Nilpotency indices of prime factor matrices

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Let the natural numbers $\mathbb{N}$ be $\{1, 2, 3, \dots\}$ here. The fundamental theorem of arithmetic tells us that each natural n can be written as a unique product of primes:

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \cdots .$$

Here, p_i is the ith prime, e.g. $p_1 = 2$, $p_4 = 7$. The e_i are then the exponents in the prime factor expansion of n. For example,

$$2025 = 2^0 \cdot 3^3 \cdot 5^2 \cdot 7^0 \cdot 13^0 \cdot \cdots ,$$

with $e_1 = 0$, $e_2 = 3$, $e_3 = 2$, etc. Only finitely many e_i are nonzero for any given n.

So each natural has an associated list of exponents. Further, since $p_n > n$ for all $n \in \mathbb{N}$, at most the first n exponents for the expansion of n can be nonzero. In practice this bound is quite weak, but you'll see why I chose it soon! We can assemble the first n prime-factor-expansion-exponent-lists into a matrix. That is, define F_n to be the matrix with

$$(F_n)_{ij} = \text{the exponent } e_j \text{ in the prime factor expansion of } i.$$

For example,

$$F_1 = \begin{pmatrix} 0 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad F_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As you can see, most of the entries of these matrices are 0. In fact, since $p_n > n$ always, all the F_n are strictly lower triangular: the only nonzero entries are below the diagonal.

A fact about $n \times n$ strictly lower triangular matrices is that they are *nilpotent of index at most n*. That is, such a matrix A has the property that $A^n = 0$ the zero matrix. The least $k \in \mathbb{N}$ such that $A^k = 0$ is called the *nilpotency index of A*.

Proposition. *The nilpotency index of F_n is the least k such that $\pi^k(n) = 0$, where $\pi(m) =$ the number of primes less than or equal to m, and where π^k just means π iterated k times.*

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A while ago I thought I found a proof for this (I think it did work, but I don't remember the details off the top of my head). The proof used the following function, which I think is rather interesting. Define $\nu : \mathbb{N} \to \mathbb{N}$ by

$$\begin{aligned}\nu(p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \dots) &= 1^{e_1} \cdot 2^{e_2} \cdot 3^{e_3} \cdot \dots \\ &= \pi(p_1)^{e_1} \cdot \pi(p_2)^{e_2} \cdot \pi(p_3)^{e_3} \cdot \dots .\end{aligned}$$

This behaves a bit like π, but notably it is multiplicative (which π isn't): $\nu(nm) = \nu(n)\nu(m)$.

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I am by no means a number theorist! This all might be obvious, well known, or totally uninteresting to number theorists. But I think it's interesting. :)

I haven't seen this result anywhere else. Joram Soch in [1] looks at very similar matrices. The ν function appears in the OEIS as [2], and the sequence of the nilpotency indices of the F_n is in the OEIS too, alternatively defined as the 'number of edges in the rooted tree with Matula-Goebel number n' [3].

References

[1] Soch, Joram (2017), *Linear Algebraic Number Theory, Part I: Foundations*, arXiv, `https://arxiv.org/abs/1709.05959`.

[2] OEIS Foundation Inc. (2025), Entry A064989 in The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A064989`, retrieved 2025-04-15.

[3] OEIS Foundation Inc. (2025), Entry A196050 in The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A196050`, retrieved 2025-04-15.