Lawvere theories

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Definitions

A Lawvere theory is one of many category theoretic ways to describe 'algebraic theories', such as the theories of groups, rings, and real vector spaces.

What is a group?

Definition. An (internal) group in **Set** is an object $G \in \mathbf{Set}$, and a diagram

$$1 \xrightarrow{\;e\;} \underset{\substack{\circlearrowleft \\ i}}{G} \xleftarrow{\;m\;} G^2, \qquad (\dagger)$$

such that some diagrams commute:

$$\begin{array}{ccc} G^3 & \xrightarrow{m\times 1_G} & G^2 \\ {\scriptstyle 1_G\times m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ G^2 & \xrightarrow[m]{} & G, \end{array} \qquad \begin{array}{ccccc} 1\times G & \xrightarrow{e\times 1_G} & G^2 & \xleftarrow{1_G\times e} & G\times 1 \\ & {\scriptstyle\sim}\searrow & \big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle\sim} & \\ & & G, & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(1_G,i)} & G^2 & \xleftarrow{(i,1_G)} & G \\ & {\scriptstyle 1_G}\searrow & \big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle 1_G} & \\ & & G, & & \end{array}$$

(associativity, identity, inverse; note that these conditions can indeed be written entirely as commutative diagrams).

We might want to use this to define a group as a functor from (†), thought of as an indexing category, to **Set**, such that the diagrams hold for the image. We'd need to add in all posible compositions of arrows in (†). But we can't ask for the diagrams to commute in (†), because e.g. we don't have G^3 or $(1_G, i)$. If we also add in all finite products of objects, and all finite products of maps and all product-induced maps (maps like $(1_G, i)$), then we *can* ask for the diagrams to commute in this extended version of (†)! And then, a group is just a finite product preserving functor from this category into **Set** – the diagrams hold by definition.

Lawvere theories generalise this sort of definition to cover all algebraic theories.

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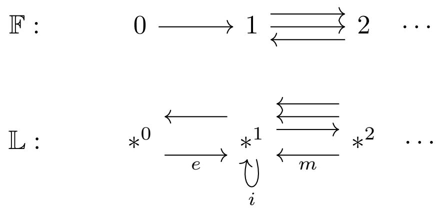
Notation. $\mathbb{F}$ is a skeleton of the category of finite sets. Note this is a finite coproduct category.

Definition. A **Lawvere theory** is a finite product category $\mathbb{L}$, along with a finite product preserving functor $I : \mathbb{F}^{\text{op}} \to \mathbb{L}$ that is an isomorphism on classes of objects.

Example (theory of groups)**.**



There are more maps in $\mathbb{L}$ than in the above: composites and finite products of existing maps, e.g.

$$*^2 \to *^1,$$
$$(x, y) \mapsto x^3 y^{-2} x.$$

The diagrams in the definition of group must hold. E.g. $m \circ (e \times 1_*) = 1_*$.

We can say that the theory of groups is the **free finite product category containing a group**. That is, we take the smallest (the universal) finite product category that contains a group object.

This might appear to be an evil definition, due to its use of a skeleton. But it's just for convenience: we could avoid using a skeleton if we wanted to.

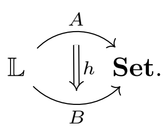
So how does this encode algebraic theories?

Definition. An **algebra** for a Lawvere theory $\mathbb{L}$ is a finite product preserving functor $\mathbb{L} \to$ **Set**.

Example (theory of groups)**.** This picks out an underlying set, then by product preservation, all other objects are specified. The maps inherited from $\mathbb{F}^{\text{op}}$ are forced, and the others give the group structure.

This definition is nice, because we immediately get a notion of homomorphisms, and in fact an entire category of algebras.

Definition. Let $\mathbb{L}$ be a Lawvere theory. A **homomorphism** between algebras $A, B : \mathbb{L} \to$ **Set** is a natural transformation



Example. For groups, this gives maps between the group objects in **Set**:

$$\begin{array}{ccccc} & & \overset{i}{\circlearrowleft} & & \\ 1 & \xrightarrow{e} & G(*) & \xleftarrow{m} & G(*)^2 \\ {\scriptstyle 1_1}\downarrow & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*^2} \\ 1 & \xrightarrow[e]{} & H(*) & \xleftarrow[m]{} & H(*)^2. \\ & & \underset{i}{\circlearrowright} & & \end{array}$$

Definition. Let $\mathbb{L}$ be a Lawvere theory. Then $\mathbf{FinProd}(\mathbb{L}, \mathbf{Set})$ is the **category of algebras** for $\mathbb{L}$.

This category comes with a forgetful functor down to **Set**. This is given by evaluation at the object $*$:

$$\begin{array}{c} \mathbf{FinProd}(\mathbb{L}, \mathbf{Set}) \\ \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathbf{Set}. \end{array}$$

This forgetful functor has a left adjoint, which gives free algebras. We'll return to this.

Lawvere theories are also nice because we can easily change the base category.

Example. Instead of working with groups in **Set**, we could work with groups in **Top** – topological groups – just by considering finite product preserving functors $\mathbb{L} \to \mathbf{Top}$.

We can do this for any finite product category as the base.

Features

Now to see how this all ties in with some more advanced stuff!

First, it is a fact that the forgetful functor from a category of algebras for a Lawvere theory (working over **Set**) is always monadic. Furthermore, it is always finitary. Furthermore, the category of Lawvere theories is equivalent to the category of finitary monads on **Set**.

Definition. A map of Lawvere theories is a diagram of the form

$$\begin{array}{ccc} & \overset{I}{\nearrow} & \mathbb{L} \\ \mathbb{F}^{\mathrm{op}} & & \downarrow{\scriptstyle \alpha} \\ & \underset{I'}{\searrow} & \mathbb{L}' \end{array}$$

where α is a finite product preserving functor. Again, a straightforward and reasonable definition.

The proof that ev_* is always monadic is interesting. It can be shown using a crude monadicity theorem, and the following two key facts:

- Limits and colimits in functor categories are computed pointwise.
- In **Set**, reflexive coequalisers commute with finite products. (A **reflexive pair** is a pair of parallel arrows with a common section, and a **reflexive coequaliser** is a coequaliser of a reflexive pair.)

Now, let's reframe the definition of Lawvere theory slightly.
Firstly, note that **Set** is equivalent to $[\mathbf{1}, \mathbf{Set}]$.
Now, note that **FinProd** lives in a 2-adjunction above **CAT**:

$$\begin{array}{c} \mathbf{FinProd} \\ F \Big\uparrow \dashv \Big\downarrow U \\ \mathbf{CAT}. \end{array}$$

Considering **Set** as a finite product category, and thus $U\mathbf{Set}$ as an ordinary category (where we have 'forgotten' that it has finite products), we thus have an equivalence of hom-categories

$$[\mathbf{1}, U\mathbf{Set}] \simeq \mathbf{FinProd}(F\mathbf{1}, \mathbf{Set}).$$

Important fact: $F\mathbf{1} \simeq \mathbb{F}^{\mathrm{op}}$! $\mathbb{F}^{\mathrm{op}}$ is the free finite product category on an object! (And $\mathbb{F}$ is the free finite coproduct category on an object!)

We then have that the forgetful functor is just composition with I:

$$\begin{array}{c} \mathbf{FinProd}(\mathbb{L}, \mathbf{Set}) \\ \Big\downarrow {\scriptstyle - \circ I =: I^*} \\ \mathbf{FinProd}(F\mathbf{1}, \mathbf{Set}) \\ \wr\wr \\ [\mathbf{1}, \mathbf{Set}]. \end{array}$$

Note that I is finite product preserving, so composing with it gives another finite product preserving functor.

What we are doing is composing as in the following diagram:

$$\begin{array}{ccc} \mathbb{F}^{\mathrm{op}} & \xrightarrow{A \circ I} & \mathbf{Set}. \\ {\scriptstyle I}\Big\downarrow & \nearrow_{A} & \\ \mathbb{L} & & \end{array}$$

But hang on... we can sort of undo this, using Kan extensions! Indeed, if we start with a finite product preserving functor $\mathbb{F}^{\mathrm{op}} \to \mathbf{Set}$, or equivalently, an ordinary functor $\mathbf{1} \to U\mathbf{Set}$, we can take the left Kan extension to obtain a finite product preserving functor $\mathbb{L} \to \mathbf{Set}$.

$$\begin{array}{ccc} \mathbb{F}^{\mathrm{op}} & \xrightarrow{D} & \mathbf{Set}. \\ {\scriptstyle I}\Big\downarrow & \nearrow_{\mathrm{Lan}_I D =: I_! D} & \\ \mathbb{L} & & \end{array}$$

This gives our left adjoint to the forgetful functor! The fact that the left Kan extension is indeed finite product preserving is very non-trivial.

Generalisations

I have been trying to generalise the definition of Lawvere theory. It turns out, this is hard.

One way that we can generalise them (I did not discover this) is with *essentially algebraic theories*. These are like Lawvere theories, but we take finite limit categories and finite limit preserving functors, instead of just dealing with finite products.

This lets us define small categories as essentially algebraic structures! Indeed: the definition of internal category looks like a diagram

$$C_0 \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}} C_1 \longleftarrow C_1 \times_{C_0} C_1,$$

subject to some diagrams holding.

We end up with the following diagram:

$$\begin{array}{ccc} \mathbf{FinLim}(\mathbb{L}, \mathbf{Set}) & \simeq & \mathbf{Cat} \\ \big\downarrow{\scriptstyle I^*} & & \\ [(\bullet \leftleftarrows \bullet), \mathbf{Set}] & \simeq & \mathbf{Graphs}. \end{array}$$

That is, the category of small categories lives over the category of (multi di-) graphs (and is in fact monadic over it).

Coda

Lawvere theories are named after F. William 'Bill' Lawvere, who was the first to write about them, in his PhD thesis. Lawvere was involved in political activism throughout his life. In 1971, he lost his job at Dalhousie University after he protested against the Vietnam war, and against the related actions of the government of Canada, actions which curtailed civil liberties in Canada. I draw attention to how closely this parallels recent events in Scotland and the rest of the UK.

I've produced some accidentally quite detailed notes for this talk, available via my website, or if you let me know your email address I can send you a copy. (You're reading them now!) These notes are in the public domain (or under the Unlicense or Creative Commons Zero version 1.0 if desired).