f_2]{} & Y, \end{array}$$

  the pullback $X_1 \times_Y X_2$ is (isomorphic to) the set $\{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$.

This next lemma is particularly important:

Lemma 2.2.1. *Monads on* **Set** *preserve monics.*

That is, recalling that the monics in **Set** are precisely the injections, if T is a monad on **Set** and $f : X \hookrightarrow Y$ is an injection, then $Tf : TX \to TY$ is an injection also.

Proof. Any functor on **Set** preserves split monics. Indeed: A split monic $A \to B$ is a morphism $A \xrightarrow{m} B$ such that there exists $B \xrightarrow{r} A$ such that the composition $A \xrightarrow{m} B \xrightarrow{r} A$ equals the identity on A. (Note that all split monics are monics.)

The only monics in **Set** that are not split are the maps $! : 0 \to X$ for X non-empty. So we want to show for any **Set**-monad T, for all non-empty sets X, that $T!$ is injective (injectivity is equivalent to monicness in **Set**).

If $T0 \cong 0$ then we are done. Otherwise, we can choose some $q : X \to T0 = UF0$, where $F \dashv U$ is the free–forgetful adjunction for T, and this adjunction gives us $\bar{q} : FX \to F0$. So we have

$$F0 \underset{\bar{q}}{\overset{F!}{\rightleftarrows}} FX$$

in T-**Alg**.

But $F0$ is initial in T-**Alg** (left adjoints preserve colimits), so $\bar{q} \circ F! = \mathrm{id}_{F0}$. So $F!$ is a split monic, so $T! = UF!$ is a (split) monic. □

This next fact is not specific to **Set**, but we will only need it for when the domain of said forgetful functor is **Set**:

Lemma 2.2.2. *Faithful functors reflect epimorphisms. That is: let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let $e : A \to B$ be a morphism in $\mathcal{C}$. Then if $Fe : FA \to FB$ is an epimorphism (in $\mathcal{D}$), then e is an epimorphism (in $\mathcal{C}$) also.*

Proof. Let $f, g : B \to C$ be morphisms in $\mathcal{C}$ such that $fe = ge$. That is, we have a fork

$$A \xrightarrow{e} B \underset{g}{\overset{f}{\rightrightarrows}} C.$$

Applying F, we get

$$FA \xrightarrow{Fe} FB \underset{Fg}{\overset{Ff}{\rightrightarrows}} C,$$

which is a fork in $\mathcal{D}$. Since Fe is epic, we have that $Ff = Fg$. And F is faithful, so we have $f = g$. So e is epic. □

Dually (consider $\mathcal{C}^{\mathrm{op}}$), faithful functors also reflect monomorphisms. Now, some more definitions and facts about monads on **Set**.

Definition 2.2.3. Let T be a monad on **Set**. I will call a T-algebra **trivial** if its underlying set has cardinality 0 or 1. I will call it **nontrivial** otherwise.

All monads on **Set** have an algebra whose underlying set has cardinality 1, and this algebra is terminal (hence it is unique up to isomorphism). We might write the terminal algebra as $\mathbf{1}$. We say **the empty algebra** to refer to the algebra whose underlying set has cardinality 0, when it exists (when it does, it is unique up to isomorphism). We might write the empty algebra as $\mathbf{0}$.

Definition 2.2.4. Let T be a monad, and let $\mathbf{A}$ be a T-algebra. A **quotient of A** is a T-algebra homomorphism $\mathbf{A} \xrightarrow{f} \mathbf{B}$ to some T-algebra $\mathbf{B}$ such that f is a regular epimorphism.

Recall that a *regular epimorphism* is an epimorphism that is the equaliser of a parallel pair of arrows.

Lemma 2.2.5. *Let T be a monad on* **Set***, and let f be a T-algebra homomorphism. Then f is a regular epimorphism if and only if Uf is a surjection.*

A proof of this can be found as Corollary 3.5.3 in Francis Borceux's *Handbook of Categorical Algebra 2*, [Bor94]. To apply that corollary here, note that the surjections in **Set** are precisely the regular epimorphisms.

Definition 2.2.6. Let T be a monad on **Set**, and let **A** be a T-algebra. We call **A simple** when **A** is nontrivial, and for each quotient $\mathbf{A} \xrightarrow{f} \mathbf{B}$ of **A**, either f is invertible, or **B** is trivial.

Many monads, certainly many commonly encountered ones, have the property that the components of their units are always monic. But not all do:

Definition 2.2.7. Let T be a monad. We call T **degenerate** when the components of T's unit η are not all monic. We call T **nondegenerate** when it is not degenerate.

A monad T on **Set** is certainly degenerate if there exists a set X with $X > TX$. Contrapositively, given a nondegenerate monad T on **Set**, $TX \geq X$ for all sets X.

I will now classify all degenerate monads on **Set**. I will use the following result, which is Exercise 2.3.11 in [Lei16].

Lemma 2.2.8. *Let T be a monad on* **Set** *such that there exists a T-algebra with at least two elements. Then all components of the unit of T are injective.*

Proof. Let X be a set. Any function with domain isomorphic to 0 or 1 is always monic; assume now that $X \geq 2$, and take distinct $x, x' \in X$. I claim that $\eta_X(x) \neq \eta_X(x')$. Indeed: take a T-algebra **A** such that $U\mathbf{A} \geq 2$. As a property of **Set**, we can take a function $f : X \to U\mathbf{A}$ such that $f(x) \neq f(x')$.

I would now like to find a function $UFX \to U\mathbf{A}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & UFX \\ & \searrow_{f} & \downarrow \\ & & U\mathbf{A} \end{array}$$

commutes, and by properties of adjunctions, $U\bar{f} : UFX \to U\mathbf{A}$ works, where $\bar{f}$ is the transpose of f across the $F \dashv U$ adjunction. Since $f(x) \neq f(x')$, it must thus be true that $\eta_X(x) \neq \eta_X(x')$, and we are done. □

Corollary 2.2.9. *Let T be a degenerate monad on* **Set**. *Then $TX \cong 1$ for all sets $X \geq 1$, and either $T0 \cong 0$ or $T0 \cong 1$.*

That is, there are at most two degenerate monads on **Set**.

Proof. The contrapositive of Lemma 2.2.8 states that if T is degenerate, then each T-algebra has at most 1 element. This implies that $TX = UFX \leq 1$ for all sets X. Because there is a function $\eta_X : X \to TX$ for all sets X, and a function with empty codomain must have empty domain, we must have that $TX \cong 1$ for all sets $X \geq 1$. A priori, we might have $T0 \cong 0$ or $T0 \cong 1$. □

Proposition 2.2.10. *There are precisely two nondegenerate monads on* **Set** *up to isomorphism, namely*

- *$S0 \cong 0$ and $SX \cong 1$ for all sets $X \geq 1$, and*
- *$TX \cong 1$ for all sets X.*

Proof. Both S and T define monads on **Set**: they are functors, and their units and multiplications are given by the only possible functions. So by Corollary 2.2.9, there are precisely two degenerate monads on **Set** (up to isomorphism of monads). □

With S and T as in the above proposition, S is the theory of 'sets with all elements equal', or 'sets with at most one element', with S-**Alg** $\simeq \mathbf{2}$. Similarly, T is the theory of 'pointed sets with all elements equal', or 'sets with precisely one element', and has T-**Alg** $\simeq \mathbf{1}$.

We will also need the following later.

Definition 2.2.11. Let T be a monad on **Set**, and let $n \in \mathbb{N}$. Then we say that a T-algebra **A** is **generated by an n-element set** when there exists a quotient $Tn \xrightarrow{h} \mathbf{A}$.

2.3 Algebraic theories

There are multiple equivalent formulations of the notion of algebraic theories. Those considered in this thesis are finitary monads, defined in this section, and the classical operations-and-equational-laws setup, defined in the next section.

Definition 2.3.1. A category is **finite** when it has finitely many objects, and each of its hom-sets is finite.

Equivalently, a category is finite when it has finitely many morphisms. A finite category can be thought of as a category internal to the category of finite sets.

Definition 2.3.2. A category $\mathcal{I}$ is **filtered** when, for all finite categories **J**, all diagrams in $\mathcal{I}$ indexed over **J** admit a cocone.

That is, for each diagram $F : \mathbf{J} \to \mathcal{I}$ in a filtered category $\mathcal{I}$, where **J** is finite, there exists a cocone $F \to u$ for some object $u \in \mathcal{I}$ (where the symbol u is also used to denote the constant functor at u).

There is a generalisation of this definition: categories can be *κ-filtered* for a *regular cardinal* κ. I will not go into the details of this here.

Lemma 2.3.3. *Let $\mathcal{I}$ be a category. Then $\mathcal{I}$ is filtered if and only if the following three conditions hold.*

- *$\mathcal{I}$ is nonempty (i.e. the empty diagram in $\mathcal{I}$ has a cocone).*
- *For each $i, j \in \mathcal{I}$, there exists a diagram*

$$\begin{array}{ccc} i & & \\ & \searrow & \\ & & u. \\ & \nearrow & \\ j & & \end{array}$$

- *For each diagram*

$$i \rightrightarrows j$$

in $\mathcal{I}$, there exists a fork

$$i \rightrightarrows j \longrightarrow u.$$

Proving this is nontrivial. For a proof, see for example Lemma VII.6.1 of Saunders Mac Lane and Ieke Moerdijk's book [MM94].

Definition 2.3.4. A monad on **Set** is **finitary** if it preserves filtered colimits.

That is, for T a monad on **Set**, we say that T is finitary when, for all filtered categories $\mathcal{I}$ and diagrams $F : \mathcal{I} \to \mathbf{Set}$, if $\operatorname{colim} F$ exists, then $\operatorname{colim} TF$ exists and the induced map

$$\operatorname{colim} TF \longrightarrow T \operatorname{colim} F$$

is an isomorphism.

Definition 2.3.5. An **finitary algebraic theory** is a finitary monad on the category of sets.

Throughout this thesis, unless otherwise specified, I will assume that all algebraic theories are finitary. It would not be unreasonable to define *algebraic theory* to mean any monad on **Set**, or perhaps any monad on **Set** with rank. We need not worry about what *rank* means in this thesis.

As will be shown in the next section, algebras for an algebraic theory are examples of *models* of a (logical) theory. In the literature, the word *model* is sometimes used instead of algebra.

2.4 Classical universal algebra

Here I will give some exposition of some standard classical universal algebra constructions and results. The canonical reference for a more in-depth introduction to this field is Stanley H. Burris and Hanamantagouda P. Sankappanavar's *A Course in Universal Algebra* [BS12].

Definition 2.4.1. A **type** is a set F along with a function $\operatorname{arity} : F \to \mathbb{N}$.

The intuition for the above definition is that F is a set of symbols, and the function arity assigns to each symbol a natural number arity, such that each symbol of F with arity n will specify that each model with underlying set A of the theory that F represents has a function $A^n \to A$ as part of its data. Equivalently, we could define a type to be a sequence of sets $(F_n)_{n\in\mathbb{N}}$: here, each F_n represents the symbols in F of arity n.

Definition 2.4.2. Let F be a type. An **algebra $\mathbf{A}$ of type** F is a set A along with, for each $f \in F$, a function $f_{\mathbf{A}} : A^{\operatorname{arity} f} \to A$. For $f \in F$, we call $f_{\mathbf{A}}$ an **operation of arity** $\operatorname{arity} F$.

Note the difference between f and $f_{\mathbf{A}}$. Comparing with the monad theoretic approach, f corresponds to an element of $T(\operatorname{arity} f)$ for the corresponding monad T, whereas $f_{\mathbf{A}}$ is bundled up into the algebra structure of the algebra $\mathbf{A} \in \operatorname{ob}(T\textbf{-Alg})$, and does not really appear on its own in the monad theoretic approach. I will be careful in this thesis to use the word *term* to refer to 'abstract' elements $f \in Tn$, and *operation* to refer to 'realised' functions of the form $A^n \to A$.

At the end of the day, however, as long as we are careful to be clear with what we mean, it is never absolutely necessary to work just with terms or just with operations, as Proposition 2.4.13, which we will see later, shows.

Example 2.4.3. All groups are algebras of type $\{m, i, e\}$, where m is the group multiplication with arity $m = 2$, i takes the inverse of an element with arity $i = 1$, and e picks out the identity element with arity $e = 0$.

The definitions of homomorphism familiar from the theories of groups and rings generalise to this setting:

Definition 2.4.4. Let F be a type, and let $\mathbf{A}$ and $\mathbf{B}$ be algebras of type F, with underlying sets A and B respectively. A **homomorphism $\mathbf{A} \to \mathbf{B}$** is a function $\phi : A \to B$ such that, for each $n \in \mathbb{N}$ and n-ary term $f \in F$, for each $a_1, \ldots, a_n \in A$, we have that

$$\phi(f_{\mathbf{A}}(a_1, \ldots, a_n)) = f_{\mathbf{B}}(\phi(a_1), \ldots, \phi(a_n)).$$

Universal algebraists like working with various classes of algebras. And there are a couple of important flavours of such classes. We need some preliminary definitions in order to build up to the definitions of these.

Definition 2.4.5. Let F be a type, and let X be a set. For this definition, extend arity to have domain $X \sqcup F$ via arity $x = 0$ for each $x \in X$. Define F_X to be the largest subset of $\coprod_{n \in \mathbb{N}} (X \sqcup F)^n$ such that for each $n \in \mathbb{N}$, for each $t \in (X \sqcup F)^n$, writing $t = (t_1, t_2, \ldots, t_n)$, we have that

- $\sum_{i=1}^{n} ((\operatorname{arity} t_i) - 1) = 0$, and
- for each $m < n$, $\sum_{i=1}^{m} ((\operatorname{arity} t_i) - 1) > 0$.

We call an element of F_X a **term of F in free variables X**.

This is just a formal way of defining what we think of when we say *term*. For example, for the type with a single binary operation f, the tuple (f, a, f, b, a), which we might write in more familiar notation as $f(a, f(b, a))$, is a term in free variables $\{a, b, c\}$. We could perhaps more intuitively instead define terms recursively, but the above avoids the hassle of worrying about identifying terms defined at different points in a recursive process.

Definition 2.4.6. Let F be a type. Then an **equation in F** is a set X along with two elements l and r of F_X. We might write the equation as '$l = r$'.

For example, $f(a, f(b, a)) = f(f(a, b), a)$ is an equation in the above setup.

Note that each F-term t in free variables X induces an operation $t_{\mathbf{A}} : A^X \to A$ on an each F-algebra $\mathbf{A}$ with underlying set A.

Definition 2.4.7. Let F be a type. Let $\mathbf{A}$ be an algebra of type F, and let $l = r$ be an equation in F. Then we say that **$\mathbf{A}$ satisfies** $l = r$ when $l_{\mathbf{A}} = r_{\mathbf{A}}$.

Definition 2.4.8. Let F be a type. A class of algebras of type F is called an **equational class** when there exists a set of equations E in F such that each algebra in the class satisfies each equation in E, and each F-algebra that satisfies each equation in E is isomorphic to some algebra in the class.

Definition 2.4.9. Let F be a type, and let $\mathscr{K}$ be a class of algebras of type F. We write

- $\mathbb{H}(\mathscr{K})$ for the class of all algebras of type F where each is isomorphic to the image of some algebra in $\mathscr{K}$ under some homomorphism,
- $\mathbb{S}(\mathscr{K})$ for the class of all algebras of type F where each is isomorphic to a subalgebra of some algebra in $\mathscr{K}$, and
- $\mathbb{P}(\mathscr{K})$ for the class of all algebras of type F where each is isomorphic to a set-indexed product of some algebras in $\mathscr{K}$.

Some sources write e.g. $\mathbb{H}$ as H or $\mathbf{H}$. Some sources only consider e.g. $\mathbb{S}(\mathscr{K})$ to be the class of all subalgebras of algebras in $\mathscr{K}$ (i.e. not up to isomorphism). Taking a homomorphic image of an algebra is, by the First Isomorphism Theorem (Theorem 6.12 of [BS12]), the same as taking a quotient of it. Note that homomorphic images, subalgebras, and set-indexed products of F-algebras are always F-algebras, inheriting their F-algebra structures from the F-algebras they are built from.

Definition 2.4.10. Let F be a type, and let $\mathscr{K}$ be a class of algebras of type F. We say that $\mathscr{K}$ is **closed under**

- **homomorphic images** when $\mathbb{H}(\mathscr{K}) = \mathscr{K}$,
- **subalgebras** when $\mathbb{S}(\mathscr{K}) = \mathscr{K}$, and
- **products** when $\mathbb{P}(\mathscr{K}) = \mathscr{K}$.

Category theorists would not worry about precise equality of classes here, preferring to work up to a notion of equivalence much the same as that for categories: $\mathbb{H}(\mathscr{K})$ is meaningfully the same as $\mathscr{K}$ here when each algebra of $\mathbb{H}(\mathscr{K})$ is isomorphic to some algebra of $\mathscr{K}$, etc.

Definition 2.4.11. Let F be a type. A class of algebras of type F is called a **variety** when it is closed under homomorphic images, subalgebras, and products.

It is a theorem attributed to Alfred Tarski that a class $\mathscr{V}$ of algebras of type F is a variety if and only if $\mathscr{V} = \mathbb{HSP}(\mathscr{V})$. A proof can be found as Theorem 9.5 of [BS12].

Garrett Birkhoff in 1935 showed the following important result.

Theorem 2.4.12 (Birkhoff's HSP Theorem). *A class of algebras is an equational class if and only if it is a variety.*

I will not prove this here. See for example Theorem 11.9 of [BS12]. We also have the following.

Proposition 2.4.13. *Let* $\mathscr{V}$ *be a variety of algebras of type* F*. Then for all* $f, f' \in F$ *such that* $\operatorname{arity} f = \operatorname{arity} f'$*,* $f = f'$ *if and only if for all* $\mathbf{A} \in \mathscr{V}$*,* $f_{\mathbf{A}} = f'_{\mathbf{A}}$.

The logical system of universal algebra is called *equational logic*, or sometimes *zeroth order logic*, and the above proposition follows from a completeness result for equational logic attributed to Birkhoff. For a proof, see for example Theorem 14.19 of [BS12], or Theorem 3.5.14 of Franz Baader and Tobias Nipkow's book [BN12].

Finally for this chapter, and crucially for this thesis, equational classes *are* finitary algebraic theories. See for example Proposition 4.6.2 of [Bor94] for a proof. This equivalence works as expected: algebras are algebras, and homomorphisms are homomorphisms, in the different definitions. Terms of arity n in the classical universal algebraic sense correspond to elements of Tn in the monad theoretic definition. I will mostly work with monads for the rest of this thesis, commenting on the syntactic meaning of results where appropriate.

Chapter 3

The classification theorem

In this chapter, I will give an overview of the theorem classifying the algebraic theories all of whose algebras are free. I will include a high-level overview of Kearnes, Kiss, and Szendrei's proof [KKS18]: this will lead in to the topics of the subsequent chapters.

3.1 Statement

For convenience, I will make the following definition.

Definition 3.1.1. An algebraic theory is **totally free** when all of its algebras are free.

Example 3.1.2. The theory of groups is not totally free: the cyclic group with two elements is not freely generated by a set.

In Proposition 2.2.10, I classified all degenerate algebraic theories. I will now introduce the other totally free theories. Firstly, sets and pointed sets.

Remark 3.1.3. The identity functor on **Set** has a monad structure (via identity natural transformations), and is the monad for the theory of sets. Sets, when considered as algebras, have no operations. This theory has **Set** as its category of algebras.

Definition 3.1.4. The functor $T : \mathbf{Set} \to \mathbf{Set}$, $TX = X + 1$ has a monad structure via $\eta_X = \mathrm{copr}_1 : X \to X + 1$ and $\mu_X = X + \binom{1_1}{1_1} : X + 1 + 1 \to X + 1$. This is the monad for the theory of **pointed sets**. It has $\mathbf{Set}_*$, the category of pointed sets and basepoint-preserving functions, as its category of algebras.

Now, division rings, which are not themselves totally free, but I will use them when stating the classification theorem for totally free theories.

Definition 3.1.5. A **division ring** is a ring such that each nonzero element of the ring has a two-sided multiplicative inverse.

Example 3.1.6. The fields are precisely the commutative division rings.

The final definitions before the classification theorem are affine spaces and vector spaces.

Definition 3.1.7. The theory of **affine spaces over the semiring R** is the theory S defined as follows. Let R be the underlying set of $\mathbf{R}$. Write $\operatorname{supp} f$ for the **support** of a function from a set X to R, that is, the set of elements of X on which f is nonzero.

- SX is the set of functions $f : X \to R$ with finite support that satisfy $\sum_{x \in \operatorname{supp} f} f(x) = 1$.
- The unit of S sends $x \in X$ to the indicator function at x, i.e. the function $\varphi : X \to R$ with $\varphi(x) = 1$, $\varphi(y) = 0$ for $y \neq x$.
- The multiplication of S sends $\Phi \in SSX$ to the function $\varphi : X \to R$ such that $\varphi(x) = \sum_{\theta \in \operatorname{supp} \Phi} \Phi(\theta) \cdot \theta(x)$, where $\cdot$ is the multiplication operation of $\mathbf{R}$. This sum is finite by definition of SX.

We write $\mathbf{Aff}_{\mathbf{R}}$ for S-**Alg**.

The theory of affine spaces over $\mathbf{R}$ is the *subtheory* of the theory of vector spaces over $\mathbf{R}$ consisting only of the terms whose 'coefficients sum to 1'. Intuitively, where a vector space is generated by taking linear combinations of generating vectors, an affine space is generated by taking linear interpolations of generating elements (and allowing these to extend beyond the generators). That is, if we have two elements of an affine space, we can obtain the line passing through them; if we have three elements, we can obtain the plane passing through them. Thus the free real affine space generated by the set 3 is $\mathbb{R}^2$. A vector space is just an affine space with a distinguished element: the origin.

Definition 3.1.8. The theory of **vector spaces over the semiring R** is the theory T defined as follows. Let R be the underlying set of $\mathbf{R}$. Write $\operatorname{supp} f$ as above.

- TX is the set of functions $f : X \to R$ with finite support.
- The unit of T sends $x \in X$ to the indicator function at x, i.e. the function $\varphi : X \to R$ with $\varphi(x) = 1$, $\varphi(y) = 0$ for $y \neq x$.
- The multiplication of T sends $\Phi \in TTX$ to the function $\varphi : X \to R$ such that $\varphi(x) = \sum_{\theta \in \operatorname{supp} \Phi} \Phi(\theta) \cdot \theta(x)$, where $\cdot$ is the multiplication operation of $\mathbf{R}$. This sum is finite by definition of TX.

We write $\mathbf{Vect}_{\mathbf{R}}$ for T-**Alg**.

Technically, since $\mathbf{R}$ is not necessarily a field, we should talk about $\mathbf{R}$-*modules* as opposed to $\mathbf{R}$-vector spaces. It is, however, standard in the literature on this topic to say vector spaces, and so I shall mostly do the same here.

We can now state the big theorem:

Theorem 3.1.9 (Classification of totally free algebraic theories)**.** *Let T be an algebraic theory all of whose algebras are free. Then T is isomorphic to a degenerate algebraic theory, the theory of sets, the theory of pointed sets, the theory of affine spaces over a division ring, or the theory of vector spaces over a division ring.*

A proof of Theorem 3.1.9 was first given by Steven Givant in the 1970s [Giv78][1], using *universal Horn classes* and other parts of logic. A proof using classical universal algebra was given by Keith A. Kearnes, Emil W. Kiss, and Ágnes Szendrei in the 2010s [KKS18], which uses past universal algebra literature such as Kearnes' 2000 paper [Kea00].

3.2 Discussion of Kearnes, Kiss, and Szendrei's proof

Kearnes, Kiss, and Szendrei in fact prove a stronger result in [KKS18] than the above. They prove that the only varieties whose *finitely generated* algebras are free are the six classes. They split their proof into two cases: when such a variety has no nullary terms, and when it has at least one. This corresponds to the distinction I will make in Chapter 4 between *affine* and *pointed* theories (although what I will mean by *affine* is not what Kearnes, Kiss, and Szendrei mean by *affine* in their paper).

We have not been able to translate Kearnes, Kiss, and Szendrei's proof into the language of category theory. That being said, this might not be desirable. Their proof makes heavy use of congruences, which are commonplace in classical universal algebra. There are however some areas in which we do have category theoretic proofs of results used by Kearnes, Kiss, and Szendrei.

Notably, Kearnes, Kiss, and Szendrei's proof of the classification theorem, in both cases, takes a nondegenerate (they use the word *nontrivial*) minimal subvariety, proves that it is either sets or affine spaces over a division ring, or pointed sets or vector spaces over a division ring (depending on the case), and then shows that such a minimal subvariety must actually be the whole variety after all. In proving this last part, they make use of the fact that all of these varieties have a notion of dimension. In Chapter 5, I will prove that all nondegenerate totally free theories are minimal, and in Chapter 6, I will prove that all nondegenerate totally free theories have a notion of dimension. Neither of these proofs will rely on knowing what the totally free theories are: they are purely from the definition of totally free. As far as we have been able to see, they are alone not enough to prove the classification theorem (in category theoretic language).

One result used in Kearnes, Kiss, and Szendrei's proof is Magari's Theorem, a classic classical universal algebra result.

Theorem 3.2.1 (Magari's Theorem)**.** *Each nondegenerate algebraic theory has a simple algebra.*

Recall that a finitary algebraic theory is essentially the same thing as a variety of algebras. A classical universal algebraic proof of Magari's Theorem can be found as Theorem 10.13 of [BS12].

The proof of Magari's theorem uses the axiom of choice in an essential way. The theorem can be used, with a process of elimination, to show that for a totally free theory, either $F1$ in the pointed case, or $F2$ in the affine case, must

[1] There is a small typographical error in at least one version of this paper. The paper was published in 1978, not 1979.

be simple – but remarkably we know of no choice-free proof of the simplicity of $F1$ and $F2$ respectively.

Kearnes, Kiss, and Szendrei's proof of depends on Corollary 2.10 of Kearnes' 2000 paper [Kea00], which states that a minimal idempotent variety is either the variety of sets, the variety of semilattices, a variety of affine modules over a simple ring, or a congruence distributive variety. Kearnes' paper includes many lemmas about congruences, and additionally depends on Proposition 2.8 of Szendrei's 1986 book [Sze86]. This proposition states that all *affine* algebras, in a universal algebraic sense which is not what I mean by *affine* in most of this thesis, are in fact modules, and gives some more information about the particular form of module that they must be. I will not consider the specifics of this proposition in this thesis, but I will mention (classical universal algebraically) affine algebras briefly in Appendix A, when comparing terminologies.

3.3 Properties of the six classes of theories

Discussion of the properties mentioned above (affineness vs. pointedness, minimality, and dimension) will form most of the rest of this thesis. For each property, we can also ask which non-totally-free theories (if any) also have that property. When that is hard to fully determine, we can also ask simpler questions, for example: for which rings $\mathbf{R}$ does the free $\mathbf{R}$-vector space/-affine space monad have that property? We can also ask this for when $\mathbf{R}$ is only a *semiring*, i.e. a 'ring without additive inverses'.

Chapter 4

Affine vs. pointed

The six classes of theories can be split up into *affine* and *pointed* classes. For example, theories of affine spaces are affine, whereas theories of vector spaces are pointed. In this chapter, I will define and describe features of these properties, including exhibiting an adjunction between the categories of affine monads and of pointed monads on **Set**.

4.1 Preliminaries

Definition 4.1.1. A monad S is **affine** when $S1 \cong 1$. We call a monad T **pointed** when $T0 \cong 1$.

I will often use S to denote an affine monad, and T to denote a pointed monad. Note that *pointed endofunctor* is in other literature used to mean an endofunctor F on a category $\mathcal{C}$ that is equipped with a natural transformation $1_{\mathcal{C}} \to F$. Since all monads are by definition pointed in this way, hopefully no confusion will arise.

Also, note that pointed theories have precisely one point. That is, $T0 \cong 1$ precisely, i.e. there is precisely one constant in the theory. We have a corresponding syntactic condition for affine theories, as follows. In this chapter, I will write id to denote an identity term.

Proposition 4.1.2. *A monad S is affine if and only if, for all $n \in \mathbb{N}$, for all $\varphi \in Sn$, $\varphi(x, \ldots, x) = x$.*

Proof. The condition $\varphi(x, \ldots, x) = x$ is equivalent to $\varphi(x, \ldots, x) = \mathrm{id}(x)$, i.e. $S!_n(\varphi)(x) = \mathrm{id}(x)$, which is just another way of writing $S!_n(\varphi) = \mathrm{id}$. (Here I am writing $!_n : n \to 1$ for the unique map.) If S is affine then this condition holds, because $S!_n(\varphi), \mathrm{id} \in S1$, and $S1 \cong 1$ since S is affine.

If alternatively we assume that $S!_n(\varphi) = \mathrm{id}$ for all n and φ, then in particular it holds for $n = 1$: in this case, $S!_n = S1_1 = 1_{S1}$, so this is saying that $\varphi = \mathrm{id}$ for all $\varphi \in S1$, i.e. $S1 \cong 1$. □

Lemma 4.1.3. *Each totally free theory T is either affine or pointed (possibly both).*

Proof. Since T is totally free, there is some set X such that $TX \cong 1$. Further, T preserves monics, so we cannot have $T0 > 1$, else there could be no X such that $TX \cong 1$.

So $T0 \cong 1$, in which case T is pointed, or $T0 \cong 0$, and the same argument but now with $T1$ shows that $T1 \cong 1$ and T is affine. Note that $T1$ can never be 0, because of the existence of the map $\eta_1 : 1 \to T1$. $\square$

For example, we can see that the six classes of totally free theories fall into one of these two categories each: the theories with categories of algebras **2**, **Set**, and $\mathbf{Aff_R}$ are all affine; and the theories with categories of algebras **1**, $\mathbf{Set}_*$, and $\mathbf{Vect_R}$ are all pointed. The theory T with $TX \cong 1$ for all X (which has T-**Alg** $\simeq$ **1**) is both pointed *and* affine; this is the only algebraic theory with this property. That an affine and pointed theory can exist appears to just be a quirk of the definition, and it does not cause us any problems.

4.2 Affine monads

Consider the category of monads on **Set**, which I will write as **Mnd**. We can take the full subcategory **AffMnd** whose objects are the affine monads. I will show that **AffMnd** forms a coreflective subcategory of **Mnd**:

$$\mathbf{AffMnd} \underset{(\ )^{\mathrm{aff}}}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathbf{Mnd}.$$

Here, $(\ )^{\mathrm{aff}}$ sends a monad to its *affinisation*, which I will now define. What we would like to do is take the submonad of T which has precisely those of T's terms that are idempotent – that is, the terms φ such that $\varphi(x, x, \ldots, x) = x$. Universal algebraists might call this a *reduct* of T.

Definition 4.2.1. Let T be a monad on **Set**. We define **the affinisation of** T, written T^{aff}, to act on an object X as the pullback in the following diagram:

$$\begin{array}{ccc} T^{\mathrm{aff}}X & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_1} \\ TX & \xrightarrow[T!_X]{} & T1. \end{array} \tag{4.1}$$

Pullbacks are limits, hence functorial, so this also defines the action on morphisms.

By definition, $T^{\mathrm{aff}}X \cong \{\varphi \in TX : (T!_X)(\varphi) = \mathrm{id}\}$: this is the elementwise characterisation of a set-valued pullback from Section 2.2. That is, $T^{\mathrm{aff}}n$ is the idempotent n-ary terms of T. So given a monad T on **Set**, we get an endofunctor T^{aff} on **Set**. We can put a monad structure on this, but it requires a bit of setup.

Lemma 4.2.2. *Let S be a monad on a category $\mathcal{E}$, and let $\mathbf{A} = (A, a)$ be an S-algebra. Then we get a monad, which I shall write as $S/\mathbf{A}$, on the slice category*

$\mathcal{E}/A$, with action on objects as follows:

$$\begin{array}{ccc} & & SX \\ X & & \downarrow{\scriptstyle Sf} \\ \downarrow{\scriptstyle f} & \mapsto & SA \\ A & & \downarrow{\scriptstyle a} \\ & & A, \end{array}$$

and action on morphisms as follows:

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & A & \end{array} \quad \mapsto \quad \begin{array}{ccccc} SX & & \xrightarrow{Sp} & & SY \\ & {\scriptstyle Sf}\searrow & & \swarrow{\scriptstyle Sg} & \\ & SA & & SA & \\ & & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle a} & & \\ & & A, & & \end{array}$$

with unit and multiplication those of S. We further have $(\mathcal{E}/A)^{S/\mathbf{A}} \cong \mathcal{E}^S/\mathbf{A}$, the category of homomorphisms into $\mathbf{A}$, i.e. $(S/\mathbf{A})$-$\mathbf{Alg} \cong S$-$\mathbf{Alg}/\mathbf{A}$.

Proof. First, $S/\mathbf{A}$ is a functor: it inherits this from S. For $S/\mathbf{A}$'s unit,

$$\begin{array}{ccccc} X & & \xrightarrow{\eta^S_X} & & SX \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle Sf} & \\ {\scriptstyle f}\searrow & & A \xrightarrow{\eta^S_A} SA & & \\ & & \downarrow{\scriptstyle 1_A} \quad \swarrow{\scriptstyle a} & & \\ & & A & & \end{array}$$

commutes – the top quadrilateral by naturality of η^S, and the right-hand triangle since (A, a) is an S-algebra – so η^S_X is a morphism in $\mathcal{E}/A$. For the multiplication,

$$\begin{array}{ccccc} SSX & & \xrightarrow{\mu^S_X} & & SX \\ & {\scriptstyle SSf}\searrow & & & \\ & SSA & & & \downarrow{\scriptstyle Sf} \\ & {\scriptstyle Sa}\searrow & \xrightarrow{\mu^S_A} & & \\ & SA & & SA & \\ & & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle a} & & \\ & & A & & \end{array}$$

commutes – the top quadrilateral by naturality of μ^S, and the bottom quaderilateral since (A, a) is an S-algebra – so μ^S_X is a morphism in $\mathcal{E}/A$. The monad laws hold because they do for S. So $S/\mathbf{A}$ is a monad.

What does an $(S/\mathbf{A})$-algebra look like? It is an object of $\mathcal{E}/A$, i.e. a diagram of the form

$$\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ A, \end{array}$$

along with a map $(SX, a \circ Sf) \xrightarrow{x} (X, f)$ in $\mathcal{E}/A$, i.e. a diagram of the form

$$\begin{array}{ccccc} SX & & \xrightarrow{\quad x \quad} & & X \\ & \searrow^{Sf} & & & \\ & & SA & & \swarrow_{f} \\ & & & \searrow^{a} & \\ & & & & A. \end{array}$$

This is just saying that f is an S-algebra homomorphism (in $\mathcal{E}^S$) from an S-algebra (X, x) to $\mathbf{A} = (A, a)$. Thus we have $(\mathcal{E}/A)^{S/\mathbf{A}} \cong \mathcal{E}^S/\mathbf{A}$. I.e. the category of $(S/\mathbf{A})$-algebras is the category of homomorphisms into $\mathbf{A}$: $(S/\mathbf{A})$**-Alg** $\cong$ S-**Alg**/$\mathbf{A}$. ☐

Lemma 4.2.3. *Let $\mathcal{E}$ be a category with pullbacks, and let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{E}$. Write $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ for the 'pulling back along f' functor: that is, we have the following diagram for each morphism $h : W \to Y$ in $\mathcal{E}$:*

$$\begin{array}{ccc} f^*W & \longrightarrow & W \\ {\scriptstyle f^*h}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

We then have the following adjunction:

$$\mathcal{E}/X \underset{f^*}{\overset{f \circ -}{\rightleftarrows}} \mathcal{E}/Y. \quad (\bot)$$

This is a standard result. A version of it is discussed in the context of topos theory in Theorem IV.7.2 of [MM94]. The notations f^*W and f^*h are just notations for the various pieces of data that make up the object

$$f^* \left(\begin{array}{c} W \\ \downarrow{\scriptstyle h} \\ X \end{array} \right) \quad = \quad \begin{array}{c} f^*W \\ \downarrow{\scriptstyle f^*h} \\ X \end{array}$$

of $\mathcal{E}/X$. Note that f^*Z depends on g: I will only use this notation for f^*Z where it is convenient whilst still unambiguous.

Proof. I will give the unit and the counit of this adjunction, and show that they satisfy the required identities. The counit of the adjunction will have components that are maps in $\mathcal{E}/Y$ of the form $f(f^*h) \to h$, for $W \xrightarrow{h} Y$ an object of $\mathcal{E}/Y$. So our counit, call it ε, will have components

$$\begin{array}{ccccc} f^*W & & \xrightarrow{\quad \varepsilon_h \quad} & & W \\ & {\scriptstyle f^*h}\searrow & & & \\ & & X & & \swarrow{\scriptstyle h} \\ & & & {\scriptstyle f}\searrow & \\ & & & Y. & \end{array}$$

We just define ε_h to be the top arrow in the pullback diagram

$$\begin{array}{ccc} f^*W & \xrightarrow{\varepsilon_h} & W \\ {\scriptstyle f^*h}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Note that this square commuting shows that ε_h is a morphism $f(f^*h) \to h$ in $\mathcal{E}/Y$.

The unit of the adjunction will have components that are maps in $\mathcal{E}/X$ of the form $g \to f^*(fg)$, for $Z \xrightarrow{g} X$ an object of $\mathcal{E}/X$. So our unit, call it η, will have components of the form

$$\begin{array}{ccc} Z & \xrightarrow{\eta_g} & V \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f^*(fg)} & \\ & X, & \end{array}$$

where the arrow $f^*(fg)$ is defined as the left side of the square

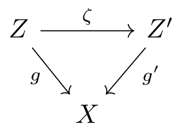
$$\begin{array}{ccc} V & \longrightarrow & Z \\ & \lrcorner & \downarrow{\scriptstyle g} \\ {\scriptstyle f^*(fg)}\downarrow & & X \\ & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

We can take the following cone over this diagram, and we get η_g given by the unique map that makes the triangles in the diagram

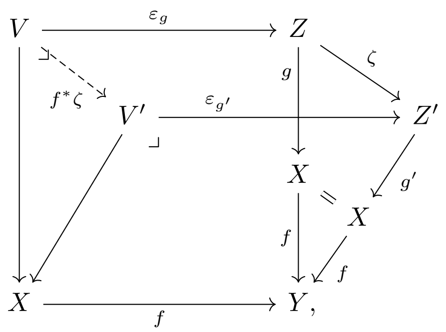
$$\begin{array}{cccc} Z & & \xrightarrow{\qquad 1_Z \qquad} & \\ & \overset{\eta_g}{\searrow} & & \\ & V & \longrightarrow & Z \\ {\scriptstyle g} & & \lrcorner & \downarrow{\scriptstyle g} \\ & \downarrow{\scriptstyle f^*(fg)} & & X \\ & & & \downarrow{\scriptstyle f} \\ \searrow & X & \xrightarrow[f]{} & Y. \end{array}$$

commute. The components of η are maps in $\mathcal{E}/X$: they make the left-hand triangle in the above diagram commute by definition.

I now need to show that η and ε are natural. I will start with ε: let

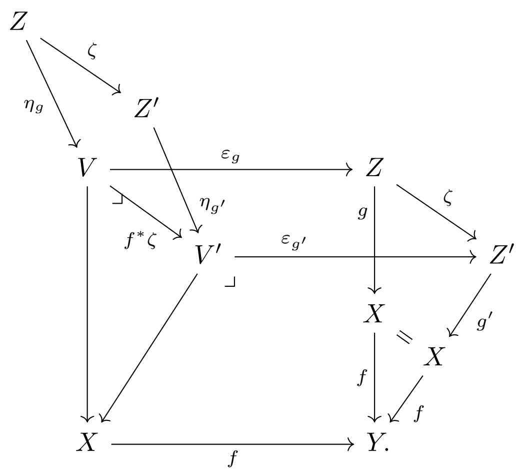


be a morphism in $\mathcal{E}/X$. Then we have the diagram

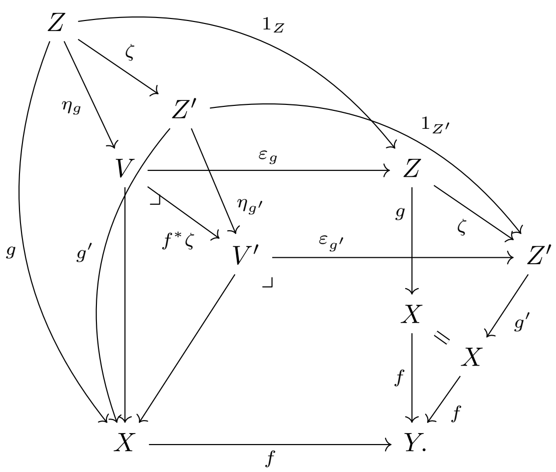


all parts of which commute, where $f^*\zeta$ is the map induced from the cone with vertex V over the pullback with vertex V'. The top trapezium in the above diagram is the naturality square for ε (at ζ).

To show that η is natural, I will adjoin a naturality square at ζ to the above diagram, giving



We can then fill in the cone arrows from the definition of η to get the diagram



Note that I have still not yet shown that the η naturality square commutes.

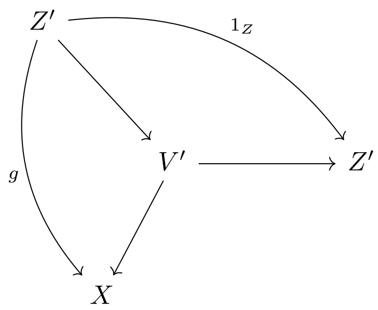
We now have two different maps $Z \to V'$, namely the composites

$$Z \xrightarrow{\zeta} Z' \xrightarrow{\eta_{g'}} V'$$

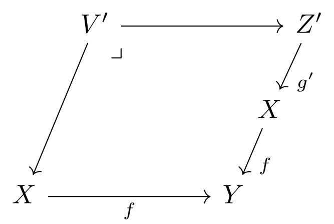
and

$$Z \xrightarrow{\eta_g} V \xrightarrow{f^*\zeta} V',$$

which both make the subdiagram (I use this word informally)

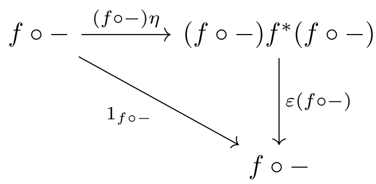


commute (by definition of η and $f^*\zeta$). But since



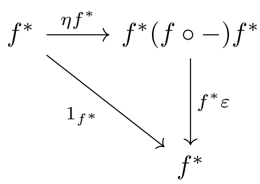
is a pullback, the map $Z \to V'$ is unique. So the naturality square for η (at ζ) commutes.

By definition of η, we have that $\varepsilon_{fg}\eta_g = 1_Z$ as maps in $\mathcal{E}$. Thus in $\mathcal{E}/Y$ we have that $\varepsilon_{fg} \circ (f \circ \eta_g) = 1_{fg}$, i.e. the triangle identity

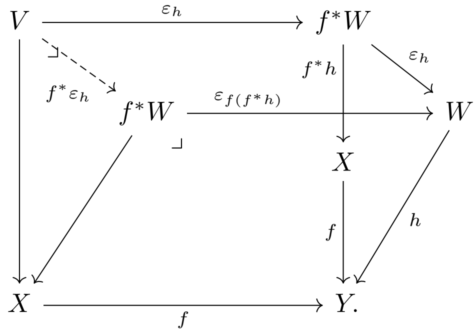


(which is a diagram in the functor category $[\mathcal{E}/X, \mathcal{E}/Y]$) commutes.

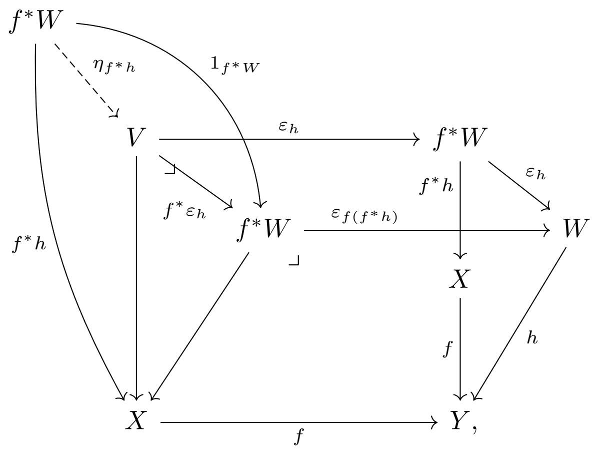
The other triangle identity we need to show commutes is



(this is a diagram in the functor category $[\mathcal{E}/Y, \mathcal{E}/X]$). Consider the diagram



We can augment it to obtain



and this demonstrates $f^*\eta_h \circ \eta_{f^*h} = 1_h$ holding in $\mathcal{E}/X$, i.e. the triangle diagram holding at an arbitrary object $W \xrightarrow{h} X$ of $\mathcal{E}/X$. We are now done: the adjunction exists as desired. $\square$

Construction 4.2.4. Consider the following diagram of categories and functors, where M is a monad:

$$\begin{array}{c} \mathcal{B} \circlearrowleft M \\ F \uparrow \dashv \downarrow G \\ \mathcal{A}. \end{array}$$

Then we get a monad GMF on $\mathcal{A}$: consider

$$\begin{array}{c} M\text{-}\mathbf{Alg} \\ \uparrow \dashv \downarrow \\ \mathcal{B} \end{array}$$

and compose adjunctions.

Construction 4.2.5. To construct the monad structure on T^{aff}, we take $\mathcal{E} = \mathbf{Set}$, $S = T$, and $\mathbf{A} = F^T 1 = (T1, \mu_1^T)$ in Lemma 4.2.2, to obtain a monad $T/F^T 1$ on $\mathbf{Set}/T1$. We then take $f = \eta_1 : 1 \to T1$ in Lemma 4.2.3, to give an adjunction

$$\mathbf{Set} \cong \mathbf{Set}/1 \underset{(\eta_1^T)^*}{\overset{\eta_1^T \circ -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Set}/T1.$$

By Construction 4.2.4, we can take the monad $T/T1$ on $\mathbf{Set} \cong \mathbf{Set}/T1$ and build a monad

$$(\eta_1^T)^* \circ T/F^T 1 \circ (\eta_1^T \circ -)$$

on $\mathbf{Set}/1 \cong \mathbf{Set}$. This is T^{aff}, with its monad structure.

We could work in more generality than just **Set** here: any category with a terminal object would do. But we would need to return to **Set** so quickly that I shall not bother.

And note that this does indeed give the same endofunctor T^{aff} as Definition 4.2.1. The definition of $T^{\mathrm{aff}} X$ as

$$(\eta_1^T)^* \circ T/F^T 1 \circ (\eta_1^T \circ -)X$$

expands as follows. We consider X as $!_X : X \to 1$, and then $(\eta_1^T \circ -)X$ is the composite

$$X \xrightarrow{!_X} 1 \xrightarrow{\eta_1^T} T1.$$

Applying $T/F^T 1$ gives

$$TX \xrightarrow{T!_X} T1 \xrightarrow{T\eta_1^T} TT1 \xrightarrow{\mu_1^T} T1.$$

Simplifying, since $\mu_1^T \circ T\eta^T 1 = 1_{T1}$ by the definition of monad, we obtain

$$TX \xrightarrow{T!_X} T1 \xrightarrow{1_{T1}} T1,$$

that is,

$$TX \xrightarrow{T!_X} T1.$$

Once we apply $(\eta_1^T)^*$, this is precisely the pullback of Definition 4.2.1.

I should also check the following.

Lemma 4.2.6. *Let T be a monad on* **Set**. *Then T^{aff} is affine.*

Proof. The definition of $T^{\text{aff}}1$ is given by the following pullback:

$$\begin{array}{ccc} T^{\text{aff}}1 & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_1} \\ T1 & \xrightarrow[T!_1]{} & T1. \end{array}$$

But $!_1 = 1_1$ the identity of $1 \in \textbf{Set}$, so $T!_1 = T1_1 = 1_{T1}$. So the pullback collapses down to give $T^{\text{aff}}1 \cong 1$. So T^{aff} is affine. ☐

Let us now show that **AffMnd** is indeed a coreflective subcategory of **Mnd**. Our functor $\textbf{Mnd} \to \textbf{AffMnd}$ is $(\ )^{\text{aff}}$, and our unit's components $\eta_S : S \to S^{\text{aff}}$, mapping from already affine monads to their affinisations, will be identities. What is the counit? It will have components $\varepsilon_T : T^{\text{aff}} \to T$ mapping to a monad from its affinisation. And we already have these maps: they are the natural transformations with components $T^{\text{aff}}X \to TX$ that are given as part of the data of the pullback that defines $T^{\text{aff}}X$ in diagram 4.1. Since we are working over **Set**, we can write what this map does on elements: for a monad T and a set X, $(\varepsilon_T)_X$ is the function $T^{\text{aff}}X \to TX$ that includes the affine terms in TX into TX. Category theoretically, this map is an injection – equivalently (since we are in **Set**) a monic – because it is a pullback of a monic, namely η_1^T.

4.3 Pointed monads

Looking at pointed monads now, we find that the full subcategory of **Mnd** of pointed monads, **PtMnd**, is a reflective subcategory of **Mnd**:

$$\textbf{Mnd} \underset{\hookleftarrow}{\overset{(\ )^{\text{pt}}}{\underset{\perp}{\longrightarrow}}} \textbf{PtMnd}.$$

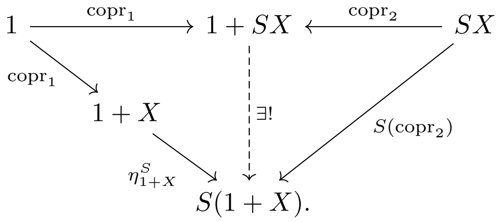
To send a monad to its reflection, we would like to adjoin a point. But we have a problem: it might already have a point, in fact it might have more than one. So we adjoin a point *that commutes with all terms*. Formally, $S^{\text{pt}} := S \otimes (1 + -)$, where $\otimes$ is the *(commutative) tensor product* of monads. For a definition of this, see Definition 3.2 of Martin Hyland and John Power's paper [HP07], noting that the authors give the definition in terms of Lawvere theories. In the case where S is affine, this simplifies to $S^{\text{pt}} \cong S(1 + -)$, the monad given by the standard distributive law of S over $1 + -$. I will not go into the definition of distributive law here; all we need to know is that $S(1+-)$ has unit components $\eta_X^{S(1+-)} : X \to S(1 + X)$ given by the following composite, where copr_2 is the second coproduct injection,

$$X \xrightarrow{\text{copr}_2} 1 + X \xrightarrow{\eta^S_{1+X}} S(1 + X)$$

and multiplication components $\mu^{S(1+-)} : S(1 + S(1 + X)) \to S(1 + X)$ given by the composite

$$S(1 + S(1 + X)) \xrightarrow{S\lambda_{1+X}} S(S(1 + 1 + X)) \xrightarrow{\mu^S_X\left(\binom{1_1}{1_1}+1_X\right)} S(1 + X),$$

where λ is the distributive law (all we need to know is that it is a natural transformation $(1+-)S \to S(1+-)$) given as the unique map given by the coproduct in the diagram



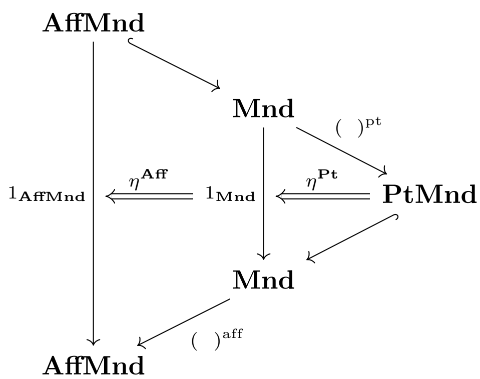
When S is an affine monad, $S(1+-)$ is indeed pointed: $S(1+0) \cong S(1) \cong 1$.

4.4 The adjunction between affine and pointed monads

Since we have a coreflective subcategory of **Set** and a reflective subcategory of **Set**, we can compose the adjunctions. This gives the following adjunction, where I am suppressing the inclusions:

$$\mathbf{AffMnd} \underset{(\ )^{\mathrm{aff}}}{\overset{(\ )^{\mathrm{pt}}}{\rightleftarrows}} \mathbf{PtMnd}. \quad (\bot)$$

We can write the unit and counit of this adjunction using the units and counits of the two adjunctions we built it from. Indeed: letting $\eta^{\mathbf{Aff}}$ be the unit for the **AffMnd**–**Mnd** adjunction, $\varepsilon^{\mathbf{Pt}}$ be the counit for the **Mnd**–**PtMnd** adjunction, etc. and writing just η for the unit of the composed adjunction etc., we have the diagram



in **CAT**, the 2-category of categories, and hence η is the composite natural transformation in the above, namely

$$1_{\mathbf{AffMnd}} \xrightarrow{(\eta^{\mathbf{Pt}})^{\mathrm{aff}}\eta^{\mathbf{Aff}}} ((\ )^{\mathrm{pt}})^{\mathrm{aff}}.$$

Syntactically, for S an affine algebraic theory and $m \in \mathbb{N}$,

$$\begin{aligned}(S^{\mathrm{pt}})^{\mathrm{aff}} m &\cong \{\varphi \in S(1+m) : (S(1+!_m))(\varphi) = (S\mathrm{copr}_2)(\mathrm{id})\} \\ &\cong \{\varphi \in S(1+m) : \varphi(x_0, x, \ldots, x) = x\}.\end{aligned}$$

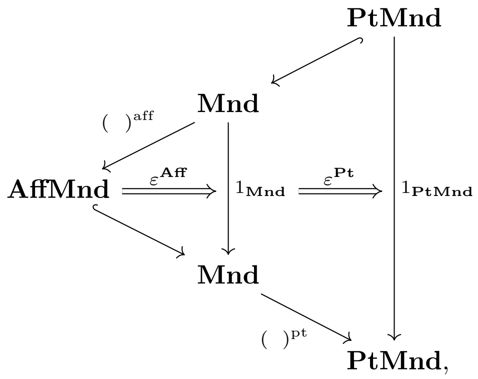
And $(\eta_S)_m$ acts on m-ary S-terms as follows:

$$(\eta_S)_m : Sm \to (S^{\mathrm{pt}})^{\mathrm{aff}} m,$$
$$\varphi \mapsto \psi$$

where

$$\psi(x_0, x_1, \ldots, x_m) = \varphi(x_1, \ldots, x_m).$$

Similarly, the diagram in **CAT** for ε is



so ε is given by

$$((\ \ )^{\mathrm{aff}})^{\mathrm{pt}} \xrightarrow{(\varepsilon^{\mathbf{Aff}})^{\mathrm{pt}}\varepsilon^{\mathbf{Pt}}} 1_{\mathbf{PtMnd}}.$$

Syntactically, for T a pointed algebraic theory and $n \in \mathbb{N}$,

$$\begin{aligned}(T^{\mathrm{aff}})^{\mathrm{pt}} n &\cong T^{\mathrm{aff}}(1+n) \\ &\cong \{\theta \in T(1+n) : (T!_{1+n})(\theta) = \mathrm{id}\} \\ &\cong \{\theta \in T(1+n) : \theta(x, x, \ldots, x) = x\},\end{aligned}$$

and $(\eta_T)_n$ acts on n-ary T-terms as follows:

$$(\varepsilon_T)_n : (T^{\mathrm{aff}})^{\mathrm{pt}} n \to Tn,$$
$$\theta \mapsto \theta(c, -, \ldots, -),$$

where $c \in T0 \cong 1$ is the 'point' of T.

Proposition 4.4.9 below states that each totally free theory is in the fixed subcategory of the **AffMnd**–**PtMnd** adjunction. That is, the components of the unit or counit of the adjunction respectively at each totally free theory are natural isomorphisms. Since we are working over **Set**, this can be done by appealing to the fact that an isomorphism in **Set** is a function that is both injective and surjective. So it is useful to investigate when the components of the components of η and ε are injective or surjective. It is also interesting. I will now state syntactic conditions for these properties.

Remark 4.4.1. Let S be an affine theory, and let $m \in \mathbb{N}$. Let $\varphi, \varphi' \in Sm$, and recall that $(\eta_S)_m(\varphi), (\eta_S)_m(\varphi') \in (S^{\mathrm{pt}})^{\mathrm{aff}}$ are defined such that

$$((\eta_S)_m(\varphi))(x_0, x_1, \ldots x_m) = \varphi(x_1, \ldots, x_m)$$

and

$$((\eta_S)_m(\varphi'))(x_0, x_1, \ldots x_m) = \varphi'(x_1, \ldots, x_m).$$

So $(\eta_S)_m$ is injective precisely when $(\eta_S)_m(\varphi) = (\eta_S)_m(\varphi')$ implies $\varphi = \varphi'$.

But, by transitivity of equality of S-terms, this is always true.

Lemma 4.4.2. *Let η be the unit of the affine-pointed adjunction. Then for all affine monads S, all components of η_S are injective.*

Remark 4.4.3. Let S be an affine theory, and let $m \in \mathbb{N}$. Then $(\eta_S)_m$ is surjective precisely when, for all $\psi \in S(1+m)$ such that $\psi(x_0, x, \ldots, x) = x$, there exists some $\varphi \in Sm$ such that $\psi(x_0, x_1, \ldots, x_m) = \varphi(x_1, \ldots, x_m)$.

Unlike injectivity, there exist affine theories S where the components of η_S are not all surjective.

Example 4.4.4. The components of the components of η are not always surjective. Indeed: let S be the affinisation of the theory of groups, and consider the term $\varphi(x_0, x_1, x_2) = x_1 x_0 x_1 x_2^{-1} x_0^{-1}$. This is a term in the affinisation of the theory of groups because 'its exponents sum to 1'. It also has the property that $\varphi(x_0, x, x) = x$, hence $\varphi \in (S^{\mathrm{pt}})^{\mathrm{aff}}2$, but φ certainly 'depends on x_0', that is, there is no $\theta \in S2$ such that $\varphi(x_0, x_1, x_2) = \theta(x_1, x_2)$. So $(\eta_S)_2$ is not surjective.

Similarly, there exist counterexamples to injectivity and surjectivity of the components of the components of ε.

Remark 4.4.5. Let T be a pointed theory, and let $n \in \mathbb{N}$. Let $\theta, \theta' \in T(1+n)$ such that $\theta(x, x, \ldots, x) = \theta'(x, x, \ldots, x) = x$. Then $(\varepsilon_T)_n$ is injective precisely when $\theta(c, -, \ldots, -) = \theta'(c, -, \ldots, -)$ implies that $\theta = \theta'$, where $c \in T0$ is the constant term of T.

Example 4.4.6. The components of the components of ε are not always injective. Indeed: let T be the theory of modules over the semiring $(\{0,1\}, \max, \min)$ (i.e. addition in the semiring is max, and multiplication is min). This semiring is also the bounded lattice with two elements. A *semiring* is a 'ring without additive inverses', also called a *rig*, because it has no *n*egatives. By *module over a semiring*, I mean a commutative monoid acted upon by a semiring. Note that T is indeed pointed, 'by' the commutative monoid's unit.

So: consider the distinct terms $0 \cdot x_0 + 1 \cdot x_1$ and $1 \cdot x_0 + 1 \cdot x_1$ in $(T^{\mathrm{aff}})^{\mathrm{pt}}1$, where I am writing $\cdot$ for the semiring action for extra clarity. These are both affine terms: 'setting $x_0 = x_1 = x$ makes them both equal to x'. But applying $(\varepsilon_T)_1$ sends them both to the term $1 \cdot x_1$ in $T1$. So $(\varepsilon_T)_1$ is not injective.

Finally for these properties, surjectivity of the components of the components of ε:

Remark 4.4.7. Let T be a pointed theory, and let $n \in \mathbb{N}$. Then $(\varepsilon_T)_n$ is surjective precisely when for each $\psi \in Tn$, there exists $\theta \in T(1+n)$ such that $\theta(x, x, \ldots, x) = x$ and $\theta(c, -, \ldots, -) = \psi$, where $c \in T0$ is the constant term of T.

Example 4.4.8. The components of the components of ε are not always surjective. Indeed: let T be the theory of modules over the semiring $(\mathbb{N}, +, \times)$. Then there is no element of $(T^{\mathrm{aff}})^{\mathrm{pt}}1$ that $(\varepsilon_T)_1$ maps to $2 \cdot x_1 \in T1$, since each $\lambda_0 \cdot x_0 + \lambda_1 \cdot x_1 \in (T^{\mathrm{aff}})^{\mathrm{pt}}1$ satisfies $\lambda_0 + \lambda_1 = 1$, which happens precisely when either $(\lambda_0, \lambda_1) = (1, 0)$ or $(\lambda_0, \lambda_1) = (0, 1)$.

Now, the above can be used to show the following.

Proposition 4.4.9. *All totally free algebraic theories are in the fixed subcategory of the* **AffMnd**–**PtMnd** *adjunction.*

It would be nice to be able to prove this without using the classification theorem, but I do not know how to do this. To prove the above proposition using the classification theorem, one can use the syntactic characterisations of injectivity and surjectivity of the components of the components of η and ε. I will not write out the details here, for the sake of time.

4.5 Further questions

An object being in the fixed subcategory means that the component of the unit or counit (as appropriate) of the adjunction at that object is an isomorphism. So for a totally free theory T, the unit η or the counit ε of the adjunction, as appropriate, has its component at T being an isomorphism.

Are the totally free theories the only theories in this fixed subcategory? If so, are they the only *monads* in it? We do not know the answer to either of these questions. But we can investigate which components of η and ε are isomorphisms. We can also ask for which theories T are the components of η_T surjective? How about injectivity and surjectivity of the components of ε_T? Note that the degenerate theory given by $TX \cong 1$ for all X, which is pointed and affine, is in the fixed subcategory both when considered as a pointed monad (as above), but also when considered as an object of **AffMnd**.

And indeed: can it be shown that all totally free theories are in the fixed subcategory of the **AffMnd–PtMnd** adjunction without using the classification theorem?

Chapter 5

Minimality

Consider the theory of $\mathbf{R}$-vector spaces, for some division ring $\mathbf{R}$ with underlying set R. Say we choose some equation in the theory, and want to see which vector spaces satisfy this theory. Without loss of generality, all equations in the theory are of the form

$$\sum_{i=1}^{n} \lambda_i x_i = 0 \tag{$\star$}$$

for $n \in \mathbb{N}$, $\lambda_i \in R$, and free variables x_i. For a vector space to satisfy this equation, it must satisfy it for all vectors x_i, so we can choose any $k \in \{1, \dots, n\}$, and set $x_j = 0$ for each $j \neq k$. Thus, for every vector x_k, we must have

$$\lambda_k x_k = 0.$$

If $\lambda_k = 0$, then this automatically holds. If instead $\lambda_k \neq 0$, then λ_k is invertible, since $\mathbf{R}$ is a division ring. But then we have that $x_k = 0$, and since this must hold for all vectors, this will only hold for the zero vector space.

So, going through each $k \in \{1, \dots, n\}$, we either have that $\lambda_k = 0$ for each $k \in \{1, \dots, n\}$, or the only vector space satisfying ($\star$) is the zero vector space. So, for the theory of $\mathbf{R}$-vector spaces, if we impose an equation, it either does nothing, or reduces the theory down to the degenerate theory with only one algebra.

5.1 Preliminaries

The above example generalises to a sense in which the nondegenerate totally free theories are minimal. From a classical universal algebraic view, this is saying that these varieties have no proper nontrivial subvarieties. By Birkhoff's HSP Theorem (Theorem 2.4.12: the varieties are the equational classes), this says that adding in any additional equations to one of these varieties either does nothing, or collapses the variety down to a variety of only trivial algebras.

From the category theoretic view, we can view this with the language of quotients of monads.

Definition 5.1.1. Let T be an algebraic theory. A **quotient of** T is an algebraic theory S and a componentwise epic monad map $T \to S$.

From a classical universal algebra point of view, taking a quotient of a theory is precisely imposing more equations onto the theory.

Definition 5.1.2. A monad map $\alpha : T \to S$ gives rise to a functor U^α : S-**Alg** $\to$ T-**Alg**, via

$$\begin{array}{c} SA \\ \big\downarrow a \\ A \end{array} \quad \mapsto \quad \begin{array}{c} TA \\ \big\downarrow \alpha_A \\ SA \\ \big\downarrow a \\ A \end{array}$$

and

$$\begin{array}{ccc} SA & \xrightarrow{Sf} & SB \\ a\big\downarrow & & \big\downarrow b \\ A & \xrightarrow[f]{} & B \end{array} \quad \mapsto \quad \begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \alpha_A\big\downarrow & & \big\downarrow \alpha_B \\ SA & & SB \\ a\big\downarrow & & \big\downarrow b \\ A & \xrightarrow[f]{} & B, \end{array}$$

such that the diagram

$$\begin{array}{ccc} S\text{-}\mathbf{Alg} & \overset{U^\theta}{\hookrightarrow} & T\text{-}\mathbf{Alg} \\ & {\scriptstyle U^S}\searrow \quad \swarrow{\scriptstyle U^T} & \\ & \mathbf{Set} & \end{array}$$

commutes.

To see that the penultimate above diagram commutes, add in the arrow $SA \xrightarrow{Sf} SB$. Note that U^α does indeed define a functor.

Lemma 5.1.3. *Let T and S be monads on* **Set**, *and let $\alpha : T \to S$ be a monad map. Then U^α : S-**Alg** $\to$ T-**Alg** has a left adjoint, and in fact U^α is monadic.*

A functor being *monadic* means that it is the forgetful functor of some monad. A proof of Lemma 5.1.3 follows directly from Chapter 3, Theorem 7.3(b) of Michael Barr and Charles Wells' book [BW05].

Lemma 5.1.4. *Let T and S be monads on any category, and let $\alpha : T \to S$ be a componentwise epic monad map. Then U^α is full and faithful.*

In fact U^α is faithful for all monad maps α, since it is monadic: faithfulness of U^α is straightforward to see directly also.

Definition 5.1.5. Let T be an algebraic theory, and let $\alpha : T \to S$ be a quotient of T. We call α **proper** when its components are not all isomorphisms. We call α **nontrivial** when S is not a degenerate theory.

If $T \ncong S$, then certainly all quotients $T \to S$ are proper, however there do exist monads T with proper quotients $T \to T$: consider for example $T : X \mapsto X + E$ for $E \in$ **Set** infinite.

The central idea for this chapter:

Definition 5.1.6. Let T be an algebraic theory. We say that T is **minimal** when it is nondegenerate and has no proper nontrivial quotients.

This is saying that, for any subcategory of T-**Alg** that is a category of algebras in its own right, albeit in a way compatible with U^T, this subcategory must in fact be either all of T-**Alg** or trivial.

We need some preliminary results about finitary monads and cardinality before showing that all totally free theories are minimal. Recall also the important Lemma 2.2.1 which says that monads on **Set** preserve monics.

The first two are about quotients of theories. They are both equivalent to fragments of the important classical universal result Birkhoff's HSP Theorem (Theorem 2.4.12).

Lemma 5.1.7. *Let T be an algebraic theory, and let $T \overset{\alpha}{\twoheadrightarrow} S$ be a quotient of T. Then S has all set-indexed T-products. This means the following. Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of S-algebras indexed by some set I. Then there exists some S-algebra $\mathbf{P}$ such that $\prod_{i \in I} \mathbf{A}_i \cong U^\alpha \mathbf{P}$.*

Proof. By Lemma 5.1.3, U^α is monadic. Monadic functors create all limits in their codomains, and products are limits. □

For an explanation of what monadic functors are, and a proof that they create all limits in their codomains, see for example Definition 5.3.1 and Theorem 5.6.5 respectively of [Rie16].

The statement of the following lemma is in much the same vein as the previous one:

Lemma 5.1.8. *Let T be an algebraic theory, and let $T \overset{\alpha}{\twoheadrightarrow} S$ be a quotient of T. Then S has all T-subalgebras. This means the following. Let $\mathbf{A}$ be an S-algebra, and let $\mathbf{B}$ be a T-subalgebra of $U^\alpha \mathbf{A}$: that is, there exists an injection $U^T \mathbf{B} \overset{i}{\hookrightarrow} U^T U^\alpha \mathbf{A}$ such that in fact i is a T-algebra homomorphism $\mathbf{B} \to U^\alpha \mathbf{A}$. Then in fact there exists some S-algebra $\mathbf{B}'$ such that $\mathbf{B} \cong U^\alpha \mathbf{B}'$.*

The proof of this lemma is a different approach to that of the previous one: subobjects are related to limits, but are not themselves limits.

Proof. Let $\mathbf{A}$, $\mathbf{B}$, and i be as in the hypothesis of the lemma, and let

$$\mathbf{A} \quad = \quad \begin{array}{c} SA \\ \big\downarrow{\scriptstyle a} \\ A \end{array}$$

and

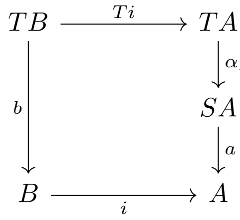
$$\mathbf{B} \quad = \quad \begin{array}{c} TB \\ \big\downarrow{\scriptstyle b} \\ B. \end{array}$$

I will show that there exists some S-algebra structure

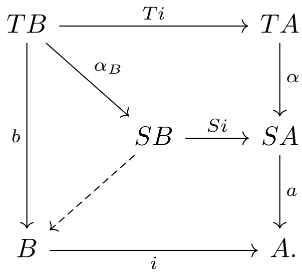
$$\mathbf{B}' \quad = \quad \begin{array}{c} SB \\ \big\downarrow{\scriptstyle b'} \\ B \end{array}$$

such that $\mathbf{B} \cong U^{\alpha}\mathbf{B}'$.

Consider the commuting diagram

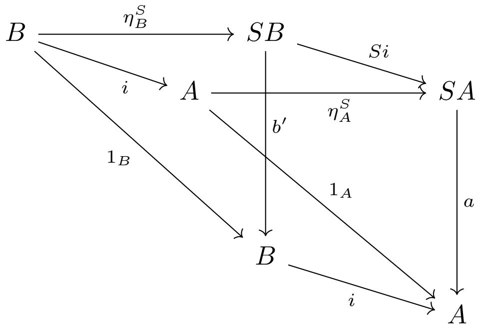


in **Set**. Augment it, to give the diagram

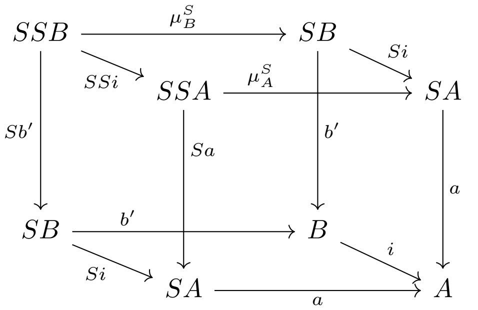


Then there exists an arrow $b' : SB \to B$, taking the place of the dashed arrow in the above diagram, that makes the bottom quadrilateral in the above commute, if and only if, for all $\xi \in SB$, we have that $a((Si)(\xi)) \in iB$, where iB denotes the image of B under i. Since α_B is surjective by assumption, this condition is equivalent to, for all $\nu \in TB$, having $a((Si)(\alpha_B(\nu))) \in iB$. Since the top quadrilateral commutes (it is the naturality square for α at i), this condition is equivalent to, for all $\nu \in TB$, having $a(\alpha_A((Ti)(\nu))) \in iB$. Since the outside square commutes by assumption, this condition is equivalent to, for all $\nu \in TB$, having $i(b(\nu)) \in iB$. But this is always true. Unravelling the equivalent statements, this gives us a function $b' : SB \to B$ making the bottom quadrilateral commute.

I will now show that b' is indeed an S-algebra structure. First, note that the bottom quadrilateral of the above diagram commutes, since α_B is epic. The unit law for b' holds since all other faces of the diagram



commute, and i is monic. Similarly, the multiplication law for b' holds since all other faces of the diagram



commute, and i is monic. Note that $U^\alpha \mathbf{B'} = U^\alpha(B, b') = (B, b) = \mathbf{B}$. So $\mathbf{B'} = (B, b')$ exists as required. □

The rest of these preliminary results are about how algebraic theories interact with cardinality.

Lemma 5.1.9. *Let T be a finitary, nondegenerate monad on* **Set**. *Then for all sets X, if $X \geq \sum_{n \in \mathbb{N}} Tn$ then $TX \cong X$.*

I.e. T fixes the cardinalities of all sufficiently large sets.

Proof. Note that $\sum_{n\in\mathbb{N}} Tn$ is infinite: each Tn is nonempty for $n \geq 1$, containing at least a projection term. Since T is finitary,

$$TX \cong \left(\sum_{n\in\mathbb{N}} Tn \times X^n \right) \Big/ \sim$$

for some equivalence relation $\sim$. Thus

$$\begin{aligned} TX &\leq \sum_{n\in\mathbb{N}} Tn \times X^n \\ &\leq \left(\sum_{n\in\mathbb{N}} Tn \right) \times X \\ &\cong X, \end{aligned}$$

with the second inequality following from X being infinite (so $X^n \cong X$ for all $n \in \mathbb{N}$), and the final isomorphism following from the assumption $X \geq \sum_{n\in\mathbb{N}} Tn$. We have $TX \leq X$: since T is nondegenerate, we also have that the unit η^T_X is an injection $X \hookrightarrow TX$. So $TX \cong X$ by the Cantor–Bernstein Theorem (which says that for sets A and B, we have that $A \leq B$ and $B \leq A$ together imply that $A \cong B$). □

The following lemma is a counterpart to the previous one.

Lemma 5.1.10. *Let T be a finitary, nondegenerate monad on* **Set**. *Then for all sets Y, if $TY > T(\sum_{n\in\mathbb{N}} Tn)$ then $Y \cong TY$.*

Proof. We prove the contrapositive. Take any set Y such that $Y \ncong TY$. Then by (the contrapositive of) Lemma 5.1.9, we have $Y < \sum Tn$. Since monads on **Set** preserve monics, we thus obtain that $TY \leq T(\sum Tn)$, proving the contrapositive. □

We can combine these two lemmas to the more crude result that just says 'if at least one of Z or TZ is sufficiently big, then in fact $TZ \cong Z$':

Corollary 5.1.11. *Let T be a finitary, nondegenerate monad on* **Set***, and let Z be a set. Then*

$$\max(Z, TZ) \geq T(\sum Tn) \implies TZ \cong Z.$$

I.e. T (as an endofunctor *and* (straightforwardly) as a monad) restricts to the full subcategory of **Set** of sets larger than A, for each sufficiently large A. Even further: it restricts to any full subcategory of such categories (up to equivalence of categories/isomorphism of monads). Note that $T(\sum Tn) \geq \sum Tn$ since T is nondegenerate.

5.2 Nondegenerate totally free theories are minimal

I believe that the following result is an original contribution.

Theorem 5.2.1. *All nondegenerate totally free algebraic theories are minimal.*

The lemmas from Section 5.1 do some heavy lifting in the following proof. I will not assume the classification theorem for this proof.

Proof. Let T be a nondegenerate totally free algebraic theory. Since T is nondegenerate, we may assume that it has an algebra $\mathbf{P}$ with $U\mathbf{P} \geq 2$, and let S be a nontrivial quotient of T. We may assume further that $\mathbf{P}$ is in S-**Alg** also (by which I – being formal – actually mean that there exists an algebra $\mathbf{P'}$ in S-**Alg** such that $U^\theta \mathbf{P'} \cong \mathbf{P}$, where U^θ is the inclusion S-**Alg** $\hookrightarrow$ T-**Alg** induced by the quotient), since S is a nontrivial quotient of T. The idea for the remainder of this proof is to use the fact that S-**Alg** has all T-subalgebras to show that it is downward closed in a certain sense, and the fact that it has all set-indexed T-products to show that it has no upper bound. This will allow us to show that S-**Alg** is in fact all of T-**Alg**.

For the first part, consider some $F^S Y \in S$-**Alg**. Take any set $X \leq Y$ and an injection $m : X \hookrightarrow Y$. I shall show that $F^S X$ is a subalgebra of $F^T Y$ in T-**Alg**, hence is an object of S-**Alg** since S-**Alg** has all subalgebras. I will start by making clear what it is here to be a subalgebra.

A subalgebra of $F^S Y$ in T-**Alg** is (for our purposes here) an object of T-**Alg** whose underlying set is a subset of $U^T F^T Y = TY$. (Recall, we are working up to isomorphism.) Since T preserves monics, $U^T F^T X = TX \leq TY = U^T F^T Y$, so $F^T X$ is a subalgebra of $F^T Y$ in T-**Alg**. Since S-**Alg** has all subalgebras, this means that $F^T X \in S$-**Alg** also. So S-**Alg** is downward closed.

We now show that there exist algebras with generating sets having arbitrarily large cardinality in S-**Alg**. Combined with the above point this will be sufficient to prove the theorem, as we will be able to see each algebra in T-**Alg** as a subalgebra of an algebra in S-**Alg** generated as an algebra of T-**Alg** by a sufficiently large set.

Recall that we have $\mathbf{P} \in T$-**Alg** such that $U^T\mathbf{P} \geq 2$. Consider

$$\prod_Y \mathbf{P} =: \mathbf{P}^Y$$

the product of Y-many copies of $\mathbf{P}$. This is an object of S-**Alg** as $\mathbf{P}$ is an S-algebra also, and S-**Alg** has all set-indexed products. The underlying set of $\mathbf{P}^Y$ is $(U^T\mathbf{P})^Y \geq 2^Y$ (noting that U^T, as a right adjoint, preserves products, and that $A \geq B$ implies $A^Y \geq B^Y$ for sets A, B, Y). Also, since all algebras in T-**Alg** are free, we can write $\mathbf{P}^Y \cong F^T Q$ for some set Q.

So $U^T F^T Q = TQ$ can be made arbitrarily large, and in particular we can make it large enough to invoke Lemma 5.1.10: there exists some set Z such that for all $Y \geq Z$ we have $Q \cong TQ$. Since $TQ \geq 2^Y$, we thus have algebras of S-**Alg** generated as algebras of T-**Alg** by arbitrarily large sets. And so we are done. □

Which properties of **Set** did we use here? We used that monads on **Set** preserve monics, and various other properties of the cardinality ordering $\leq$ on **Set**. We also used Birkhoff's theorem, although not that much of it: only that equational classes are closed under subalgebras and products.

5.3 Equivalent conditions to minimality

The following result is likely well-known.

Proposition 5.3.1. *Let T be a nondegenerate theory. Then the following are equivalent.*

(i) *T is minimal.*

(ii) *For all $n \in \mathbb{N}$, distinct n-ary operations θ and θ', and nontrivial T-algebras $\mathbf{B}$, there exist $b_1, \ldots, b_n \in U\mathbf{B}$ such that $\theta(b_1, \ldots, b_n) \neq \theta'(b_1, \ldots, b_n)$.*

(iii) *For all sets X, distinct $a, a' \in FX$, and nontrivial T-algebras $\mathbf{B}$, there exists a T-algebra homomorphism $f : FX \to \mathbf{B}$ such that $f(a) \neq f(a')$.*

Note that (ii) is saying that the whole theory is the theory of any particular algebra (the theory given by precisely the equations that the algebra satisfies). No equational laws in any algebra that do not hold in some other algebra. For hands-on intuition, one can consider what this result means in the example of **R**-vector spaces from the beginning of this chapter.

Proof. ((i) $\implies$ (ii)) Let S be a quotient of T obtained by adjoining the equation $\theta = \theta'$. By minimality, S is degenerate, so $\mathbf{B}$ is not an S-algebra. So $\theta_{\mathbf{B}} \neq \theta'_{\mathbf{B}}$.

((ii) $\implies$ (iii)) Let $a, a' \in FX$ with $a \neq a'$. By finitariness, there exists $n \in \mathbb{N}$ and $j : \{1, \ldots, n\} \to X$ such that $a, a' \in \mathrm{im}(Fj : F\{1, \ldots, n\} \to FX)$. Say $a = (Fj)(\theta)$ for some $\theta \in F\{1, \ldots, n\}$, and similarly $a' = (Fj)(\theta')$. Then use (ii).

((iii) $\implies$ (ii)) Take X to be finite, and note that the transpose of $f : FX \to \mathbf{B}$ is a map $X \to U\mathbf{B}$, which, when X is an n-element set, just picks out n elements of $U\mathbf{B}$, i.e. our b_i.

((ii) $\implies$ (i)) Take a proper quotient S of T. Then there exists $n \in \mathbb{N}$ and $\theta, \theta' \in Tn$ such that $\theta \neq \theta'$ in Tn, but θ and θ' are identified in S. Every S-algebra $\mathbf{B}$ is a T-algebra such that $\theta_B = \theta'_B$. So by (ii), $\mathbf{B}$ is trivial. Hence S is degenerate. □

Corollary 5.3.2. *In a totally free theory T, for all T-algebras $\mathbf{A}$ and $\mathbf{B}$ and $a, a' \in U\mathbf{A}$, there exists a T-algebra homomorphism $f : \mathbf{A} \to \mathbf{B}$ satisfying $f(a) \neq f(a')$ as long as $a \neq a'$ and $\mathbf{B}$ is nontrivial.*

Proof. All totally free algebraic theories are minimal by Theorem 5.2.1: apply (i) $\implies$ (iii) in Proposition 5.3.1. ☐

In the statement of this corollary, we could replace $\mathbf{B}$ with the smallest nontrivial algebra, which will be $F1$ in the pointed case and $F2$ in the affine case. This in fact gives an equivalent statement to the corollary as stated. Indeed: we can map e.g. for the pointed case

$$\mathbf{A} \longrightarrow F1 \longrightarrow \mathbf{B}$$

for any nontrivial $\mathbf{B}$, where $F1 \to \mathbf{B}$ is the transpose of an inclusion $1 \hookrightarrow X$, where $\mathbf{B} = FX$ for some set X since T is totally free. Thus, since $UF1 \to U\mathbf{B}$ is just T applied to the inclusion $1 \hookrightarrow X$, and monads on **Set** preserve monics (Lemma 2.2.1), we have that the composite – call it f – satisfies $f(a) \neq f(a')$. Equivalence in the other direction is immediate.

5.4 Further questions

Does minimality still hold when we replace 'totally free finitary algebraic theory' with 'totally free algebraic theory with rank κ', for κ some fixed cardinal? How about 'totally free algebraic theory with rank' (i.e. arbitrarily big, but still cardinal rank)? How about monads on **Set** that don not have rank?

Which theories are minimal but are not totally free? Can we classify them all?

Chapter 6

Dimension

By definition, an algebra for a totally free theory is freely generated by some set. In the case of vector spaces over a field, it is a familiar result from linear algebra that any minimal generating sets of a given vector space must have the same cardinality, and this notion of dimension in fact generalises to all nondegenerate totally free algebraic theories. In this chapter, I will give a proof of this, without relying on the classification theorem for totally free theories.

6.1 Nondegenerate totally free theories have dimension

The following result, including all proofs in this chapter, is due to Tom Leinster.

Theorem 6.1.1. *Let T be a nondegenerate totally free algebraic theory. Then for all sets X and Y, $FX \cong FY$ implies $X \cong Y$.*

I will give a proof of this later in this chapter (Theorem 6.2.12).

This property of a theory having a well-defined notion of dimension is in a certain way 'dual' to the notion of a theory being totally free. Indeed: take an algebraic theory T, and consider its free functor F. Then T being totally free is equivalent to saying that F is essentially surjective on objects: every T algebra is isomorphic to F applied to some set.

The notion of an algebraic theory T having dimension is equivalent to its free functor F being essentially *injective* on objects. By this, I mean that for sets X and Y, we have that $FX \cong FY$ implies $X \cong Y$.

Theorem 6.1.1 says that all totally free theories have dimension, but the converse is not true.

Example 6.1.2. The endfunctor $2 + - : \mathbf{Set} \to \mathbf{Set}$ has a monad structure given by $\eta_X = \mathrm{copr}_2 : X \to 2 + X$, the second coprojection; and

$$\mu_X = \left(\begin{smallmatrix} 1_2 \\ 1_2 \end{smallmatrix}\right) + 1_X : 2 + 2 + X \to 2 + X,$$

the map that 'flattens both 2s together and leaves X alone'. This is an example of an *exception monad*. It is not totally free, but it does have dimension. Indeed, it is a consequence of commonly used foundations of set theory that for any sets X and Y, if $2 + X \cong 2 + Y$, then $X \cong Y$.

6.2 Proof

The proof of Theorem 6.1.1 is a generalisation of the proof that every (possibly infinite-dimensional) vector space has a basis. Here I give it broken up into some lemmas. The general idea that we will work with is that we should be able to think of dimension for totally free theories as the maximum number of times that we can repeatedly take proper quotients of the theory. For convenience in the rest of this chapter, I will make the following definition.

Definition 6.2.1. Let T be an algebraic theory with forgetful functor U. Let f be a T-algebra homomorphism. I will call f a **surjection**, or say that f is **surjective**, when Uf is a surjection. I may write such a surjection with the arrow $\twoheadrightarrow$: indeed, surjections are epic in T-**Alg**, so this does not cause a conflict of notation.

Since a T-algebra homomorphism $f : \mathbf{A} \to \mathbf{B}$ is *defined* to be a map $U\mathbf{A} \to U\mathbf{B}$ satisfying certain properties (Definition 2.1.6), hence U is faithful, f and Uf are often conflated. But I stress that formally, subsequent to the definition, they are different mathematical objects.

Recall that Uf is a surjection if and only if f is a regular epimorphism (Lemma 2.2.5). Also, note that not always are the epic morphisms in T-**Alg** all surjections.

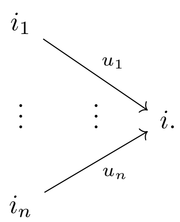
Let us begin the proof. For the remainder of this section, all algebraic theories will be assumed to be finitary.

Lemma 6.2.2. *Let T be an algebraic theory, let $\mathcal{I}$ be a filtered category, and let $D : \mathcal{I} \to T$-**Alg** be a diagram of T-algebras where all homomorphisms in the diagram are surjections. Let D have colimit $\mathbf{C}$. Let $n \in \mathbb{N}$, and suppose that $\mathbf{C}$ is generated by some n-element set. Then so is Dk for some $k \in \mathcal{I}$.*

Proof. Since $\mathbf{C}$ is a colimit, we can write it as

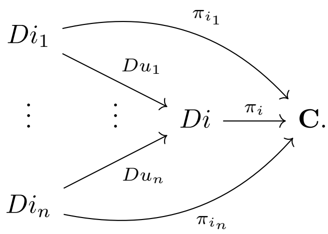
$$\left(\sum_{i\in\mathcal{I}} Di\right)\Big/ \sim$$

for some equivalence relation $\sim$. Take generators $c_1, \ldots, c_n$ of $\mathbf{C}$, and write $Di \xrightarrow{\pi_i} \mathbf{C}$ for the coprojections. There must exist $i_p \in \mathcal{I}$ and $a_p \in UDi_p$ such that $\pi_{i_p}(a_p) = c_p$ for each $p \in \{1, \ldots, n\}$, and we can take a cocone in $\mathcal{I}$ as follows, since $\mathcal{I}$ is filtered:



Since $\mathbf{C}$ is the colimit of D, we can without loss of generality take $i_1 = \cdots =$

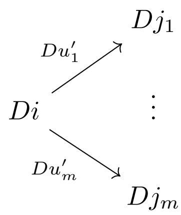
$i_n = i$, since the Di factor through π_i:



That is: by filteredness, there exists $i \in \mathcal{I}$ and $a_1, \ldots a_n \in UDi$ such that $\pi_i(a_p) = c_p$ for each $p \in \{1, \ldots, n\}$.

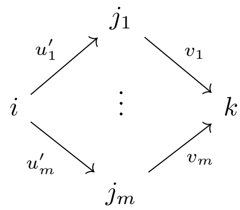
Since Di is finitely generated, there exists $m \in \mathbb{N}$ and generators $\alpha_1, \ldots, \alpha_m$ of Di.

For each $r \in \{1, \ldots, m\}$, we have $\pi_i(\alpha_r) \in \mathbf{C}$, so $\pi_i(\alpha_r) = \theta_r(c_1, \ldots, c_n)$ for some n-ary operation θ_r. So $\pi_i(\alpha_r) = \theta_r(\pi_i(a_1), \ldots, \pi_i(a_n)) = \pi_i(\theta_r(a_1, \ldots, a_n))$ (since π_i is a homomorphism), i.e. $\alpha_r \sim \theta_r(a_1, \ldots, a_n)$. So there exist maps $i \xrightarrow{u'_r} j_r$ such that $Du'_r(\alpha_r) = Du'_r(\theta_r(a_1, \ldots, a_n))$. We thus have



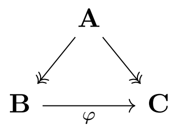
in T-**Alg**.

By filteredness, there exists a commutative diagram



in $\mathcal{I}$. If $m \neq 0$, let $w = v_1 u'_1 = \cdots = v_m u'_m : i \to k$, and if $m = 0$, let $k = i$ and $w = 1_i$. Then $Dw(\alpha_r) = Dw(\theta_r(a_1, \ldots, a_n))$ for each r. Since $Dw : Di \to Dk$ is surjective, we have that $Dw(\alpha_1), \ldots, Dw(\alpha_m)$ generate Dk. For each r, we have that $Dw(\alpha_r) = Dw(\theta_r(a_1, \ldots, a_n)) = \theta_r(Dw(a_1), \ldots, Dw(a_n))$, so $Dw(a_1), \ldots, Dw(a_n)$ generate Dk. □

Definition 6.2.3. Let T be an algebraic theory, and let $\mathbf{A}$ be a T-algebra. Write $\mathrm{Qt}(\mathbf{A})$ for the **category of quotients of A**. The objects of this category are the quotients $(\mathbf{A} \twoheadrightarrow \mathbf{B})$ of $\mathbf{A}$, and a morphism from $(\mathbf{A} \twoheadrightarrow \mathbf{B})$ to $(\mathbf{A} \twoheadrightarrow \mathbf{C})$ is a map $\varphi : \mathbf{B} \to \mathbf{C}$ in T-**Alg** such that the triangle



commutes. Equivalently, $\mathrm{Qt}(\mathbf{A})$ is the full subcategory of $\mathbf{A}/T\text{-}\mathbf{Alg}$ whose objects are the surjections.

This category is essentially the *lattice of congruences of* $\mathbf{A}$ of classical universal algebra. If such a φ exists as in the diagram above, it must be both unique and surjective (surjective as a T-algebra map): uniqueness follows from surjectivity of the map $\mathbf{A} \twoheadrightarrow \mathbf{B}$, and surjectivity follows from surjectivity of the map $\mathbf{A} \twoheadrightarrow \mathbf{C}$.

Lemma 6.2.4. *Let T be an algebraic theory, and let $\mathbf{A}$ be a T-algebra. Let $\mathcal{P}$ be a full subcategory of $\mathrm{Qt}(\mathbf{A})$ such that $\mathcal{P}$ is nonempty and closed under colimits in T-$\mathbf{Alg}$ indexed by totally ordered categories: that is, if $D : \mathbf{I} \to \mathcal{P}$ is a diagram in $\mathcal{P}$ with $\mathbf{I}$ a totally ordered category, the map $\mathbf{A} \twoheadrightarrow \operatorname{colim}_{i\in\mathbf{I}} Di$ (where the colimit is taken in T-$\mathbf{Alg}$) in T-$\mathbf{Alg}$ given by the composite*

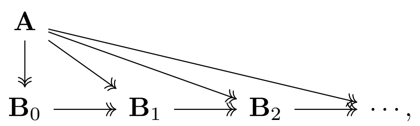
$$\mathbf{A} \twoheadrightarrow Dj \twoheadrightarrow \operatorname{colim}_{i\in\mathbf{I}} Di,$$

for any $j \in \mathbf{I}$, is an object of $\mathcal{P}$.

Then there exists an object $(\mathbf{A} \twoheadrightarrow \mathbf{B})$ of $\mathcal{P}$ that is minimal in the following sense: if there exists an object $(\mathbf{A} \twoheadrightarrow \mathbf{C})$ of $\mathcal{P}$ such that there exists a map φ from $(\mathbf{A} \twoheadrightarrow \mathbf{B})$ to $(\mathbf{A} \twoheadrightarrow \mathbf{C})$ in $\mathcal{P}$, then φ is an isomorphism.

Note that 'minimal' here refers to the quotient *algebra* $\mathbf{B}$. One might reasonably prefer to call the quotient *map* φ 'maximal' in that it 'identifies a maximal number of elements together'. In the above statement of the lemma, all $j \in \mathbf{I}$ will give the same composite $\mathbf{A} \twoheadrightarrow Dj \twoheadrightarrow \operatorname{colim}_{i\in\mathbf{I}} Di$, since $\operatorname{colim}_{i\in\mathbf{I}} Di$ is a colimit.

An example of a diagram indexed by a totally ordered category is



which is indexed by $\mathbb{N}$, considered as a category. In this case, the map $\mathbf{A} \twoheadrightarrow \operatorname{colim}_i \mathbf{B}_i$ is given by the composite

$$\begin{aligned}(\mathbf{A} \twoheadrightarrow \mathbf{B}_0 \twoheadrightarrow \operatorname{colim}_i \mathbf{B}_i) &= (\mathbf{A} \twoheadrightarrow \mathbf{B}_1 \twoheadrightarrow \operatorname{colim}_i \mathbf{B}_i)\\ &= (\mathbf{A} \twoheadrightarrow \mathbf{B}_2 \twoheadrightarrow \operatorname{colim}_i \mathbf{B}_i)\\ &= \cdots .\end{aligned}$$

Proof of Lemma 6.2.4. This result follows from Zorn's Lemma (which is equivalent to the axiom of choice) holding in $\mathbf{Set}$. Indeed: Zorn's Lemma says that in a partially ordered set in which all chains have an upper bound, there exists at least one minimal element; in our case, our poset is $\operatorname{ob}\mathcal{P}$, with $(\mathbf{A} \twoheadrightarrow \mathbf{B}) \le (\mathbf{A} \twoheadrightarrow \mathbf{C})$ if and only if there exists a morphism form $(\mathbf{A} \twoheadrightarrow \mathbf{B})$ to $(\mathbf{A} \twoheadrightarrow \mathbf{C})$ in $\mathrm{Qt}(\mathbf{A})$. The collection of isomorphism classes of objects of $\mathrm{Qt}(\mathbf{A})$ forms a set and inherits the poset structure, so we can apply Zorn. □

Lemma 6.2.5. *Let T be a nondegenerate, totally free algebraic theory, and let X and Y be sets such that $X \ge Y$. Assume that we are not in the case where both T is affine and $Y \cong 0$. Then there exists a surjection $FX \twoheadrightarrow FY$ in T-$\mathbf{Alg}$. Further, if $X > Y$, then there exists a non-invertible surjection $FX \twoheadrightarrow FY$ in T-$\mathbf{Alg}$.*

Proof. If $Y \ncong 0$, then we can take a surjection $f : X \twoheadrightarrow Y$. This gives that Ff is a surjection (this can be seen, for example, syntactically). If in this case $X > Y$, then there exist $x, x' \in X$ such that $f(x) = f(x')$. Thus $(UFf)(x) = (UFf)(x')$, so UFf is non-invertible, thus Ff is non-invertible.

If $Y \cong 0$, then by assumption T is pointed, recalling that totally free theories are either affine or pointed as shown in Lemma 4.1.3. So $FY \cong F0 \cong 1$, so the unique map $FX \to 1 \cong FY$ is a surjection. If in this case $X > Y$ also, then $UFX > X$ since finitary monads preserve monics (Lemma 2.2.1), and so $UFX > Y$. Thus any function $UFX \to Y$ is non-invertible, thus the unique map $FX \to 1 \cong FY$ is non-invertible. ☐

Definition 6.2.6. Let T be an algebraic theory, and let $\mathbf{A}$ be a T-algebra. I will say that $\mathbf{A}$ has **height** n when n is the largest natural number such that there exist non-invertible surjections

$$\mathbf{A} = \mathbf{A}_n \twoheadrightarrow \mathbf{A}_{n-1} \twoheadrightarrow \mathbf{A}_{n-2} \twoheadrightarrow \cdots \twoheadrightarrow \mathbf{A}_1 \twoheadrightarrow \mathbf{A}_0 \cong \mathbf{1}.$$

I will write $\operatorname{ht}\mathbf{A}$ to denote the height of $\mathbf{A}$.

An algebra has height 0 if and only if it is isomorphic to $\mathbf{1}$. An algebra has height 1 if and only if it is simple. We might say that an empty algebra has 'height $-\infty$'. Not every algebra has height, for example infinite-dimensional real vector spaces.

Example 6.2.7. Let T be the real vector space monad. Then $T3 \cong \mathbb{R}^3$ and has height 3, via for example

$$T3 \cong \mathbb{R}^3 \xrightarrow{(x,y,z)\mapsto(x,y)} \mathbb{R}^2 \xrightarrow{(x,y)\mapsto x} \mathbb{R}^1 \xrightarrow{x\mapsto 0} 0.$$

Note that $T0 \cong 0$, the zero vector space, and that 0 is isomorphic to $\mathbf{1}$, the algebra with one element, here.

Example 6.2.8. Let S be the free real affine space monad. Then $S3 \cong \mathbb{R}^2$ (consider three points forming the vertices of a triangle, generating $\mathbb{R}^2$). So $S3 \cong \mathbb{R}^2$ has height 2 here, via for example

$$S3 \cong \mathbb{R}^2 \xrightarrow{(x,y)\mapsto x} \mathbb{R}^1 \xrightarrow{x\mapsto 0} 0.$$

Again, note that 0, the zero affine space, is isomorphic to $\mathbf{1}$. Note however that $\mathbf{1} \cong S1$ (not $S0$) here.

The following is an important lemma, and the proof is quite long.

Lemma 6.2.9. *Let T be a nondegenerate totally free algebraic theory, and let n be a natural number.*

(i) *Let X be a set. If T is pointed, then the T-algebra FX has height n if and only if $X \cong n$. If T is affine, then FX has height n if and only if $X \cong n+1$.*

(ii) *Let X be a set. If there exists a surjective T-algebra map $Fn \twoheadrightarrow FX$, then $n \geq X$.*

(iii) *All surjective T-algebra maps $Fn \twoheadrightarrow Fn$ are isomorphisms.*

Proof. We prove by induction on (i), (ii), and (iii) simultaneously. That is; we prove that each part holds for $n = 0$; then assume that each holds for $n \leq k$ for some arbitrary $k \in \mathbb{N}$, and prove that each must hold for $n = k + 1$. The base case is straightforward. The base case depends on whether T is affine or pointed: a decent amount of the induction step does not.

Now we assume that (i), (ii), and (iii) each hold for all $n \leq k$ for some arbitrary $k \in \mathbb{N}$: I now will prove that they hold for $n = k + 1$.

First I will show that (i) holds for $n = k + 1$. Let $\mathcal{P}$ be the category of quotients of $F(k+1)$ that can be properly quotiented further to Fk: that is, $\mathcal{P}$ is the full subcategory of $\mathrm{Qt}(F(k+1))$ consisting of objects

$$\begin{array}{c} F(k+1) \\ \downarrow\!\!\!\!\downarrow \\ \mathbf{B}, \end{array}$$

$\mathbf{B} \in T\text{-}\mathbf{Alg}$, such that there exists a non-invertible surjection

$$\begin{array}{c} \mathbf{B} \\ \downarrow\!\!\!\!\downarrow \\ Fk. \end{array}$$

The identity map $1_{F(k+1)}$ is an object of $\mathcal{P}$ (noting that there exists a non-invertible surjection $F(k+1) \twoheadrightarrow Fk$ by Lemma 6.2.5): $\mathcal{P}$ is nonempty.

Take some indexing category $\mathbf{I}$ representing a totally ordered set (i.e. $\mathbf{I}$ is small; has at most one morphism in each hom-set; and for each $i, j \in \mathbf{I}$, there exists a morphism $i \to j$ or there exists a morphism $j \to i$). Consider an arbitrary diagram

$$\left(\begin{array}{c} F(k+1) \\ \downarrow\!\!\!\!\downarrow \\ \mathbf{B}_i \end{array} \right)_{i \in \mathbf{I}}$$

in $\mathcal{P}$, and let $\mathbf{C}$ be the colimit *in* T-$\mathbf{Alg}$ of this diagram. Then $\mathbf{C}$ is the colimit of a filtered diagram of algebras and surjections.

Since T is totally free, $\mathbf{C}$ is isomorphic to FX for some set X. Assume towards contradiction that there exists some $m \in \mathbb{N}$, $m = |X|$, $m \leq k$. By Lemma 6.2.2, there exists some $j \in \mathbf{I}$ such that $\mathbf{B}_j$ is generated by an m-element set also. By definition of the $\mathbf{B}_i$, we have a diagram

$$F(k+1) \xrightarrow{g} \mathbf{B}_j \xrightarrow{h} Fk,$$

where h is non-invertible: by (ii), $m \geq k$, so $m = k$. So by (iii), hg is an isomorphism. So $U(hg) = (Uh)(Ug)$ is an isomorphism in $\mathbf{Set}$ i.e. a bijection, so Ug is an injection. But Ug is a surjection also, so Ug is a bijection, so Uh is a bijection i.e. an isomorphism. Since U reflects isomorphisms, h is an isomorphism in T-$\mathbf{Alg}$. Contradiction.

So $X > k$. Thus by Lemma 6.2.5, there exists a non-invertible surjection $\mathbf{C} \twoheadrightarrow Fk$, and thus the composite $F(k+1) \twoheadrightarrow \mathbf{B}_i \to \mathbf{C}$ (for any $i \in \mathbf{I}$) is an object of $\mathcal{P}$. So we can apply Lemma 6.2.4. This gives an object of $F(k+1) \twoheadrightarrow \mathbf{D}$ of $\mathcal{P}$ such that for all diagrams

$$\mathbf{D} \xrightarrow{g} \mathbf{E} \xrightarrow{h} Fk, \qquad (*)$$

$\mathbf{E} \in T\text{-}\mathbf{Alg}$, either g or h is an isomorphism.

Since T is totally free, there exists a set Y such that $\mathbf{D} \cong FY$. We cannot have $Y > k+1$, since then there would exist a diagram

$$\mathbf{D} \twoheadrightarrow F(k+1) \twoheadrightarrow Fk,$$

of non-invertible surjections, contradicting $(*)$. So $Y \leq k+1$. Since $F(k+1) \twoheadrightarrow \mathbf{D}$ is an object of $\mathcal{P}$, there exists a non-invertible surjection $\mathbf{D} \twoheadrightarrow Fk$. So we cannot have $Y \cong k$, since in that case by (iii) all surjections $\mathbf{D} \cong Fk \twoheadrightarrow Fk$ are invertible. So $Y \cong k+1$. So for any diagram

$$F(k+1) \xrightarrow{g} \mathbf{E} \xrightarrow{h} Fk, \qquad (\dagger)$$

of surjections, at least one of g or h is an isomorphism. We also have a non-invertible surjection $F(k+1) \twoheadrightarrow Fk$.

Now I will show that $F(k+1)$ has height $k+1$ when T is pointed. In this case, we have a diagram

$$F(k+1) \twoheadrightarrow Fk \twoheadrightarrow \cdots \twoheadrightarrow F0 \cong \mathbf{1}$$

of non-invertible surjections, with arrows given by Lemma 6.2.5. This shows that $\mathrm{ht}(F(k+1)) \geq k+1$. Consider also an arbitrary diagram

$$F(k+1) \twoheadrightarrow \mathbf{A}_m \twoheadrightarrow \mathbf{A}_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathbf{A}_1 \twoheadrightarrow \mathbf{A}_0 \cong \mathbf{1}$$

of non-invertible surjections in T-$\mathbf{Alg}$, for some $m \in \mathbb{N}$. Write $\mathbf{A}_m \cong FZ$ for some set Z. Assume towards contradiction that $Z \geq k+1$. We then have a surjection $\mathbf{A}_m \cong FZ \twoheadrightarrow Fk$ by Lemma 6.2.5, and thus have a diagram

$$F(k+1) \twoheadrightarrow \mathbf{A}_m \twoheadrightarrow Fk.$$

By $(\dagger)$, noting that the map $F(k+1) \twoheadrightarrow \mathbf{A}_m$ is by definition non-invertible, we have that the map $\mathbf{A}_m \twoheadrightarrow Fk$ is invertible. We then have an inverse $Fk \to \mathbf{A}_m$, such that the composite $\mathbf{A}_m \twoheadrightarrow Fk \to \mathbf{A}_m$ is the identity $1_{\mathbf{A}_m}$. Applying U, we have that $U\mathbf{A}_m \twoheadrightarrow UFk \to U\mathbf{A}_m$ is the identity $1_{U\mathbf{A}_m}$ in $\mathbf{Set}$. So $UFk \to U\mathbf{A}_m$ is a surjection, so $Fk \to \mathbf{A}_m \cong FZ$ is a surjection. By (ii), we thus have $Z \cong k$: contradiction.

So $Z \leq k$. So by (i), $\mathbf{A}_m$ has height $|Z|$. So, since $\mathrm{ht}(F(k+1)) \leq \mathrm{ht}\,\mathbf{A} + 1$, we have $\mathrm{ht}(F(k+1)) \leq k+1$. So, since $\mathrm{ht}(F(k+1)) \geq k+1$, we have that $F(k+1)$ has height precisely $k+1$.

In the case where T is affine, the above argument works in exactly the same way other than that we start with a diagram

$$F(k+1) \twoheadrightarrow Fk \twoheadrightarrow \cdots \twoheadrightarrow F1 \cong \mathbf{1}.$$

The argument thus shows that, in this case, $F(k+1)$ has height k. Note that at all points when invoking Lemma 6.2.5 in the affine case, the set passed to the Y parameter of that lemma is nonempty.

Now I will show that in the case where T is pointed, if X is a set such that the T-algebra FX has height $k+1$, then $X \cong k+1$. If $X > k+1$, then, noting that $k+1 \ncong 0$, by Lemma 6.2.5 there exists a non-invertible surjection $FX \twoheadrightarrow F(k+1)$, and thus we have a diagram

$$FX \twoheadrightarrow F(k+1) \twoheadrightarrow Fk \twoheadrightarrow \cdots \twoheadrightarrow F0 \cong \mathbf{1}$$

of non-invertible surjections. So $\mathrm{ht}(FX) > k+1$: contradiction. So $X \leq k+1$. If $X \leq k$ then by (iii), $\mathrm{ht}(FX) = |X| \leq k$: contradiction. So $X \cong k+1$.

To finish proving (i), I will show that in the case where T is affine, if X is a set such that FX has height $k+1$, then $X \cong k+2$. If $k \cong 0$, then $FX \cong F1$. If $X > k+2$, then, noting that $k+1 \ncong 0$, by Lemma 6.2.5 there exists a non-invertible surjection $\mathbf{A} \twoheadrightarrow F(k+1)$, and thus we have a diagram

$$FX \twoheadrightarrow F(k+1) \twoheadrightarrow Fk \twoheadrightarrow \cdots \twoheadrightarrow F1 \cong \mathbf{1}$$

of non-invertible surjections. So $\mathrm{ht}(FX) > k+1$: contradiction. So $X \leq k+2$. If $X \leq k+1$ then by (iii), $\mathrm{ht}(FX) = |X| \leq k$: contradiction. So $X \cong k+2$. This completes the part of the induction step for (i).

Now I will prove that (ii) holds when $n = k+1$. Take a set X and a surjection $f : F(k+1) \twoheadrightarrow FX$. Then $\mathrm{ht}(F(k+1)) \geq \mathrm{ht}(FX)$, and by (i), $\mathrm{ht}(F(k+1)) = k+1$ when T is pointed and $\mathrm{ht}(F(k+1)) = k+2$ when T is affine. Either way, by (i) again, we have that $X \leq k+1$.

Finally, I will prove that (iii) holds for $n = k+1$. Consider a surjective T-algebra homomorphism $F(k+1) \twoheadrightarrow F(k+1)$, and assume towards contradiction that it is non-invertible. Note that, by (i), $F(k+1)$ has finite height. Thus $\mathrm{ht}(F(k+1)) > \mathrm{ht}(F(k+1))$: contradiction.

This completes the induction step, and we are now done with the proof of this lemma. □

Say that a T-algebra homomorphism f is *injective* when Uf is injective. It is true that any injective T-algebra homomorphism $Fn \to Fn$, where $n \in \mathbb{N}$, is an isomorphism. But Tom Leinster and I have been unable to prove this. Looking for a potential proof is an area for further research.

Lemma 6.2.10. *Let T be a nondegenerate algebraic theory. Let Y be a set, and let $i : Z \hookrightarrow Y$ be a subset. Suppose that the subset $Ti : TZ \hookrightarrow TY$ is in fact all of TY, i.e. Ti is an isomorphism. Then Z is all of Y, i.e. i is an isomorphism.*

We are implicitly using here the fact that T preserves monics (Lemma 2.2.1): Ti is a monic.

Proof. By definition, T being nondegenerate means that the components of η, the unit of T, are monic. As a general fact about adjunctions (see for example Lemma 4.5.13 of [Rie16]), this implies that F, the free functor of T, is faithful. Since U is faithful, $T = UF$ is faithful also. By Lemma 2.2.2, faithful functors reflect epics. Thus if Ti is epic, then i is also. All isomorphisms are epic, so Ti being an isomorphism means that it is epic, which means that i is epic, which, since i is also monic and the isomorphisms in **Set** are precisely the monic epics, means that i is an isomorphism. □

We now prove that all algebraic theories have dimension for their infinitely generated algebras.

Lemma 6.2.11. *Let T be a nondegenerate algebraic theory. Let X and Y be infinite sets. Then if $FX \cong FY$, then $X \cong Y$.*

Proof. Let $j : FX \to FY$ be an isomorphism. For each $x \in X$, I claim that there exists a finite subset $Y_x \subseteq Y$ such that $(Uj)(\eta_X(x)) \in TY_x \subseteq TY$. Indeed: Y is the colimit of its poset of finite subsets, which is filtered, and T, being a finitary monad, preserves filtered colimits. So TY is the colimit of the poset of subsets of TY of the form TZ for some $Z \subseteq Y$ a finite subset. So each element of TY is an element of some such TZ, in particular, $(Uj)(\eta_X(x))$ is. Alternatively, the existence of these Y_x can be seen syntactically: since T is finitary, each T-term can only contain finitely many free variables.

So $(Uj)(\eta_X(X)) \subseteq T(\bigcup_{x\in X} Y_x)$. But $(Uj)(\eta_X(X))$ is a generating subset of the underlying set of FY (considering, for example, $X \xrightarrow{\eta_X} TX \xrightarrow{Uj} TY$, and noting that η_X is an injection since T is nondegenerate). So in fact $T(\bigcup_{x\in X} Y_x) = TY$. Thus, by Lemma 6.2.10, $\bigcup_{x\in X} Y_x = Y$. We now do a cardinality calculation:

$$Y = \bigcup_{x\in X} Y_x \leq \sum_{x\in X} Y_x \leq \mathbb{N} \times X \leq X,$$

noting that each Y_x is finite, and X is infinite. So $Y \leq X$. The same argument the other way around shows that $X \leq Y$. So by the Cantor–Bernstein Theorem, $X \cong Y$. □

Finally, we have enough to prove that all totally free theories have dimension (Theorem 6.1.1). For convenience, I will recall the theorem again here.

Theorem 6.2.12. *Let T be a nondegenerate totally free algebraic theory. Then for all sets X and Y, $FX \cong FY$ implies $X \cong Y$.*

Proof. Let X and Y be sets, and let $j : FX \to FY$ be an isomorphism. If X and Y are both infinite, then $X \cong Y$ by Lemma 6.2.11.

Assume now that, without loss of generality, Y is finite. Since there is a surjective homomorphism $FY \to FX$ (namely j^{-1}), by point (ii) of Lemma 6.2.9, $X \leq Y$, and in particular X is finite. There is also a surjective homomorphism $FX \to FY$ (namely j), so the same result, noting that X is finite, shows that $Y \leq X$. So by the Cantor–Bernstein Theorem, $X \cong Y$. This completes all possible cases for the theorem. □

6.3 Further questions

As I stated in Example 6.1.2, there exist finitary algebraic theories which have dimension but are not totally free. Is it possible to classify all such theories? Are they all of the form $TX \cong X + E$ for $E \in \mathbf{Set}$?

Do there exist non-finitary algebraic theories that have dimension?

Appendix A

Terminology

Here is an overview of some of the different terminologies used by category theorists and classical universal algebraists. There are some words that universal algebraists use to mean one thing, whilst category theorists use to mean a different thing. In most of this thesis I use the category theoretic terminology.

What category theorists call *algebraic theories*, *finitary monads*, or *Lawvere theories (over* **Set***)*, universal algebraists call *varieties of algebras* or *equational theories*. (These are not the same as what algebraic geometers call *algebraic varieties*, although they have a not-entirely-dissimilar flavour.) Both category theorists and universal algebraists will tell you that within their field, their different teminologies actually refer to sligtly different ways of looking at things – e.g. a finitary monad is a particular endofunctor, and a Lawvere theory over **Set** is a monoidal functor from the category of finite sets with the categorical coproduct as the monoidal product to **Set** with the categorical product as the monoidal product). However, all of these are just different ways of looking at precisely the same objects.

What I in this thesis call *terms* of (an algebraic theory or) a variety of algebras are sometimes called *function symbols*, e.g. in Theorem 3.1 of [KKS18].

Universal algebraists sometimes require their varieties of algebras (or, more specifically, their signatures (each a list of arities of operations in a certain class of algebras of the same *type*)) to have no nullary operations. Category theorists don't do this; it's worth being aware that some results may appear to differ slighly when constants are or aren't being excluded.

What category theorists call an *affine theory*, universal algebraists call an *idempotent variety.* (Not to be confused with algebraic geometers' *affine varieties.*)

Universal algebraists call varieties of algebras *affine* when they are, up to a certain notion of sameness, a category of modules over some ring. There are multiple equivalent conditions for this, some of which involve specifying an abelian group (which will be the underlying abelian group of the modules). Note that this is not the same as the category theoretic meaning of affine theory – in fact, the only example of an affine variety in the universal algebraic sense that is an affine theory in the category theoretic sense is the theory of modules over the zero ring.

Universal algebraists call algebras *abelian* (or occasionally *TC-algebras*), or varieties *abelian*, when the following predicate holds in the algebra or variety respectively, called the *term condition*.

Definition A.1. An algebra satisfies the **term condition** when, for all n, for all $(n+1)$-ary operations θ of the algebra,

$$\theta(a, u_1, \ldots, u_n) = \theta(a, v_1, \ldots, v_n)$$

implies

$$\theta(b, u_1, \ldots, u_n) = \theta(b, v_1, \ldots, v_n).$$

I do not know of a word category theorists use for these theories.

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