A GEOMETRIC APPROACH TO BISTABLE FRONT PROPAGATION IN SCALAR REACTION-DIFFUSION EQUATIONS WITH CUT-OFF

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ABSTRACT. 'Cut-offs' were introduced to model front propagation in reaction-diffusion systems in which the reaction is effectively deactivated at points where the concentration lies below some threshold. In this article, we investigate the effects of a cut-off on fronts propagating into metastable states in a class of bistable scalar equations. We apply the method of geometric desingularization from dynamical systems theory to calculate explicitly the change in front propagation speed that is induced by the cut-off. We prove that the asymptotics of this correction scales with fractional powers of the cut-off parameter, and we identify the source of these exponents, thus explaining the structure of the resulting expansion. In particular, we show geometrically that the speed of bistable fronts increases in the presence of a cut-off, in agreement with results obtained previously via a variational principle. We first discuss the classical Nagumo equation as a prototypical example of bistable front propagation. Then, we present corresponding results for the (equivalent) cut-off Schlögl equation. Finally, we extend our analysis to a general family of reaction-diffusion equations that support bistable fronts, and we show that knowledge of an explicit front solution to the associated problem without cut-off is necessary for the correction induced by the cut-off to be computable in closed form.

1. INTRODUCTION

Front propagation in reaction-diffusion systems constitutes a fundamental topic in non-equilibrium physics. Central questions concern the propagation speed that is selected by traveling fronts, as well as the factors that influence this selection process. The subject is vast and complex, as one has to distinguish between bistable fronts propagating into metastable states versus fronts that propagate into unstable states, which may be of either 'pulled' or 'pushed' type. For a comprehensive review of these and related issues, the reader is referred to [17].

The characteristics of propagating fronts in such systems are altered substantially when 'cut-off' functions are placed on the reaction kinetics. These cut-offs, which decrease the reaction amplitude at all points in the domain at which the concentration lies below a certain threshold, were introduced by Brunet and Derrida in the pioneering study [5] to model fluctuations that arise in the large-scale limit of discrete N-particle systems, among other phenomena: with the threshold set to $\varepsilon = N^{-1}$, the reaction terms are cut-off (and oftentimes set to zero) at points where the concentration is below ε, as such concentrations are not attainable when the particles are assumed to be indivisible.

In particular, in [5], Brunet and Derrida investigated the effects of a cut-off on the dynamics of pulled fronts in the classical Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation,

$$(1.1) \qquad \phi_t = \phi_{xx} + \phi(1 - \phi^2)H(\phi - \varepsilon),$$

by introducing a Heaviside cut-off, which is defined by

$$(1.2) \qquad H(\phi - \varepsilon) \equiv 0 \quad \text{if} \quad \phi < \varepsilon \quad \text{and} \quad H(\phi - \varepsilon) \equiv 1 \quad \text{if} \quad \phi > \varepsilon,$$

at the zero rest state in (1.1). One of their principal findings was that the selected front speed in the cut-off Equation (1.1) is given by

$$\text{(1.3)} \qquad c_{\text{FKPP}}(\varepsilon) \sim 2 - \frac{\pi^2}{(\ln \varepsilon)^2} \quad \text{as} \quad \varepsilon \to 0^+,$$

to leading order in ε, which represents a substantial reduction compared to the classical propagation speed $c_{\text{FKPP}}(0) = 2$ in the corresponding equation without cut-off, even when ε is small. Moreover, it was observed numerically in [5] that the approximation provided by (1.3) is in good agreement with the front propagation speed found in discrete N-particle systems for N large, and it was conjectured that (1.3) is valid for a wide variety of cut-offs in (1.1).

In [8], we proved the existence of traveling front solutions that propagate between the rest states at 1 and 0 in Equation (1.1), and we established the validity of the conjecture of Brunet and Derrida, including a generalization of it. In particular, we considered the general class of cut-off functions Θ that satisfy

$$\text{(1.4)} \qquad \Theta(\phi, \varepsilon, \tfrac{\phi}{\varepsilon}) < 1 \quad \text{if} \quad \phi < \varepsilon \quad \text{and} \quad \Theta(\phi, \varepsilon, \tfrac{\phi}{\varepsilon}) \equiv 1 \quad \text{if} \quad \phi > \varepsilon,$$

where, moreover, Θ is bounded at $\phi = \varepsilon$ and $0 < \varepsilon \ll 1$ denotes the cut-off parameter, as before. (Examples include the Heaviside step function H defined in (1.2) as well as the linear cut-off, with $\Theta(\phi, \varepsilon, \tfrac{\phi}{\varepsilon}) = \tfrac{\phi}{\varepsilon}$ for $\phi < \varepsilon$; see [8] for details.) We gave a rigorous derivation of the leading-order ε-asymptotics of c_{FKPP} in (1.3), and we showed that the coefficient π^2 in that expansion is universal within the class of cut-off functions that satisfy (1.4). The asymptotics of c_{FKPP}, as given in (1.3), was subsequently also confirmed in [3], via a variational approach.

The present article builds on the results obtained in [8], in that we show how the geometric approach developed there, in the context of the FKPP equation with cut-off in (1.1), can be generalized to study front propagation in the broad class of cut-off reaction-diffusion equations that is given by

$$\text{(1.5)} \qquad \phi_t = \phi_{xx} + f(\phi)\Theta(\phi, \varepsilon, \tfrac{\phi}{\varepsilon}).$$

Here, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $\phi(t, x) \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ denotes a smooth reaction function which vanishes at the three rest states at ϕ^+, ϕ°, and ϕ^- in (1.5). Moreover, we assume that the corresponding equation without cut-off supports bistable front solutions that propagate from the stable rest state at ϕ^- into the metastable rest state at ϕ^+, with propagation speed c_0; these assumptions will be made precise in Section 4. (As shown *e.g.* in [4, 11], the front speed c_0 is unique in the bistable case, whereas the FKPP equation supports traveling front solutions for a continuum of speeds c in the absence of a cut-off, with $c \geq c_{\text{FKPP}}(0)$ and $c_{\text{FKPP}}(0)$ the 'critical' front speed; cf. [8] for a detailed discussion.) Finally, the cut-off Θ is as defined in (1.4); for clarity of exposition, we will only discuss the case where $\Theta = H$ (the Heaviside cut-off) in detail here. Other choices of Θ can be treated in a similar fashion; see [8] and Remark 12 below.

The propagation of traveling fronts in (1.5) is naturally studied in the framework of the associated traveling front equation

$$\text{(1.6)} \qquad u'' + cu' + f(u)H(u - \varepsilon) = 0,$$

where the prime denotes differentiation with respect to the traveling wave variable $\xi = x - ct$ and $u(\xi) = \phi(t, x)$ is the corresponding front solution; moreover, we have now set $\Theta = H$ in (1.6). In addition to proving the existence of traveling front solutions, we will calculate explicitly the ε-dependent correction $\Delta c(\varepsilon)$ to the front propagation speed c_0 that is induced by the cut-off in (1.6). In particular, we will prove that this correction is positive, *i.e.*, that the propagation speed of bistable fronts increases in the presence of a cut-off, which is in agreement with results reported previously in [4, 13]. Moreover, we will show that Δc scales with fractional powers of the cut-off parameter ε, and we will provide explicit expressions for these exponents, as well as – in certain cases – for the respective leading-order coefficients in the expansion for $\Delta c(\varepsilon)$. Finally, we emphasize that the numerical values of these coefficients will, in general, depend on the choice of cut-off Θ in (1.5), in contrast to the situation encountered in the study of

Equation (1.1) in [5, 8]; however, the corresponding powers of ε will be universal within the family of cut-offs defined in (1.4).

Our analysis of (1.6) relies heavily on the blow-up technique from dynamical systems theory, a method also known as geometric desingularization. To the best of our knowledge, this technique was first used in the study of limit cycles near a cuspidal loop in [10]. It has since been successfully applied, including in [9], as an extension of the more classical geometric singular perturbation theory to situations in which normal hyperbolicity is lost; a list of additional references can be found in [8].

Rather than applying geometric desingularization directly to (1.6), we rewrite that equation as the equivalent first-order system

(1.7a) $$u' = v,$$

(1.7b) $$v' = -cv - f(u)H(u - \varepsilon),$$

(1.7c) $$\varepsilon' = 0,$$

where we have appended the trivial ε-dynamics. In the context of (1.7), traveling front solutions of (1.5) that connect the rest states at ϕ^- and ϕ^+ correspond to heteroclinic connections between the associated equilibrium points of (1.7), which are found at $(\phi^\mp, 0, \varepsilon)$. We will denote these points by Q_ε^- and Q_ε^+, respectively, where $\varepsilon \in [0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small. (The third equilibrium point, with $u = \phi^\circ$, is of no interest to us here.) Without loss of generality, we will assume $\phi^+ = 0$ in the following.

Now, the equilibrium point $Q_\varepsilon^+ = (0, 0, \varepsilon)$ corresponding to $\phi^+ = 0$ in (1.6) is degenerate (non-hyperbolic), with a double zero eigenvalue, which is due to the presence of the cut-off H. This degeneracy can be removed by desingularizing ('blowing up') the origin to an invariant two-dimensional manifold. As will become clear in the following, the blow-up regularizes the dynamics in a neighborhood of the degenerate equilibrium at Q_ε^+, which can then be studied using standard techniques from dynamical systems theory.

In the context of (1.7), the required blow-up transformation takes the form

(1.8) $$u = \bar{r}\bar{u}, \quad v = \bar{r}\bar{v}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon};$$

see also [8]. Here, $(\bar{u}, \bar{v}, \bar{\varepsilon}) \in \mathbb{S}^2 = \left\{ (\bar{u}, \bar{v}, \bar{\varepsilon}) \,\middle|\, \bar{u}^2 + \bar{v}^2 + \bar{\varepsilon}^2 = 1 \right\}$, with $\bar{r} \in [0, r_0]$ for $r_0 > 0$ sufficiently small; in other words, the transformation in (1.8) maps the origin to the two-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. (In fact, it suffices to consider the blown-up dynamics on the quarter-sphere $\mathbb{S}_+^2$ which is defined by restricting $\mathbb{S}^2$ to $\bar{\varepsilon} \geq 0$ and $\bar{u} \geq 0$.)

To study the dynamics on (and near) $\mathbb{S}_+^2$ in the blown-up phase space that is induced by the flow of (1.7), we introduce (local) coordinate charts: we will define a phase-directional chart K_1, corresponding to $\bar{u} = 1$ in (1.8), and a rescaling chart K_2, with $\bar{\varepsilon} = 1$. In particular, we observe that $\mathbb{S}_+^2$ will be invariant under the induced dynamics in each of these charts, which will result in a regularization of the singular limit as $\varepsilon \to 0^+$ in (1.7).

Remark 1. Given any object $\square$ in the original (u, v, ε)-variables, we will denote the corresponding blown-up object by $\overline{\square}$. In charts K_i $(i = 1, 2)$, that object will be denoted by $\square_i$, as required. $\qquad\square$

For future reference, we note that the change of coordinates κ_{21} between charts K_2 and K_1 on their domain of overlap is given by

(1.9) $$r_1 = r_2 u_2, \quad v_1 = v_2 u_2^{-1}, \quad \text{and} \quad \varepsilon_1 = u_2^{-1};$$

similarly, the inverse change $\kappa_{12} : K_1 \to K_2$ satisfies

(1.10) $$u_2 = \varepsilon_1^{-1}, \quad v_2 = v_1 \varepsilon_1^{-1}, \quad \text{and} \quad r_2 = r_1 \varepsilon_1.$$

Remark 2. While the blow-up transformation defined in (1.8) is homogeneous in $\bar{r}$, we remark that one may, more generally, make a quasi-homogeneous Ansatz of the form $u = \bar{r}^\alpha \bar{u}$, $v = \bar{r}^\beta \bar{v}$, and $\varepsilon = \bar{r}^\gamma \bar{\varepsilon}$, where α, β, and γ are positive integers; see *e.g.* [7]. These integers are then determined by the requirement that the leading-order terms in the resulting equations for $\bar{u}$, $\bar{v}$,

and $\bar{\varepsilon}$ scale with the same power of $\bar{r}$. In the context of (1.7), we thus recover (1.8), since $\alpha = \beta$ by (1.7a), as well as $\alpha = \gamma$, due to the linear dependence in the argument of H on u and ε. $\quad\square$

Finally, we outline our strategy for proving the existence of traveling fronts in (1.6), using geometric desingularization. Substituting the blow-up transformation in (1.8) into (1.7), one finds that the phase space of the resulting blown-up equations is naturally decomposed into three regions: an outer region, where $u = O(1)$, an inner region defined by $u < \varepsilon$, and an intermediate region, where $\varepsilon < u < O(1)$. The corresponding analysis will be carried out in two steps. First, we will construct a singular heteroclinic connection Γ in the singular limit as $\varepsilon \to 0^+$ in (1.7). Then, we will demonstrate that there exists a unique value $c(\varepsilon)$ of c so that Γ persists, for $\varepsilon \in (0, \varepsilon_0]$ sufficiently small. We remark that Γ and the persistent heteroclinic will lie on and near $\mathbb{S}_2^+$, respectively, and that they will traverse all three regions (outer, intermediate and inner) in connecting Q_ε^- to Q_ε^+. Our persistence proof will be constructive, in that we will track the unstable manifold $\mathcal{W}^u(Q_\varepsilon^-)$ and the stable manifold $\mathcal{W}^s(Q_\varepsilon^+)$ through the outer and inner regions, respectively. Then, we will show that these two manifolds coincide in the intermediate region, for $c = c(\varepsilon)$, to form the desired persistent heteroclinic connection in (1.7). The required 'matching' procedure will also directly yield the leading-order ε-asymptotics of $c(\varepsilon)$, completing our argument. In particular, it will follow that the fractional powers of ε arising in that asymptotics are given by the ratio of two of the eigenvalues of the linearized blown-up dynamics at an equilibrium point on the equator of $\mathbb{S}_+^2$, *i.e.*, in chart K_1.

This article is organized as follows. In Section 2, we discuss the propagation of bistable fronts in the presence of a cut-off in the Nagumo equation, with $f(\phi) = \phi(1 - \phi)(\phi - \gamma)$ for $\gamma \in (0, \frac{1}{2})$ in (1.5), which represents a prototypical example of a reaction-diffusion system that supports bistable front propagation into a metastable state. Then, in Section 3, we apply the results obtained in the previous section to the Schlögl equation with cut-off, which is equivalent to the Nagumo equation under a coordinate transformation, to calculate the asymptotics of the corresponding front speed $c(\varepsilon)$ to leading order. Finally, in Section 4, we generalize the results of Sections 2 and 3: we study bistable fronts propagating into metastable states in the general family of cut-off reaction-diffusion equations in (1.5). In particular, we show that knowledge of an exact solution to the corresponding problem without cut-off is necessary for the correction induced by the cut-off to be computable in closed form.

2. The cut-off Nagumo equation

In this section, we study bistable front propagation into a metastable state in the cut-off Nagumo equation

$$(2.1) \qquad \phi_t = \phi_{xx} + \phi(1 - \phi)(\phi - \gamma)H(\phi - \varepsilon).$$

Here, $0 < \gamma < \frac{1}{2}$ is a fixed parameter and H denotes the Heaviside cut-off, as before.

The following theorem is the main result of this section:

Theorem 2.1. *Let $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, and let $\gamma \in (0, \frac{1}{2})$. Then, there exists a unique value $c(\varepsilon)$ of c (dependent on γ) such that Equation (2.1) possesses a unique traveling front solution propagating between $\phi^- = 1$ and $\phi^+ = 0$. Moreover, $c(\varepsilon) = c(0) + \Delta c(\varepsilon)$, where $c(0) = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ (the propagation speed in the absence of a cut-off) and Δc is a positive, $\mathcal{C}^1$-smooth function in ε (including at $\varepsilon = 0$) and γ that satisfies*

$$(2.2) \qquad \Delta c(\varepsilon) = K_\gamma \varepsilon^{1+2\gamma} + o(\varepsilon^{1+2\gamma}),$$

with

$$(2.3) \qquad K_\gamma = \frac{\Gamma(4)}{\Gamma(1 + 2\gamma)\Gamma(3 - 2\gamma)} \frac{\sqrt{2}\gamma}{(1 + 2\gamma)^{2\gamma}}$$

a positive constant.

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(Here and in the following, $\Gamma(\cdot)$ denotes the standard Gamma function [1, Section 6.1]; moreover, the dependence of $c(\varepsilon)$ on the parameter $\gamma \in (0, \frac{1}{2})$ is suppressed for convenience of notation.)

Remark 3. In fact, the function $\Delta c(\varepsilon)$ will be obtained as the solution of a relation of the form $K_\gamma \varepsilon^{1+2\gamma} = \Delta c[1 + \theta(\varepsilon, \Delta c, \Delta c \ln(\Delta c), \gamma)]$; cf. Equation (2.30) below. Here, θ is $\mathcal{C}^\infty$-smooth in Δc, $\Delta c \ln(\Delta c)$, ε, and γ, including at $(0, 0, 0, \gamma)$, with $\theta(0, 0, 0, \gamma) = 0$. In particular, the logarithmic Δc-dependence translates into $\mathcal{C}^1$-smoothness when Δc is considered as a function of ε and γ alone; see the proof of Proposition 2.2 below. $\qquad\square$

The proof of Theorem 2.1 will follow the general procedure outlined in the previous section. The traveling front equation corresponding to (2.1) may be expressed as the equivalent first-order system

(2.4a) $$u' = v,$$

(2.4b) $$v' = -cv - u(1 - u)(u - \gamma)H(u - \varepsilon),$$

(2.4c) $$\varepsilon' = 0;$$

cf. (1.7). The points $Q_\varepsilon^- = (1, 0, \varepsilon)$ and $Q_\varepsilon^+ = (0, 0, \varepsilon)$ are hyperbolic saddle equilibria, in (u, v), of the system of equations

(2.5a) $$u' = v,$$

(2.5b) $$v' = -cv - u(1 - u)(u - \gamma)$$

that is obtained from (2.4) in the absence of a cut-off. (The eigenvalues of the corresponding linearization are given by $\lambda_\pm^- = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 + 4(1 - \gamma)}$ and $\lambda_\pm^+ = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 + 4\gamma}$, respectively.) In a first step, we desingularize the origin in (2.4) by applying the blow-up transformation defined in (1.8). Then, we construct a singular heteroclinic connection Γ between Q_0^- and Q_0^+ in (2.4); the construction is performed in the blown-up vector field that is induced by (2.4) on $\mathbb{S}_+^2$. The phase space of (2.4) naturally decomposes into three regions, an outer region, an inner region, and an intermediate region that represents the transition between the former two.

Finally, we prove that the singular heteroclinic orbit Γ will persist as a heteroclinic connection between Q_ε^- and Q_ε^+ for a unique value $c(\varepsilon)$ of c in (2.4) and each $\varepsilon > 0$ sufficiently small. That connection will correspond precisely to the sought-after front solution of (2.1) propagating with speed $c(\varepsilon)$. The corresponding persistence proof will also yield the leading-order ε-asymptotics of $c(\varepsilon)$, thus showing (2.2), as claimed.

Remark 4. The expansion for $\Delta c(\varepsilon)$ in (2.2) agrees with results obtained previously, via a variational principle, in Section V.A of [13] and in [4]. In particular, [4, Equation (9)], which implies $\Delta c \sim -K f'(0)\varepsilon^{1+\lambda}$, is equivalent to (2.2), with $f'(0) = -\gamma$ and $\lambda = 2\gamma$. However, the numerical value of K_γ, as stated in (2.3), differs from that reported for K in [4] by a multiplicative factor of $(1 + 2\gamma)^{-2\gamma}$. While the reason for this discrepancy warrants further investigation, we are confident that (2.3) is correct. Our analysis is also supported by numerical simulations of the first-order system in (2.4): evaluating (numerically) the distance between $\mathcal{W}^u(Q_\varepsilon^-)$ and $\mathcal{W}^s(Q_\varepsilon^+)$ in the hyperplane $\{u = \varepsilon\}$ for a range of values of K and ε, we found that the minimum of that distance is attained for $K = K_\gamma$, as expected (data not shown). $\qquad\square$

2.1. **Construction of Γ.** In this section, we perform the construction of the singular heteroclinic connection Γ between Q_0^- and Q_0^+, as outlined above.

2.1.1. *'Outer' region.* In the outer region, where $u = O(1)$, the system in (2.4) reduces precisely to (2.5), as $H \equiv 1$ there. The corresponding solution of the equivalent traveling front equation $u'' + cu' + u(1 - u)(u - \gamma) = 0$ (without cut-off) that connects the rest states at 1 and 0 is known explicitly in this case:

(2.6) $$u(\xi) = \frac{1}{1 + e^{\frac{1}{\sqrt{2}}(\xi - \xi^-)}},$$

with arbitrary phase ξ^-. (We note that, for $\xi \geq \xi^-$ large, $u \sim u(\xi^-)e^{-\frac{1}{\sqrt{2}}(\xi-\xi^-)}$, in agreement with classical 'mode counting' arguments which require that (2.6) can have no linearly growing modes [11].)

The propagation speed of the front solution defined in (2.6) is given by $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ [2]. The associated orbit in the context of the first-order system in (2.5) can then be written as

$$(2.7) \qquad v(u, c_0) = \frac{1}{\sqrt{2}}u(u-1),$$

as can be seen directly from (2.6). In the framework of (2.5), that orbit is precisely the unstable manifold $\mathcal{W}^u(Q_\varepsilon^-)$ of the point Q_ε^-, for $\varepsilon = 0$, since (2.6) implies that $(u, u') \to (1, 0)$ as $\xi \to -\infty$.

We now write $c = c_0 + (c - c_0) = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma + \Delta c$, with $\Delta c = o(1)$. Then, noting that the manifold $\mathcal{W}^u(Q_\varepsilon^-)$ is analytic in the state variables u and v (at least as long as $u \geq \varepsilon$), as well as in the parameter c, we may assume an expansion of the form

$$(2.8) \qquad v(u, c) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j v}{\partial c^j}(u, c_0)(\Delta c)^j$$

for $\mathcal{W}^u(Q_\varepsilon^-)$.

Remark 5. While the expansion in (2.8) depends explicitly on u and Δc, it is only implicitly ε-dependent: the structure of (2.5) implies that any ε-dependence in v can only enter through c. Correspondingly, the unstable manifold $\mathcal{W}^u(\ell^-)$ of the line $\ell^- = \{(1, 0, \varepsilon) \,|\, \varepsilon \in [0, \varepsilon_0]\}$, which is a foliation in ε with fibers $\mathcal{W}^u(Q_\varepsilon^-)$, only depends on ε in a trivial fashion. $\square$

As will become clear in the following, only the first two terms in (2.8) play a role to the order considered here: the leading-order term $v(u, c_0)$ is again given by (2.7), while the next-order term in Δc can be found from the variational equation associated to (2.5), taken along $v(u, c_0)$. That equation is obtained as follows: we first rewrite (2.5) with u as the independent variable; then, we differentiate the resulting equation with respect to c and substitute in c_0 and $v(u, c_0)$, cf. (2.7), which gives

$$(2.9) \qquad \frac{\partial}{\partial u}\left(\frac{\partial v}{\partial c}(u, c_0)\right) = -1 + 2\frac{u - \gamma}{u(1 - u)}\frac{\partial v}{\partial c}(u, c_0).$$

Equation (2.9) can be solved in closed form:

Lemma 2.1. *For $u \in (0, 1]$, the unique solution $\frac{\partial v}{\partial c}(u, c_0)$ to (2.9) that satisfies $\frac{\partial v}{\partial c}(1, c_0) = 0$ is given by*

$$(2.10) \qquad \frac{\partial v}{\partial c}(u, c_0) = \frac{1}{3 - 2\gamma}u^{-2\gamma}(1 - u)F(3 - 2\gamma, -2\gamma; 4 - 2\gamma; 1 - u),$$

where $F(\cdot, \cdot; \cdot; \cdot)$ denotes the hypergeometric function [1, Section 15]. In particular, $\frac{\partial v}{\partial c}(u, c_0)$ is strictly positive for any $u \in (0, 1)$.

The proof of Lemma 2.1 can be found in Appendix A.

Remark 6. The hypergeometric function F also occurs in the bounds for the front propagation speed in the cut-off Nagumo equation that were obtained by Méndez, *et al.*, via a 'generalized variational approach;' see [13, Section V]. However, the precise relationship between their analysis and ours remains to be clarified. $\square$

Remark 7. The result of Lemma 2.1 implies that the solution $\frac{\partial v}{\partial c}(u, c_0)$ of the variational equation in (2.9) has a branch point at $u = 0$ and, hence, that it becomes unbounded at that point, for any $\gamma \in (0, \frac{1}{2})$. In the limit as $\gamma \to 0^+$ in (2.9), the singularity disappears, *i.e.*, it follows from (2.10) that the solution of the corresponding equation remains bounded at 0. $\square$
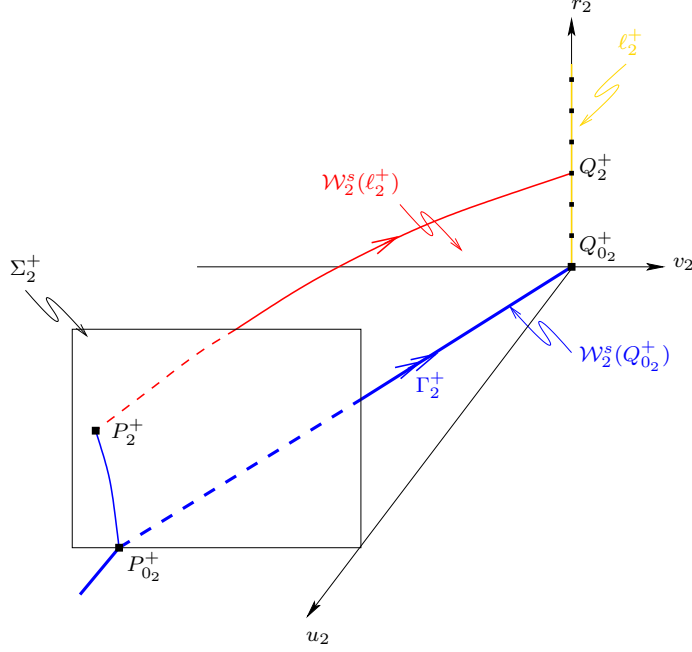
FIGURE 1. The geometry in chart K_2.

Finally, we introduce the following notation: for ρ positive and small, with $\rho \geq \varepsilon_0$, we denote the hyperplane $\{u = \rho\}$ in (u, v, ε)-space by Σ^-, and we write P_0^- for the point of intersection of $\mathcal{W}^u(Q_0^-)$ with Σ^-. (Here and in the following, we will suppress the ρ-dependence of Σ^- and P_0^-, for the sake of brevity.) We remark that Σ^- defines a section for the flow of (2.4), and that the segment of $\mathcal{W}^u(Q_0^-)$ located between Q_0^- and P_0^-, which we label Γ^-, gives precisely the portion of the singular heteroclinic connection Γ that lies in this outer region, i.e., in $\{u \geq \rho\}$.

2.1.2. 'Inner' region. In the inner region, the dynamics of (2.4) is governed by the corresponding cut-off equations, since $H \equiv 0$ for $u < \varepsilon$. We study these equations in the rescaling chart K_2, where the blow-up transformation in (1.8) is given by

(2.11) $$u = r_2 u_2, \quad v = r_2 v_2, \quad \text{and} \quad \varepsilon = r_2.$$

Substituting (2.11) into (2.4), we find the equivalent system

(2.12a) $$u_2' = v_2,$$

(2.12b) $$v_2' = -c v_2,$$

(2.12c) $$r_2' = 0$$

in (u_2, v_2, r_2)-space. We note that, for $r_2 (= \varepsilon)$ fixed, all points on the u_2-axis are equilibria of (2.12). However, since only points on the line $\ell_2^+ = \{(0, 0, r_2) \,|\, r_2 \in [0, r_0]\}$ can correspond to Q_ε^+, for $\varepsilon > 0$, after blow-down (i.e., after transformation to the original (u, v, ε)-variables), we will only consider those points here, and we will collectively denote them by Q_2^+.

In the singular limit as $r_2 \to 0$, the front propagation speed $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ in (2.12) is known explicitly; see Section 2.1. The unique solution of the resulting singular equation

(2.13) $$\frac{dv_2}{du_2} = -c_0, \quad \text{with} \quad v_2(0) = 0,$$

is given by $v_2(u_2) = -c_0 u_2$; the orbit Γ_2^+ corresponding to that solution yields precisely the stable manifold $\mathcal{W}_2^s(Q_{0_2}^+)$ of $Q_{0_2}^+ = (0, 0, 0)$.

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Finally, we define the section Σ_2^+ for the flow of (2.12) by

$$(2.14) \qquad \Sigma_2^+ = \big\{ (1, v_2, r_2) \, \big| \, (v_2, r_2) \in [-v_0, 0] \times [0, \rho] \big\},$$

for $v_0 > c_0 > 0$ fixed, and we note that Σ_2^+ represents a natural boundary for the inner region: since $u = \varepsilon$ is equivalent to $u_2 = 1$, after blow-up and transformation to chart K_2, Σ_2^+ marks the transition between the regime where the dynamics of the first-order system in (2.4) is unaffected by the cut-off and the cut-off regime. The orbit Γ_2^+ intersects Σ_2^+ in the point $P_{0_2}^+ = (1, -c_0, 0)$, as $v_2(1) = -c_0$; therefore, Γ_2^+ gives the portion of the singular orbit Γ that lies in this inner region. The geometry in chart K_2 is illustrated in Figure 1.

2.1.3. *'Intermediate' region.* The intermediate region, where $\varepsilon < u < O(1)$, provides the connection between the outer and inner regions discussed in the previous two sections and is most conveniently studied in chart K_1. Here, the blow-up transformation in (1.8) is given by

$$(2.15) \qquad u = r_1, \quad v = r_1 v_1, \quad \text{and} \quad \varepsilon = r_1 \varepsilon_1.$$

Correspondingly, in the new $(r_1, v_1, \varepsilon_1)$-coordinates, (2.4) becomes

$$(2.16a) \qquad r_1' = r_1 v_1,$$

$$(2.16b) \qquad v_1' = -c v_1 - v_1^2 + \gamma - (1 + \gamma) r_1 + r_1^2,$$

$$(2.16c) \qquad \varepsilon_1' = -\varepsilon_1 v_1.$$

Since c reduces to $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ for $\varepsilon (= r_1 \varepsilon_1) = 0$, it follows that the two equilibria of (2.16) are located at $P_1^s = \big(0, -\frac{1}{\sqrt{2}}, 0\big)$ and $P_1^u = (0, \sqrt{2}\gamma, 0)$. These equilibria correspond to the stable eigendirection and the unstable eigendirection, respectively, of the linearization at Q_0^+ of the first-order system without cut-off in (2.5). (In other words, the blow-up transformation in (1.8) teases apart the asymptotics of solutions in a neighborhood of Q_ε^+ and, hence, desingularizes the cut-off dynamics of (2.4) down to $\varepsilon = 0$.) Both P_1^s and P_1^u are hyperbolic saddle equilibria for (2.16), with eigenvalues $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}(1 + 2\gamma)$, and $\frac{1}{\sqrt{2}}$, respectively, $\sqrt{2}\gamma$, $-\frac{1}{\sqrt{2}}(1 + 2\gamma)$, and $-\sqrt{2}\gamma$. The relevant equilibrium for us is P_1^s, since $v_1 = \frac{v}{u} \to -\frac{1}{\sqrt{2}}$ as $u \to 0^+$; recall (2.6).

Remark 8. We remark that the exponent of ε in the leading-order ε-asymptotics of Δc in (2.2) is given by the ratio of the second and third eigenvalues of the linearization of (2.16) at P_1^s. Moreover, we note the presence of a potential $(1, -1)$-resonance in (2.16) which involves the factor $1 + 2\gamma$. This resonance manifests itself *e.g.* for $\gamma \to 0^+$, in which case the Nagumo equation in (2.1) reduces to the so-called Zeldovich equation. The effects of a cut-off in that case were analyzed in detail in [16, Section 4], where it was also shown that the resulting asymptotics of Δc contains logarithmic 'switchback' terms in ε; see [14] for a more general discussion of logarithmic switchback and resonance, from a geometric point of view. $\qquad\square$

Next, we observe that the hyperplanes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$ are invariant for (2.16), as well as that both hyperplanes correspond to the singular limit as $\varepsilon \to 0^+$ in (2.4). The resulting, reduced dynamics determines the location of the singular heteroclinic orbit Γ in this intermediate region. Specifically, in $\{\varepsilon_1 = 0\}$, the orbit passing through $P_{0_1}^-$ (which is the image of the point P_0^- under the blow-up transformation in (1.8)) is asymptotic to P_1^s as $\xi \to \infty$. We denote this orbit by Γ_1^-, and we note that Γ_1^- corresponds to the unstable manifold $\mathcal{W}^u(Q_0^-)$ of the point Q_0^-, after blow-up and transformation to K_1. (Alternatively, Γ_1^- can be interpreted as the equivalent, in K_1, of the 'tail' of the traveling front solution in (2.6), in the absence of a cut-off.)

Similarly, in the invariant hyperplane $\{r_1 = 0\}$, the orbit through $P_{0_1}^+$ (which is the image of the point $P_{0_2}^+$ in Σ_2^+ under the coordinate transformation κ_{21} between charts K_2 and K_1) asymptotes to P_1^s in backward 'time,' *i.e.*, as $\xi \to -\infty$. We denote that orbit by Γ_1^+, and we
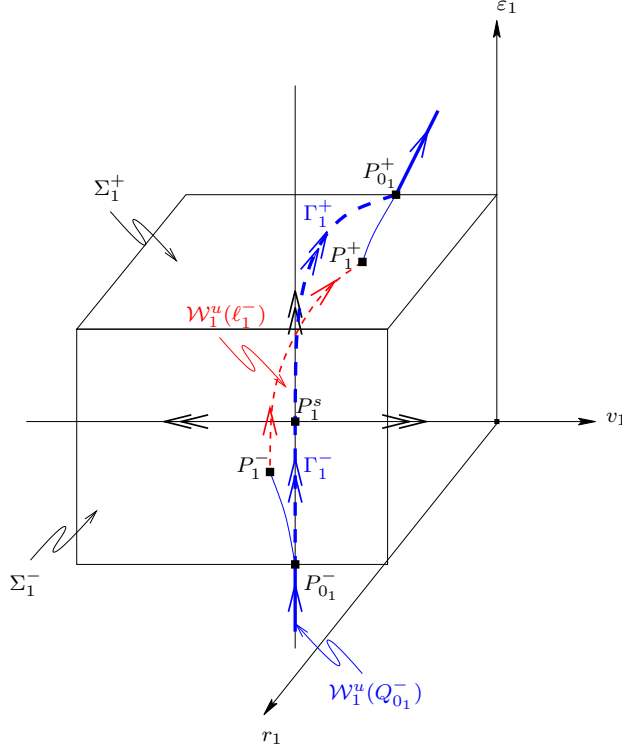
FIGURE 2. The geometry in chart K_1.

remark that it can be determined explicitly as follows: dividing (2.16b) (formally) by (2.16c) and setting $r_1 = 0$ in the resulting equation, we obtain

$$\frac{dv_1}{d\varepsilon_1} = \frac{c_0 v_1 + v_1^2 - \gamma}{\varepsilon_1 v_1}.$$

The solution (in implicit form) can be found by separation of variables:

$$(2.17) \qquad \ln \varepsilon_1 - \frac{1}{2} \ln \left| c_0 v_1 + v_1^2 - \gamma \right| - \frac{c_0}{\sqrt{4\gamma + c_0^2}} \operatorname{arctanh}\left(\frac{2v_1 + c_0}{\sqrt{4\gamma + c_0^2}}\right) \equiv \text{constant},$$

which can in principle be solved for $v_1(\varepsilon_1)$, taking into account that $v_1(1)$ must equal $v_{0_1}^+ = -c_0$ (the v_1-coordinate of $P_{0_1}^+$). Thus, we conclude that the union of Γ_1^- and Γ_1^+ constitutes the portion of Γ that is found in the intermediate region; see Figure 2 for an illustration.

2.1.4. *Summary.* In sum, the singular heteroclinic connection Γ (or, rather, the corresponding orbit $\overline{\Gamma}$ in blown-up phase space) is therefore defined as the union of the orbits $\overline{\Gamma}^-$ and $\overline{\Gamma}^+$ and of the singularities at $\overline{Q}_0^-$, $\overline{P}^s$, and $\overline{Q}_0^+$, which completes our discussion of the singular dynamics of (2.4). The resulting global geometry (in blown-up coordinates) is summarized in Figure 3.

2.2. **Existence and asymptotics of** $c(\varepsilon)$**.** In this section, we establish the persistence of the singular heteroclinic orbit Γ constructed in the previous section for ε positive and sufficiently small. To that end, we combine the dynamics obtained separately in the three regions (inner, outer, and intermediate) in Section 2.1 above.

In the outer region, the unstable manifold $\mathcal{W}^u(Q_0^-)$ of Q_0^- will persist, in an analytic fashion, as the unstable manifold $\mathcal{W}^u(Q_\varepsilon^-)$ of Q_ε^- (at least as long as $u \geq \varepsilon$, for $\varepsilon > 0$ sufficiently small); cf. Section 2.1.1. Given ε fixed, $\mathcal{W}^u(Q_\varepsilon^-)$ corresponds precisely to the sought-after persistent heteroclinic in that region. The unstable manifold $\mathcal{W}^u(\ell^-)$ of ℓ^- is then obtained as $\bigcup_{\varepsilon \in [0,\varepsilon_0]} \mathcal{W}^u(Q_\varepsilon^-)$. (In other words, that manifold is defined as a foliation in $\varepsilon \in [0, \varepsilon_0]$, with fibers $\mathcal{W}^u(Q_\varepsilon^-)$.)
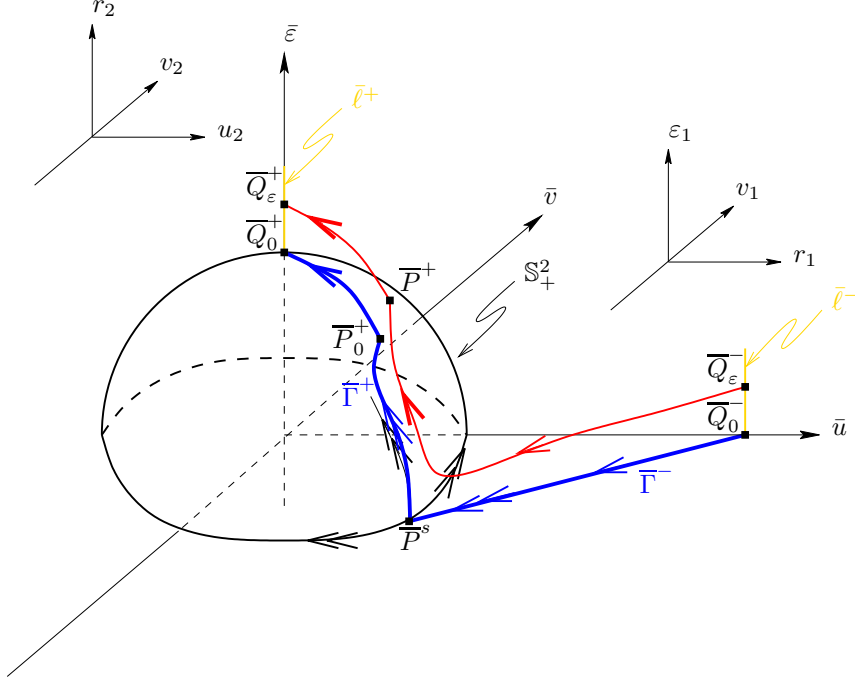
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FIGURE 3. The global geometry of the blown-up vector field.

Similarly, in the inner region, the stable manifold $\mathcal{W}_2^s(Q_{0_2}^+)$ of $Q_{0_2}^+$, which is given explicitly by $v_2(u_2) = -c_0 u_2$, cf. (2.13), will perturb analytically, for $r_2(=\varepsilon) > 0$ small and $u_2 \leq 1$, to the manifold $\mathcal{W}_2^s(Q_2^+)$ of Q_2^+, as defined in Section 2.1.2. (In fact, the persistent manifold is also known explicitly in this chart, and is given by the graph of $v_2 = -cu_2$, for $c = c_0[1 + o(1)]$.) For ε fixed, $\mathcal{W}^s(Q_\varepsilon^+)$ corresponds to the segment of the persistent heteroclinic that is located in the inner region (after blow-down). As before, the corresponding stable manifold $\mathcal{W}_2^s(\ell_2^+)$ of the line of equilibria ℓ_2^+ is retrieved as the union of these manifolds over $\varepsilon \in [0, \varepsilon_0]$.

It remains to show that the two manifolds $\mathcal{W}^u(\ell^-)$ and $\mathcal{W}^s(\ell^+)$ connect in the intermediate region for a unique value of c in (2.4) and each ε sufficiently small; the existence of that connection is equivalent to the persistence of the singular heteroclinic orbit Γ. We will henceforth denote the corresponding c-value by $c(\varepsilon)$; in particular, since we will show that $c(\varepsilon)$ reduces to c_0 in the singular limit as $\varepsilon \to 0^+$, we will identify $c(0)$ and c_0 once the existence of $c(\varepsilon)$ has been proven in Proposition 2.2 below. That proof will be carried out entirely in the intermediate region, i.e., in chart K_1. In a first step, we introduce two sections Σ_1^- and Σ_1^+ for the flow of (2.16), as follows:

$$(2.18a) \qquad \Sigma_1^- = \left\{ (\rho, v_1, \varepsilon_1) \,\middle|\, (v_1, \varepsilon_1) \in [-v_0, 0] \times [0, 1] \right\},$$

$$(2.18b) \qquad \Sigma_1^+ = \left\{ (r_1, v_1, 1) \,\middle|\, (r_1, v_1) \in [0, \rho] \times [-v_0, 0] \right\},$$

where $v_0 > 0$ is defined as before. (The restriction to the negative v_1-axis is possible due to the fact that we are only interested in the dynamics of (2.16) in a neighborhood of P_1^s; recall the discussion in Section 2.1.3.) We note that Σ_1^- corresponds to the section Σ^- introduced in Section 2.1.1, after blow-up and transformation to chart K_1; moreover, we again suppress the ρ-dependence of that section, for convenience of notation. Similarly, Σ_1^+ is equivalent to Σ_2^+ under the change of coordinates κ_{12}; see (1.10) and (2.14). Clearly, Σ_1^- separates the outer region from the intermediate region, while Σ_1^+ defines the boundary between the intermediate and inner regions.

Now, the crucial step in showing the existence and uniqueness of $c(\varepsilon)$ consists in describing the transition map $\Pi_1 : \Sigma_1^- \mapsto \Sigma_1^+$ sufficiently accurately to the order considered here. In other words, we will require that, for $\varepsilon > 0$ small enough, the point of intersection of $\mathcal{W}^u(Q_\varepsilon^-)$ with the

section Σ^-, which we denote by P^-, is mapped to the point of intersection P_2^+ of $\mathcal{W}_2^s(Q_2^+)$ with Σ_2^+ in the transition through the intermediate region. (Here, we note that the corresponding orbit constitutes the portion of the persistent heteroclinic that lies in this region; moreover, we omit the parameter dependence of the points P^- and P_2^+, for brevity.) The required persistence proof will also reveal that $\Delta c(\varepsilon) = c(\varepsilon) - c_0$ must be positive. Finally, it will provide us with the leading-order ε-asymptotics of $c(\varepsilon)$, as stated in Theorem 2.1.

2.2.1. *Preparatory analysis.* We now set out to describe the asymptotics of Π_1, as indicated above. To that end, we first recast (2.16) in a form that is more convenient, via a sequence of coordinate transformations: we write $c = c_0 + (c - c_0) = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma + \Delta c$; then, we introduce the new variable $z = v_1 + \frac{1}{2}c_0 = v_1 + \frac{1}{2\sqrt{2}}(1 - 2\gamma)$. With these transformations, the equations in (2.16) become

(2.19a) $\qquad r_1' = -\left[\frac{1}{2\sqrt{2}}(1 - 2\gamma) - z\right]r_1,$

(2.19b) $\qquad z' = \left[\frac{1}{2\sqrt{2}}(1 - 2\gamma) - z\right]\Delta c - z^2 + \frac{1}{8}(1 + 2\gamma)^2 - (1 + \gamma)r_1 + r_1^2,$

(2.19c) $\qquad \varepsilon_1' = \left[\frac{1}{2\sqrt{2}}(1 - 2\gamma) - z\right]\varepsilon_1.$

(Here, we observe that the linear v_1-terms in (2.16b) cancel due to our choice of constant in the definition of z.) Next, we divide out the factor of $\frac{1}{2\sqrt{2}}(1 - 2\gamma) - z$, which is positive in the z-regime considered here, cf. Section 2.1.3, from the right-hand sides of the vector field in (2.19):

(2.20a) $\qquad r_1' = -r_1,$

(2.20b) $\qquad z' = \Delta c - \dfrac{z^2 - \frac{1}{8}(1 + 2\gamma)^2}{\frac{1}{2\sqrt{2}}(1 - 2\gamma) - z} + \dfrac{-(1 + \gamma)r_1 + r_1^2}{\frac{1}{2\sqrt{2}}(1 - 2\gamma) - z},$

(2.20c) $\qquad \varepsilon_1' = \varepsilon_1.$

This transformation corresponds to a rescaling of ξ that leaves the phase portrait of (2.19) unchanged; correspondingly, the prime now denotes differentiation with respect to a new independent variable ζ. Moreover, since the equations in (2.20) are autonomous, we may assume without loss of generality that $\zeta^- = 0$ in Σ_1^-, independent of the choice of ξ^- in (2.6).

Now, the desired expression for the transition map Π_1 may be obtained by simplifying the equations in (2.20) appropriately. To that end, we derive a normal form system for (2.20), as follows:

Proposition 2.1. *Let* $\mathcal{V} := \left\{(r_1, z, \varepsilon_1) \,\middle|\, (r_1, z, \varepsilon_1) \in [0, \rho] \times [-z_0, 0] \times [0, 1]\right\}$, *where* $z_0 = v_0 + \frac{1}{\sqrt{2}}(1 - 2\gamma)$, *with* v_0 *as in* (2.18). *Then, there exists a* $\mathcal{C}^\infty$-*smooth coordinate transformation*

$$\psi : \begin{cases} \mathcal{V} \to \psi(\mathcal{V}), \\ (r_1, z, \varepsilon_1) \mapsto (r_1, \hat{z}, \varepsilon_1), \end{cases}$$

with $\hat{z}(z, r_1) = z + O(r_1)$, *such that* (2.20) *can be written as*

(2.21a) $\qquad r_1' = -r_1,$

(2.21b) $\qquad \hat{z}' = \Delta c - \dfrac{\hat{z}^2 - \frac{1}{8}(1 + 2\gamma)^2}{\frac{1}{2\sqrt{2}}(1 - 2\gamma) - \hat{z}},$

(2.21c) $\qquad \varepsilon_1' = \varepsilon_1.$

Proof. The result follows from standard normal form theory; see for example [6] and the references therein. In particular, we note that the r_1-dependent terms in (2.20b) are non-resonant and that they can hence be removed completely via a near-identity coordinate change ψ. Moreover, ψ can only depend on the variables r_1 and z, as (2.20b) is independent of ε_1. Therefore, $\hat{z} = z + O(r_1)$, as claimed. $\qquad\square$

2.2.2. *Uniqueness of Δc.* Let P_1^- and P_1^+ denote the points that correspond to P^- and P_2^+, respectively, after transformation to chart K_1, and let $\widehat{P}_1^-$ and $\widehat{P}_1^+$ be the respective corresponding points after application of the normal form transformation ψ defined in Proposition 2.1. Finally, let $\hat{z}^-$ and $\hat{z}^+$ denote the associated $\hat{z}$-values that are obtained from z^- and z^+, respectively. We find

Lemma 2.2. *For any $\rho \in (\varepsilon, 1)$, with $\varepsilon \in (0, \varepsilon_0]$ and Δc sufficiently small, the points $\widehat{P}_1^- = (\rho, \hat{z}^-, \varepsilon \rho^{-1})$ and $\widehat{P}_1^+ = (\varepsilon, \hat{z}^+, 1)$ satisfy*

(2.22)
$$\hat{z}^- = \hat{z}^-(\rho, \Delta c) = -\frac{1}{2\sqrt{2}}(1 + 2\gamma) + \nu(\rho, \Delta c)\Delta c, \quad with \quad \nu(\rho, 0) = \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)[1 + \nu_1(\rho)] > 0,$$

and

(2.23)
$$\hat{z}^+ = \hat{z}^+(\Delta c, \varepsilon) = -\left[\frac{1}{2\sqrt{2}}(1 - 2\gamma) + \Delta c\right] + \omega(\Delta c, \varepsilon)\varepsilon.$$

Here, $\nu(\rho, \Delta c)$ is a $\mathcal{C}^\infty$-smooth function in ρ and Δc, while ν_1 is $\mathcal{C}^\infty$-smooth down to $\rho = 0$, with $\nu_1(0) = 0$. Finally, $\omega(\Delta c, \varepsilon)$ is $\mathcal{C}^\infty$-smooth in Δc and ε, including in a neighborhood of $(0, 0)$.

Proof. Given that $u(= r_1) = \rho$ in Σ_1^-, cf. (2.18a), we evaluate the expansion in (2.8) to find

(2.24)
$$v^- := v(\rho, c) = v(\rho, c_0) + \frac{\partial v}{\partial c}(\rho, c_0)\Delta c + O[(\Delta c)^2]$$

for the v-coordinate of P^-, where the $O[(\Delta c)^2]$-terms are $\mathcal{C}^\infty$-smooth as long as ρ is positive. Substituting in $v(\rho, c_0) = \frac{1}{\sqrt{2}}\rho(\rho - 1)$, see (2.7), and noting that $v^- = \rho v_1^-$, we have

$$z^- = v_1^- + \frac{1}{2\sqrt{2}}(1 - 2\gamma) = -\frac{1}{2\sqrt{2}}(1 + 2\gamma) + \frac{\rho}{\sqrt{2}} + \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)\Delta c + O[(\Delta c)^2].$$

It remains to transform that expression into $(r_1, \hat{z}, \varepsilon_1)$-coordinates: applying the normal form transformation ψ from the proof of Proposition 2.1 and taking into account that ψ is near-identity and $\mathcal{C}^\infty$-smooth, we obtain a transformed value $\hat{z}^- = \hat{z}^-(\rho, \Delta c)$ from z^-, where

$$\hat{z}^- = -\frac{1}{2\sqrt{2}}(1 + 2\gamma) + \nu_0(\rho) + \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)[1 + \nu_1(\rho)]\Delta c + \nu_2(\rho, \Delta c)(\Delta c)^2,$$

for Δc sufficiently small. Here, ν_j, $j = 0, 1, 2$, are $\mathcal{C}^\infty$-smooth functions in their respective arguments; in particular, ν_0 and ν_1 are smooth down to $\rho = 0$, with $\nu_0(0) = 0 = \nu_1(0)$. Next, we note that $\nu_0(\rho) \equiv 0$ must hold, since $\hat{z} = \pm\frac{1}{2\sqrt{2}}(1 + 2\gamma)$ is invariant for $\Delta c = 0$ in (2.21b). (Here, we remark that these $\hat{z}$-values correspond precisely to the (rectified) stable and unstable manifolds $\mathcal{W}_1^s(\widehat{P}_1^s)$ and $\mathcal{W}_1^u(\widehat{P}_1^u)$ of $\widehat{P}_1^s$ and $\widehat{P}_1^u$, respectively, after transformation to $(r_1, \hat{z}, \varepsilon_1)$-coordinates.) Hence, we may express $\hat{z}^-$ as stated in (2.22), with $\nu(\rho, \Delta c) = \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)[1 + \nu_1(\rho)] + \nu_2(\rho, \Delta c)\Delta c$; in particular, the smoothness of ν_1 implies that $\nu(\rho, 0)$ is a $\mathcal{C}^\infty$-smooth function in ρ. Finally, $\nu(\rho, 0)$ is positive for $\rho \in (0, 1)$ small enough, as claimed, since $\frac{\partial v}{\partial c}$ is positive for any $u \in (0, 1)$, by Lemma 2.1, which establishes (2.22).

To show (2.23), we first note that $P_1^+ = \kappa_{21}(P_2^+)$ must hold for the singular heteroclinic orbit Γ to persist for some c-value $c(\varepsilon)$, with ε sufficiently small. (Here, the change of coordinates $\kappa_{21} : K_2 \to K_1$ is as defined in (1.9).) Therefore, we may obtain the desired estimate for z^+ by estimating P_2^+ first. To that end, we recall that $v_2^+ = -(\frac{1}{\sqrt{2}} - \sqrt{2}\gamma) - \Delta c$, as well as that $v_2^+ = v_1^+$, since $u_2 = 1$ in Σ_2^+, by (2.14). Hence,

$$z^+ = -\frac{1}{2\sqrt{2}}(1 - 2\gamma) - \Delta c,$$

and application of the near-identity transformation ψ defined in the proof of Proposition 2.1 to z^+ yields a corresponding value $\hat{z}^+$ that satisfies $\hat{z}^+(\Delta c, \varepsilon) = -\frac{1}{2\sqrt{2}}(1 - 2\gamma) - \Delta c + \omega_{01}(\Delta c, \varepsilon)$, where ω_{01} is $\mathcal{C}^\infty$-smooth in both Δc and ε, including in a neighborhood of $(0, 0)$. In particular,

since $\hat{z} = z + O(r_1)$, cf. Proposition 2.1, and since $r_1 = \varepsilon$ in Σ_1^+, it follows that ω_{01} vanishes in the singular limit as $\varepsilon \to 0^+$. Writing $\omega_{01}(\Delta c, \varepsilon) = \omega(\Delta c, \varepsilon)\varepsilon$ for some new function ω, we obtain (2.23), which completes the proof. $\qquad\square$

Remark 9. Since $\frac{\partial v}{\partial c}(\rho, c_0)$ will become unbounded as $\rho \to 0^+$, by Lemma 2.1, (2.22) implies that $\nu(\rho, \Delta c)$ cannot remain bounded in that limit, either. As will become clear in the following, this unboundedness will be resolved in the transition through the intermediate region, provided the ρ-dependence of Π_1 is accounted for accordingly; see Lemma 2.4 below.

In general, the function $\nu_2(\rho, \Delta c)$ defined above cannot be expected to remain bounded when $\Delta c > 0$ and $\rho \to 0^+$. However, $\nu_2(\rho, \Delta c)(\Delta c)^{-2}$ will still be uniformly bounded for ρ in a compact subset of $(0, 1)$. Correspondingly, in the statement of Lemma 2.2, it is assumed that ρ is positive. $\qquad\square$

For given Δc small, $\varepsilon \in (0, \varepsilon_0]$, and $\rho \in (\varepsilon, 1)$, we now consider the solution to (2.21) with initial $\hat{z}$-value $\hat{z}(0) = \hat{z}^-(\rho, \Delta c)$, where $\hat{z}^-$ is as in (2.22). Let $\hat{z}_-^+$ denote the corresponding value of $\hat{z}(\zeta^+)$, where $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$ is the value of ζ in Σ_1^+. (Here, ζ^+ can *e.g.* be found from $r_1(\zeta) = \rho e^{-\zeta}$, cf. (2.21a), in combination with $r_1(\zeta^+) = \varepsilon$.)

Lemma 2.3. *For $\hat{z}_-^+$ defined as above, there holds $\frac{\partial \hat{z}_-^+}{\partial c}(\rho, \Delta c) > 0$. Moreover, there can exist at most one value of Δc such that $\hat{z}_-^+(\rho, \Delta c) = \hat{z}^+(\Delta c, \varepsilon)$, where $\hat{z}^+$ is as in (2.23).*

Proof. The first statement can be seen by considering the variational equations corresponding to (2.21); in particular, along any orbit with $\varepsilon = r_1\varepsilon_1 > 0$, the equation for $\frac{\partial \hat{z}}{\partial c}$ obtained from (2.21b) is given by $\frac{\partial}{\partial \zeta}\left(\frac{\partial \hat{z}}{\partial c}\right) = 1 + Z(\zeta)\frac{\partial \hat{z}}{\partial c}$, where Z is some smooth function. Since, by Lemma 2.2, $\frac{\partial \hat{z}^-}{\partial c} > 0$ for ρ and Δc sufficiently small, any solution of this equation has to remain strictly positive for all $\zeta \geq \zeta^- = 0$. As $\frac{\partial \hat{z}^+}{\partial c} < 0$, again by Lemma 2.2, the second statement then follows trivially. $\qquad\square$

Lemma 2.3 implies, in particular, that a connection between the points $\widehat{P}^-$ and $\widehat{P}^+$ under the flow of (2.21) can exist for at most one value of Δc in (2.21b). As a consequence, persistence of the singular heteroclinic orbit Γ constructed in Section 2.1 is also only possible for at most one value of Δc.

2.2.3. *Existence and asymptotics of Δc.* We are now in a position to prove that there exists, in fact, a function $\Delta c = \Delta c(\varepsilon)$ so that the singular heteroclinic orbit Γ persists, for ε positive and sufficiently small and $c = c_0 + \Delta c(\varepsilon)$ in (2.4). To that end, we integrate the normal form equations obtained in (2.21), taking into account the estimates for $\hat{z}^-$ and $\hat{z}^+$ found in Lemma 2.2 above:

Proposition 2.2. *Let $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, and let $\gamma \in (0, \frac{1}{2})$. Then, there exists a function $c(\varepsilon) = c_0 + \Delta c(\varepsilon)$, with $\Delta c(0) = 0$, such that the singular orbit Γ persists if and only if $c = c(\varepsilon)$ in (2.4). Moreover, Δc is positive, and $\mathcal{C}^1$-smooth in ε (including at $\varepsilon = 0$) and γ.*

Proof. Given the normal form system in (2.21), we need to determine Δc so that $\widehat{P}_1^-$ is mapped to $\widehat{P}_1^+$ under Π_1. We first integrate (2.21b), using separation of variables, to obtain

$$(2.25) \quad \zeta^+ - \zeta^- - \frac{1}{2}\ln\left|2\hat{z}^2 + 2\Delta c\hat{z} - \frac{1}{\sqrt{2}}(1 - 2\gamma)\Delta c - \frac{1}{4}(1 + 2\gamma)^2\right|\Big|_{\hat{z}^-}^{\hat{z}^+}$$

$$- \frac{\frac{1}{\sqrt{2}}(1 - 2\gamma) + \Delta c}{\sqrt{\frac{1}{2}(1 + 2\gamma)^2 + \sqrt{2}(1 - 2\gamma)\Delta c + (\Delta c)^2}}$$

$$\times \operatorname{arctanh}\left(\frac{2\hat{z} + \Delta c}{\sqrt{\frac{1}{2}(1 + 2\gamma)^2 + \sqrt{2}(1 - 2\gamma)\Delta c + (\Delta c)^2}}\right)\Big|_{\hat{z}^-}^{\hat{z}^+} = 0;$$

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see also (2.17). Now, we recall that $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$ and $\zeta^- = 0$; moreover, we make use of $\hat{z}^+ = -[\frac{1}{2\sqrt{2}}(1 - 2\gamma) + \Delta c] + \omega(\Delta c, \varepsilon)\varepsilon$ and of $\hat{z}^- = -\frac{1}{2\sqrt{2}}(1 + 2\gamma) + \nu(\rho, \Delta c)\Delta c$, as given in (2.23) and (2.22), respectively. Substituting into (2.25), rewriting the hyperbolic arctangent via

$$\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x},$$

and expanding the result in terms of Δc and ε, we find

$$(2.26) \quad -\ln\frac{\varepsilon}{\rho} - \frac{1}{2}\ln\left| - 2\gamma - \sqrt{2}(1 - 2\gamma)\omega(\Delta c, \varepsilon)\varepsilon + O(2)\right|$$

$$+ \frac{1}{2}\ln\left| - \sqrt{2}[1 + (1 + 2\gamma)\nu(\rho, 0) + O(1)]\Delta c\right| - \frac{1}{2}\left\{\frac{1 - 2\gamma}{1 + 2\gamma} + \frac{8\sqrt{2}\gamma}{(1 + 2\gamma)^3}\Delta c + O(2)\right\}$$

$$\times\left\{\ln\left|2\gamma - \frac{4\sqrt{2}\gamma}{1 + 2\gamma}\Delta c + \sqrt{2}(1 + 2\gamma)\omega(\Delta c, \varepsilon)\varepsilon + O(2)\right| - \ln\left|\sqrt{2}\frac{1 + (1 + 2\gamma)\nu(\rho, 0) + O(1)}{(1 + 2\gamma)^2}\Delta c\right|\right\} = 0.$$

Here, $O(1)$ denotes terms of at least order 1 in Δc, and $O(2)$ stands for terms of at least order 2 in Δc and ε; both $O(1)$ and $O(2)$ are $\mathcal{C}^\infty$-smooth, and uniform in their respective arguments, if ρ is restricted to compact subsets of $(0, 1)$. (The uniformity is lost as $\rho \to 0^+$, as before, since $\nu(\rho, \Delta c)$ becomes unbounded in that limit; recall the proof of Lemma 2.2 and, in particular, Remark 9.)

Since, by Lemma 2.3, (2.26) can have a solution for at most one value of Δc, we will restrict ourselves to $\Delta c > 0$ and show that a solution exists in that case. That solution will then necessarily be unique.

Hence, taking into account that $\nu(\rho, 0) > 0$, by Lemma 2.2, we exponentiate (2.26) to obtain

$$(2.27) \quad \left(\frac{\varepsilon}{\rho}\right)^{2(1+2\gamma)} = (2\gamma)^{-(1+2\gamma)}\left\{\sqrt{2}\left[1 + (1 + 2\gamma)\nu(\rho, 0)\right]\Delta c\right\}^{1+2\gamma}$$

$$\times (2\gamma)^{-(1-2\gamma)}\left\{\sqrt{2}\frac{1 + (1 + 2\gamma)\nu(\rho, 0)}{(1 + 2\gamma)^2}\Delta c\right\}^{1-2\gamma}[1 + O(1)],$$

where the $O(1)$-terms are now $\mathcal{C}^\infty$-smooth in Δc, $\Delta c\ln(\Delta c)$, and ε. (Here, the occurrence of logarithmic terms in Δc is due to the $\Delta c\ln(\Delta c)$-terms in (2.26).) Clearly, the relation in (2.27) is satisfied at $(\Delta c, \varepsilon) = (0, 0)$; moreover, it is $\mathcal{C}^1$-smooth, in ε, Δc, γ, and ρ, in a uniform fashion, for ε and Δc sufficiently small (including at $(\Delta c, \varepsilon) = (0, 0)$) and γ and ρ in a compact subset of $(0, \frac{1}{2})$ and $(0, 1)$, respectively. Finally, since $1 + (1 + 2\gamma)\nu(\rho, 0) > 0$, it follows from the Implicit Function Theorem that (2.27) has a solution $\Delta c(\varepsilon, \gamma, \rho)$ which is $\mathcal{C}^1$-smooth in ε (down to $\varepsilon = 0$), γ, and ρ, as claimed.

By definition, that solution yields precisely the value of Δc for which a heteroclinic connection exists between the points Q_ε^- and Q_ε^+ in (2.4). Hence, $\Delta c = \Delta c(\varepsilon, \gamma)$ must hold, i.e., Δc cannot depend on ρ. (In other words, Δc must be independent of the definition of the section Σ^-, which is arbitrary.) Finally, to determine the leading-order ε-asymptotics of $\Delta c(\varepsilon) \equiv \Delta c(\varepsilon, \gamma)$, we solve (2.27) to leading order:

$$\Delta c(\varepsilon) = K_\gamma \varepsilon^{1+2\gamma} + o(\varepsilon^{1+2\gamma}),$$

where the constant K_γ is defined by

$$(2.28) \qquad K_\gamma = \frac{\sqrt{2}\gamma(1 + 2\gamma)^{1-2\gamma}}{1 + (1 + 2\gamma)\nu(\rho, 0)}\frac{1}{\rho^{1+2\gamma}} \equiv \frac{\sqrt{2}\gamma(1 + 2\gamma)^{1-2\gamma}}{(1 + 2\gamma)\delta(\gamma)} > 0.$$

Here,

$$(2.29) \qquad \delta(\gamma) = \left[\frac{1}{1 + 2\gamma} + \nu(\rho, 0)\right]\rho^{1+2\gamma}$$

denotes a strictly positive function that is $\mathcal{C}^\infty$-smooth in $\gamma \in (0, \frac{1}{2})$, for *any* $\rho \in (0, 1)$ fixed and sufficiently small. This completes the proof of Proposition 2.2. $\qquad\square$

***Remark* 10.** The proof of Proposition 2.2 implies that Δc is, in fact, obtained as the solution of an implicit equation of the form

$$(2.30) \qquad K(\gamma)\varepsilon^{1+2\gamma} = \Delta c[1 + \theta(\varepsilon, \Delta c, \Delta c \ln(\Delta c), \gamma)],$$

where $K(\gamma) \equiv K_\gamma$ and θ are $\mathcal{C}^\infty$-smooth in their respective arguments, with $K(\gamma)$ positive, see (2.28), and $\theta(0,0,0,\gamma) = 0$. Even though Δc is only $\mathcal{C}^1$-smooth when considered as a function of ε, the relation in (2.30) allows us to calculate the ε-asymptotics of Δc to higher order than a mere $\mathcal{C}^1$-dependence might suggest. $\qquad \square$

We emphasize that the definition of K_γ in (2.28) has to be independent of ρ, as $\Delta c(\varepsilon)$ is defined by the global condition that the singular heteroclinic orbit Γ persists, for ε sufficiently small: while the Implicit Function Theorem is applied for ρ fixed in the proof of Proposition 2.2, our argument is valid for arbitrary ρ. (To state it differently, although the function $\nu(\rho,0)$, as defined in (2.22), may depend on the definition of Σ_1^- and, hence, on ρ, that dependence must cancel, as a matter of principle, once the dynamics of (2.4) in the outer region has been taken into account.) Therefore, the function $\delta(\gamma)$ also cannot depend on ρ, and we may obtain the value of δ by evaluating (2.29) for *any* $\rho \in (0,1)$; in particular, we may pass to the zero-ρ limit. Recalling the definition of $\nu(\rho,0)$ from (2.22), we have

$$(2.31) \qquad \delta(\gamma) = \lim_{\rho \to 0^+} \left\{ \rho^{1+2\gamma} \nu(\rho,0) \right\} = \lim_{\rho \to 0^+} \left\{ \rho^{2\gamma} \frac{\partial v}{\partial c}(\rho, c_0) \right\};$$

cf. the proof of Lemma 2.2. It remains to evaluate the above limit. To that end, we make use of the explicit solution $\frac{\partial v}{\partial c}$ of the variational equation that is associated to the first-order system without cut-off in (2.5), as found in Lemma 2.1:

Lemma 2.4. *The function δ defined in (2.29) satisfies*

$$(2.32) \qquad \delta(\gamma) = \lim_{\rho \to 0^+} \left\{ \rho^{2\gamma} \frac{\partial v}{\partial c}(\rho, c_0) \right\} = \frac{\Gamma(3-2\gamma)\Gamma(2\gamma+1)}{\Gamma(4)},$$

where $\frac{\partial v}{\partial c}(u, c_0)$ is as given in (2.10) and $\Gamma(\cdot)$ denotes the standard Gamma function, as before.

Proof. Evaluating (2.10) at $u = \rho$, for ρ positive and small, and noting that the hypergeometric function F converges absolutely at $\rho = 0$ due to $\Re(1+2\gamma) > 0$ [1, Section 15.1.1], we find

$$\rho^{2\gamma} \frac{\partial v}{\partial c}(\rho, c_0) = \frac{1}{3-2\gamma}(1-\rho)F(3-2\gamma, -2\gamma; 4-2\gamma; 1-\rho).$$

Making use of the identity [1, Equation (15.1.20)]

$$(2.33) \qquad F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for} \quad c \notin \mathbb{Z}_- \quad \text{and} \quad \Re(c-a-b) > 0$$

and taking the limit as $\rho \to 0^+$ in the resulting equation, we obtain

$$(2.34) \qquad \delta(\gamma) = \frac{1}{3-2\gamma} \frac{\Gamma(4-2\gamma)\Gamma(2\gamma+1)}{\Gamma(1)\Gamma(4)}.$$

Since $\Gamma(1) = 1$ and $\Gamma(4-2\gamma) = (3-2\gamma)\Gamma(3-2\gamma)$, (2.32) follows, as claimed, which completes the proof. $\qquad \square$

2.2.4. *End of proof of Theorem 2.1.* We are now in a position to complete the proof of Theorem 2.1:

Proposition 2.3. *The constant K_γ introduced in (2.2) is given by*

$$(2.35) \qquad K_\gamma = \frac{\Gamma(4)}{\Gamma(1+2\gamma)\Gamma(3-2\gamma)} \frac{\sqrt{2}\gamma}{(1+2\gamma)^{2\gamma}};$$

cf. (2.30).

Proof. The result is immediate from (2.28) and (2.31), in combination with (2.32). $\qquad \square$

Since Equation (2.35) is precisely (2.3), this completes the proof of Theorem 2.1.

Remark 11. Clearly, K_γ reduces to zero in the limit as $f'(0) = \gamma \to 0^+$, which is in agreement with [16, Theorem 2]: there, it was shown that the correction Δc to $c_0 = \frac{1}{\sqrt{2}}$ that is induced by the cut-off in the resulting Zeldovich equation, with $f(u) = u^2(1-u)$, is of the order $O(\varepsilon^2)$; see also [4, Equation (9)]. $\qquad\square$

Remark 12. The approach developed in this section can be extended to more general choices of cut-off function Θ in (2.1). (In particular, for Θ as defined in (1.4), the dynamics of the associated traveling front equation will remain unchanged in the outer and intermediate regions, since $\Theta \equiv 1$ there.) However, while the exponent $1+2\gamma$ in the leading-order ε-asymptotics of Δc will be Θ-independent, the numerical value of K_γ will depend on the choice of Θ, in contrast to the expansion for Δc obtained in [5, 8], in the context of the cut-off Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation in (1.1), where the corresponding coefficient was universal. Since, moreover, the governing equations in the inner region will typically have no closed-form solution in the singular limit as $\varepsilon \to 0^+$, it will not be possible to evaluate K_γ in closed form for general Θ. $\qquad\square$

Remark 13. The leading-order approximation $\zeta^+ = -\ln\varepsilon + O(1)$ made in [15, Section 3], while sufficient to prove $\Delta c(\varepsilon) = O(\varepsilon^{1+2\gamma})$, is not accurate enough to give the value of the coefficient K_γ: that value can only be determined if $\delta(\gamma)$ is known; however, the definition of δ crucially depends on the factor of $\rho^{1+2\gamma}$ that was neglected in [15]. $\qquad\square$

Remark 14. Alternatively, existence and uniqueness of $c(\varepsilon)$ can be shown via a phase plane argument, as was done in the analysis of the cut-off FKPP equation in [8, Proposition 3.1]: first, $\mathcal{W}^u(\ell^-)$ is tracked in forward 'time' ξ to the hyperplane $\{u = \varepsilon\}$, where it lies inside of $\mathcal{W}^s(\ell^+)$ for $c > c(\varepsilon)$, whereas it is located outside for $c < c(\varepsilon)$. Hence, the two manifolds must coincide for some value $c(\varepsilon)$ of c close to c_0. Moreover, by that same argument, it follows that $c(\varepsilon) > c_0$. Finally, $c(\varepsilon)$ is unique, as the positions of the points of intersection of $\mathcal{W}^u(\ell^-)$ and $\mathcal{W}^s(\ell^+)$ with $\{u = \varepsilon\}$ change monotonically with c.

The ε-asymptotics of Δc, however, cannot be obtained in that manner, but has to be derived separately, via an analysis of the transition through the intermediate region, as was done in [8, Proposition 3.2] and in the proof of Proposition 2.2 above. That proof shows the existence of Δc in addition to yielding its leading-order asymptotics, rendering a separate existence argument unnecessary. $\qquad\square$

3. The cut-off Schlögl equation

In this section, we briefly discuss the Schlögl equation with cut-off:

$$(3.1) \qquad \phi_t = \phi_{xx} - [2(1-\sigma)\phi + (\sigma - 3)\phi^2 + \phi^3]H(\phi - \varepsilon).$$

Here, $0 < \sigma < 1$ is a (fixed) parameter [13], and ϕ is defined so that the metastable state, which in the usual formulation is located at $\phi^+ = -1$, lies at zero now. Additional rest states of (3.1) are found at $\phi^\circ = 1 - \sigma$ and $\phi^- = 2$. Moreover, we note that an explicit expression is known for the traveling front solution that propagates between the rest states at 2 and 0, with propagation speed $c_0 = \sqrt{2}\sigma$, in the corresponding equation without cut-off; that solution is given by

$$(3.2) \qquad u(\xi) = 1 - \tanh\left(\frac{\xi - \xi^-}{\sqrt{2}}\right),$$

with arbitrary phase ξ^-; see *e.g.* [13, 15].

The following corollary is the main result of this section:

Corollary 3.1. *Let $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, and let $\sigma \in (0, 1)$. Then, the unique (σ-dependent) value $c(\varepsilon)$ of c for which Equation (4.1) supports a unique traveling front solution propagating between $\phi^- = 2$ and $\phi^+ = 0$ is given by $c(\varepsilon) = c(0) + \Delta c(\varepsilon)$, where*

$c(0) = \sqrt{2}\sigma$ (the propagation speed in the absence of a cut-off) and Δc is a positive, $\mathcal{C}^1$-smooth function in ε (including at $\varepsilon = 0$) and σ that satisfies

$$(3.3) \qquad \Delta c(\varepsilon) = K_\sigma \varepsilon^{2-\sigma} + o(\varepsilon^{2-\sigma}),$$

for

$$(3.4) \qquad K_\sigma = \frac{\Gamma(4)}{\Gamma(2+\sigma)\Gamma(2-\sigma)} \frac{\sqrt{2}(1-\sigma)}{2^{2-\sigma}(2-\sigma)^{1-\sigma}}$$

a positive constant. (Here, $\Gamma(\cdot)$ denotes the standard Gamma function, as before.)

Proof. The result is most easily seen from the well-known equivalence of the Schlögl and Nagumo equations [4, 13]: introducing the new variables

$$\Phi = \frac{1}{2}\phi, \quad T = 4t, \quad X = 2x, \quad E = \frac{1}{2}\varepsilon, \quad \text{and} \quad \Sigma = \frac{1}{2}(1-\sigma)$$

in (3.1), we obtain

$$(3.5) \qquad \Phi_T = \Phi_{XX} + \Phi(1-\Phi)(\Phi - \Sigma)H(\Phi - E),$$

which corresponds precisely to Equation (2.1). (In particular, there holds $\Sigma \in (0, \frac{1}{2})$ for $\sigma \in (0, 1)$, as required.) Defining the new traveling wave variable Ξ by $\Xi = X - CT$, we note that $\Xi = 2\xi$, with $\xi = x - ct$, as before. Hence, $c = 2C$, and applying the result of Theorem 2.1 to (3.5), we have

$$(3.6) \qquad C(E) = C_0 + K_\Sigma E^{1+2\Sigma} + o(E^{1+2\Sigma}),$$

where $C_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\Sigma$ and K_Σ are defined as in (2.3), with γ replaced with Σ there. Rewriting (3.6) in terms of ε, c, and σ, we find (3.3) and (3.4), as claimed, which completes the proof. $\square$

To the best of our knowledge, the value of K_σ had not been calculated explicitly before. Finally, we remark that a preliminary, geometric analysis of the cut-off Schlögl equation has appeared previously in [15]. (In particular, the statement of Corollary 3.1 makes rigorous the reasoning presented in [15, Section 3].)

Remark 15. Alternatively, the result of Corollary 3.1 can also be obtained from the equivalent first-order system

$$(3.7a) \qquad u' = v,$$
$$(3.7b) \qquad v' = -cv + [2(1-\sigma)u + (\sigma - 3)u^2 + u^3]H(u - \varepsilon),$$
$$(3.7c) \qquad \varepsilon' = 0$$

corresponding to (3.1), as was done for the Nagumo equation in Section 2. In particular, the existence and uniqueness of $\Delta c(\varepsilon) = K_\sigma \varepsilon^{2-\sigma} + o(\varepsilon^{2-\sigma})$, with $K_\sigma = \frac{\sqrt{2}(1-\sigma)}{(2-\sigma)^{1-\sigma}\delta(\sigma)} > 0$ and $\delta(\sigma)$ a $\mathcal{C}^\infty$-smooth, positive function, can be shown exactly as in Section 2.2 above. To determine the value of δ, one considers the corresponding variational equation along the heteroclinic connection $v(u, c_0) = \frac{1}{\sqrt{2}}u(u - 2)$ between the equilibrium points at $Q_0^- = (2, 0, 0)$ and $Q_0^+ = (0, 0, 0)$ in (3.7):

$$(3.8) \qquad \frac{\partial}{\partial u}\left(\frac{\partial v}{\partial c}(u, c_0)\right) = -1 + 2\frac{u + \sigma - 1}{u(2-u)}\frac{\partial v}{\partial c}(u, c_0).$$

The unique solution of (3.8) that satisfies $\frac{\partial v}{\partial c}(2, c_0) = 0$ is given by

$$(3.9) \qquad \frac{\partial v}{\partial c}(u, c_0) = \frac{2^{1-\sigma}}{2+\sigma}u^{\sigma-1}(2 - u)F(2 + \sigma, -1 + \sigma; 3 + \sigma; 1 - \tfrac{u}{2}),$$

where $F(\cdot, \cdot; \cdot; \cdot)$ denotes the hypergeometric function [1, Section 15], as before. Hence, we have

$$(3.10) \qquad \delta(\sigma) = \lim_{\rho \to 0^+}\left\{\rho^{1-\sigma}\frac{\partial v}{\partial c}(\rho, c_0)\right\} = 2^{2-\sigma}\frac{\Gamma(2+\sigma)\Gamma(2-\sigma)}{\Gamma(4)},$$

which we substitute into K_σ to find (3.4), as required. □

4. General cut-off bistable dynamics

The results obtained in Sections 2 and 3 for the Nagumo and Schlögl equations, respectively, generalize to bistable front propagation into a metastable state in the more general class of cut-off reaction-diffusions equations defined in (1.5), which we restate for reference here:

$$(4.1) \qquad \phi_t = \phi_{xx} + f(\phi)H(\phi - \varepsilon),$$

where H again denotes the Heaviside cut-off, as before, and the smooth reaction function f satisfies several mild assumptions that are specified in the following.

Assumption $\mathcal{A}_1$. The function $f(\phi)$ in (4.1) has three roots, corresponding to the three rest states at ϕ^+, ϕ^-, and ϕ° in (4.1). Moreover, the stable rest state at ϕ^+ is located at the origin, i.e., f may be written as $f(\phi) = \phi g(\phi)$, where g is a smooth function, with

$$(4.2) \qquad g(0) < 0 \quad \text{and} \quad g'(0) > 0, \quad \text{as well as} \quad g(\phi^-) = 0 \quad \text{and} \quad g'(\phi^-) < 0.$$

For future reference, we note that, clearly, $g(0) = f'(0)$. We also remark that, by Assumption $\mathcal{A}_1$, the third rest state at ϕ° satisfies $g(\phi^\circ) = 0$ and $g'(\phi^\circ) \geq 0$.

Next, we require an assumption about the global dynamics of Equation (4.1) in the absence of a cut-off:

Assumption $\mathcal{A}_2$. The equation $\phi_t = \phi_{xx} + f(\phi)$ supports a traveling front solution that propagates between the rest states at ϕ^- and $\phi^+(= 0)$, with propagation speed c_0.

Finally, to ensure that the front propagation speed c_0, as defined in Assumption $\mathcal{A}_2$, is non-negative, we impose the following assumption on the integral of f, cf. e.g. [12, Section 6.2]:

Assumption $\mathcal{A}_3$. The reaction function f in (4.1) satisfies $\int_0^1 f(\phi)\, d\phi > 0$.

Remark 16. The requirement in Assumption $\mathcal{A}_3$ can be seen by considering the traveling front equation $u'' + c_0 u' + f(u) = 0$ (without cut-off): multiplying that equation with u', integrating over ξ, and taking into account that $u' \to 0$ as $\xi \to \pm\infty$, one finds

$$c_0 \int_{-\infty}^{\infty} [u'(\xi)]^2 \, d\xi = \int_0^1 f(u)\, du;$$

hence, the sign of c_0 must equal the sign of the integral of f over $(0, 1)$. □

As in Section 2, we will study front propagation in (4.1) in the framework of the first-order system

$$(4.3a) \qquad\qquad u' = v,$$
$$(4.3b) \qquad\qquad v' = -cv - ug(u)H(u - \varepsilon),$$
$$(4.3c) \qquad\qquad \varepsilon' = 0$$

that is equivalent to the traveling front equation corresponding to (4.1); see also (1.7). For ε small and fixed, the relevant equilibrium points of (4.3) are found at $Q_\varepsilon^- = (\phi^-, 0, \varepsilon)$ and $Q_\varepsilon^+ = (0, 0, \varepsilon)$; recall Assumption $\mathcal{A}_1$. These points are hyperbolic saddle equilibria, in (u, v), of the corresponding system

$$(4.4a) \qquad\qquad u' = v,$$
$$(4.4b) \qquad\qquad v' = -cv - ug(u)$$

that is obtained from (4.3) in the absence of a cut-off; the associated eigenvalues are given by $\lambda_\pm^- = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4\phi^- g'(\phi^-)}$ and $\lambda_\pm^+ = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4g(0)}$, respectively. Traveling front solutions of (4.1) propagating with speed c between the rest states at ϕ^- and 0 then correspond to heteroclinic orbits that connect Q_ε^- and Q_ε^+ in (4.3), as before. In particular, Assumption $\mathcal{A}_2$ implies that there exists a heteroclinic connection between Q_0^- and Q_0^+ for a locally unique value

of the parameter $c = c_0$ in the equations without cut-off in (4.4). Finally, c_0 is non-negative by Assumption $\mathcal{A}_3$.

The following theorem is the main result of this section:

Theorem 4.1. *Let $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small. Then, there exists a unique value $c(\varepsilon)$ of c such that Equation (4.1) possesses a unique traveling front solution propagating between ϕ^- and $\phi^+(=0)$. Moreover, $c(\varepsilon) = c(0) + \Delta c(\varepsilon)$, where $c(0) = c_0$ (the propagation speed in the absence of a cut-off) and Δc is a positive, $\mathcal{C}^1$-smooth function in ε (including at $\varepsilon = 0$) and p that satisfies*

$$
(4.5) \qquad \Delta c(\varepsilon) = K\varepsilon^p + o(\varepsilon^p),
$$

with

$$
(4.6) \qquad p = \frac{2\sqrt{c_0^2 - 4g(0)}}{c_0 + \sqrt{c_0^2 - 4g(0)}}
$$

and

$$
(4.7) \qquad K = |f'(0)|^{\frac{p}{2}} \left[\frac{-c_0 + \sqrt{c_0^2 - 4f'(0)}}{c_0 + \sqrt{c_0^2 - 4f'(0)}} \right]^{1 - \frac{p}{2}} \frac{[c_0^2 - 4f'(0)]^{\frac{1-p}{2}}}{\delta(p)}
$$

a positive constant. Here, δ is a $\mathcal{C}^\infty$-smooth, positive function that is defined as

$$
(4.8) \qquad \delta(p) = \lim_{\rho \to 0^+} \left\{ \rho^{p-1} \frac{\partial v}{\partial c}(\rho, c_0) \right\},
$$

where $\frac{\partial v}{\partial c}(u, c_0)$ denotes the solution of the variational equation corresponding to (4.3), taken along the heteroclinic orbit $v(u, c_0)$.

We remark that, in general, the value of the leading-order coefficient K in (4.5) cannot be determined in closed form, as the function $\delta(p)$ defined in (4.8) can only be evaluated exactly if a closed-form expression for $\frac{\partial v}{\partial c}(u, c_0)$ is known. As will become clear in the following, explicit knowledge of $v(u, c_0)$ is a necessary, but not a sufficient, condition for the computability of $\frac{\partial v}{\partial c}$ and, hence, of K.

***Remark* 17.** The Nagumo equation discussed in Section 2 is a special case of the very general scenario considered here, as Equation (2.1) satisfies (4.2) with $g(\phi) = (1 - \phi)(\phi - \gamma)$ and $\phi^- = 1$. Correspondingly, the leading-order ε-asymptotics of $c(\varepsilon)$, as stated in Theorem 4.1, agrees with the results of Section 2 in that case: substituting $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ and $g(0) = -\gamma$ into (4.6) and (4.7), we recover $p = 1 + 2\gamma$ and $K = K_\gamma$, as in the statement of Theorem 2.1. Similarly, the Schlögl equation discussed in Section 3 is of the form in (4.1), with $g(\phi) = -[\phi^2 + (\sigma - 3)\phi + 2(1 - \sigma)]$ and $\phi^- = 2$: since $c_0 = \sqrt{2}\sigma$ and $g(0) = -2(1 - \sigma)$, it follows that $p = 2 - \sigma$ and $K = K_\sigma$, as claimed in Corollary 3.1. $\qquad \square$

***Remark* 18.** To obtain classical bistable reaction kinetics in (4.1), one would typically require the following additional assumption; cf. again [12] for details and references.

Assumption $\mathcal{A}_1'$. The function g defined in Assumption $\mathcal{A}_1$ has exactly two roots. Moreover, $g'(\phi^\circ)$ is strictly positive.

Assuming, without loss of generality, that $\phi^- = 1$ and $\phi^\circ = \gamma$ for $\gamma \in (0, 1)$, one could rewrite (4.1) as

$$
(4.9) \qquad \phi_t = \phi_{xx} + \phi(1 - \phi)(\phi - \gamma)\tilde{g}(\phi)H(\phi - \varepsilon),
$$

where $\tilde{g}(\phi) = 1 + O(\phi)$ denotes a smooth function that is strictly positive on $(0, 1)$. Assumption $\mathcal{A}_2$ would then imply the existence of a front solution propagating between the rest states at 1 and 0 in (4.9). For the propagation speed c_0 to be non-negative, we would assume $\int_0^1 \phi(1 - \phi)(\phi - \gamma)\tilde{g}(\phi)\,d\phi > 0$, as in Assumption $\mathcal{A}_3$.

Equation (4.9) is, to leading order, precisely the Nagumo equation studied in Section 2 and can be analyzed accordingly. The proof of Theorem 4.1, however, only requires information on

g in a neighborhood of the rest states at ϕ^- and $\phi^+(= 0)$, in addition to the global requirement that a front solution connecting the two exists. Hence, the stricter Assumption $\mathcal{A}_1'$ is of no relevance to us and will not be imposed here. □

The proof of Theorem 4.1 follows the procedure outlined in the Introduction: the origin in (4.3) is desingularized via the blow-up transformation in (1.8). The singular heteroclinic connection Γ is then constructed in the blown-up vector field that is induced by (4.3) on $\mathbb{S}_+^2$. The required construction is again performed in the two charts K_2 and K_1, in which the dynamics of (4.3) is analyzed in the inner and intermediate regions, respectively; the dynamics in the outer region can conveniently be described in the original (u, v, ε)-variables, as before. Finally, persistence of Γ as a heteroclinic connection between Q_ε^- and Q_ε^+ for $\varepsilon > 0$ sufficiently small may be established in exactly the same manner as was done in Sections 2 and 3; see also [8]. As before, that connection will yield the sought-after traveling front solution of (4.1) that propagates between ϕ^- and 0, with speed $c(\varepsilon)$. Since the corresponding analysis is in many ways similar to that presented in Section 2, we will omit some of the details in the following.

4.1. Construction of Γ.

4.1.1. *Outer region.* In the outer region, which is again defined by $u = O(1)$, the governing equations are given by (4.3) with $H \equiv 1$, *i.e.*, by (4.4). (Equivalently, the dynamics in this region is governed by the corresponding traveling front equation $u'' + cu' + ug(u) = 0$; cf. Section 2.1.1.) As noted above, we assume the existence of a heteroclinic connection between $u = \phi^-$ and $u = 0$ in that equation for some (locally unique, non-negative) value c_0 of the front speed c. In the context of (4.4), this heteroclinic orbit corresponds precisely to the unstable manifold $\mathcal{W}^u(Q_0^-)$ of the point Q_0^-, since Assumption $\mathcal{A}_2$ implies $(u, v) \to (\phi^-, 0)$ as $\xi \to -\infty$. Moreover, as before, we may assume an expansion for that manifold of the form

$$(4.10) \qquad v(u, c) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j v}{\partial c}(u, c_0)(\Delta c)^j,$$

where $\Delta c = c - c_0$ is taken to be $o(1)$; cf. (2.8). For future reference, we note that the leading-order asymptotics of the lowest-order term $v(u, c_0)$ in (4.10) near $u = 0$ is given by the linear approximation $v(u, c_0) = \lambda_-^+ u + O(u^2)$, where $\lambda_-^+ = -\frac{c_0}{2} - \frac{1}{2}\sqrt{c_0^2 - 4g(0)}$, as above, with c replaced with c_0. (In fact, noting that $v(u, c_0)$ must be $\mathcal{C}^\infty$-smooth in u as $u \to 0^+$, we may make a series expansion for v, which can be used to determine the asymptotics of $v(u, c_0)$ near $u = 0$ to arbitrary order.)

The next-order term $\frac{\partial v}{\partial c}(u, c)$ is again obtained as the solution of the variational equation associated to (4.4), taken along the orbit $v(u, c_0)$ that corresponds to the traveling front solution $u(\xi)$ of the problem without cut-off; recall Section 2.1.1. Rewriting (4.4) with u as the independent variable, differentiating the resulting equation with respect to c, and evaluating at c_0, we find

$$(4.11) \qquad \frac{\partial}{\partial u}\left(\frac{\partial v}{\partial c}(u, c_0)\right) = -1 - \frac{c_0 + \frac{\partial v}{\partial u}(u, c_0)}{v(u, c_0)}\frac{\partial v}{\partial c}(u, c_0) = -1 + \frac{ug(u)}{[v(u, c_0)]^2}\frac{\partial v}{\partial c}(u, c_0).$$

(Here, the last equality follows immediately from (4.4).) While (4.11) will, in general, have no explicit solution, as not even $v(u, c_0)$ will typically be known in closed form, we can still prove that $\frac{\partial v}{\partial c}(u, c_0)$ will be strictly positive on $(0, \phi^-)$.

Lemma 4.1. *The unique solution $\frac{\partial v}{\partial c}(u, c_0)$ of (4.11) that satisfies $\frac{\partial v}{\partial c}(\phi^-, c_0) = 0$ is strictly positive for any $u \in (0, \phi^-)$.*

Proof. The proof is based on a phase plane analysis of the first-order system

$$\dot{u} = u(\phi^- - u),$$

$$\dot{w} = -u(\phi^- - u) + \frac{u^2(\phi^- - u)g(u)}{[v(u, c_0)]^2}w$$

that is equivalent to the variational equation in (4.11), with $w \equiv \frac{\partial v}{\partial c}$. Considering the linearization of the above system at the saddle equilibrium that is located at $(\phi^-, 0)$, we find

$$
\begin{bmatrix} -\phi^- & 0 \\ \phi^- & \frac{u^2(\phi^- - u)g(u)}{[v(u,c_0)]^2} \big|_{u=\phi^-} \end{bmatrix} = \begin{bmatrix} -\phi^- & 0 \\ \phi^- & \frac{(\phi^-)^2 g'(\phi^-)}{(\lambda_+^-)^2} \end{bmatrix}.
$$

As $\frac{(\phi^-)^2 g'(\phi^-)}{(\lambda_+^-)^2} < 0$, it follows that the only orbits that are asymptotic to $(\phi^-, 0)$ for $u \to \phi^-$ are those on the corresponding stable manifold. Since the slope of that manifold is negative and since, clearly, $\dot{w}|_{\{w=0\}} < 0$ for $u \in (0, \phi^-)$, w is strictly positive on this interval (and, in fact, becomes unbounded as $u \to 0^+$, which can be seen by linearizing the equations at the saddle equilibrium at the origin). $\qquad\square$

***Remark* 19.** The qualitative dynamics of Equation (4.11) in this general setting agrees with that of the exactly solvable variational equation in (2.9). However, as will become clear in the following, the limit as $u \to 0^+$ in $\frac{\partial v}{\partial c}(u, c_0)$ cannot be evaluated analytically unless $v(u, c_0)$ is known explicitly, as was the case in Section 2.1. $\qquad\square$

Finally, as in Section 2.1, let Σ^- be defined as the hyperplane $\{u = \rho\}$ in (u, v, ε)-space, with $\rho \geq \varepsilon_0$ for $\varepsilon_0 > 0$ sufficiently small, and let P_0^- denote the point of intersection of $\mathcal{W}^u(Q_0^-)$ with Σ^-. Then, the segment of Γ that is located in this outer region is given by the manifold $\mathcal{W}^u(Q_0^-)$, restricted to $\{u \geq \rho\}$. In the context of (4.3), that singular heteroclinic connection is determined precisely by the leading-order term $v(u, c_0)$ in (4.10).

4.1.2. *Inner region.* The dynamics of (4.3) in the inner region is governed by the cut-off system that is obtained by imposing $H \equiv 0$ in (4.3b). Since the blow-up transformation defined in (1.8) is again given by (2.11) in the corresponding (rescaling) chart K_2, the resulting equations are identical to those found previously in (2.12); cf. Section 2.1.2.

In particular, the line of equilibria ℓ_2^+, which is the segment of the r_2-axis obtained for $r_2 \in [0, r_0]$, again corresponds to the point Q_ε^+, after blow-up and transformation to chart K_2; given $r_2(= \varepsilon)$ fixed, we write $Q_2^+ = (0, 0, r_2) \in \ell_2^+$, as before. In the limit as $r_2 \to 0^+$, the unique solution of the resulting singular equation $\frac{dv_2}{du_2} = -c_0$, with $v_2(0) = 0$, is found as $v_2(u_2) = -c_0 u_2$. The corresponding orbit coincides with the stable manifold $\mathcal{W}_2^s(Q_{02}^+)$ of $Q_{02}^+ = (0, 0, 0)$ and intersects the section Σ_2^+ defined in (2.14) in the point $P_{02}^+ = (1, -c_0, 0)$, as before. Thus, the restriction of $\mathcal{W}_2^s(Q_{02}^+)$ to $\{u_2 < 1\}$ yields precisely the segment Γ_2^+ of the sought-after singular connection Γ that lies in this inner region; see again Figure 1 for an illustration.

4.1.3. *Intermediate region.* The system of equations corresponding to (4.3) in the intermediate region, where $\varepsilon < u < O(1)$ and $H \equiv 1$, is given by

(4.12a) $\qquad\qquad\qquad r_1' = r_1 v_1,$

(4.12b) $\qquad\qquad\qquad v_1' = -c v_1 - v_1^2 - g(0) - [g(r_1) - g(0)],$

(4.12c) $\qquad\qquad\qquad \varepsilon_1' = -\varepsilon_1 v_1,$

as the blow-up transformation in chart K_1 again reduces to (2.15); recall the equations in (2.16). (Here, we note that $g(r_1) - g(0) = O(r_1)$ may be of higher order, due to our assumptions on g in (4.2).) The two equilibria of (4.12) are located at $P_1^s = (0, \lambda_-^+, 0)$ and $P_1^u = (0, \lambda_+^+, 0)$, where $\lambda_\pm^+ = -\frac{c_0}{2} \pm \frac{1}{2}\sqrt{c_0^2 - 4g(0)}$, as before; these equilibria correspond to the stable eigendirection and the unstable eigendirection, respectively, of the linearization at Q_0^+ of (4.4), in the absence of a cut-off. The associated eigenvalues are given by λ_-^+, $-c_0 - 2\lambda_-^+$, and $-\lambda_-^+$, respectively, λ_+^+, $-c_0 - 2\lambda_+^+$, and $-\lambda_+^+$. The relevant equilibrium for us is again P_1^s, since $\frac{u'}{u} \to \lambda_-^+$ must hold as $\xi \to \infty$ along the traveling front solution of (4.1) that corresponds to the singular heteroclinic orbit Γ.

The portion of that orbit that is located in this intermediate region can be found by analyzing the dynamics of (4.12) in the invariant hyperplanes defined by $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$, as before. Specifically, we denote by Γ_1^- the singular orbit obtained for $\varepsilon_1 = 0$ that is asymptotic to P_1^s as $\xi \to \infty$, and we write Γ_1^+ for the orbit that asymptotes to P_1^s as $\xi \to -\infty$ in $\{r_1 = 0\}$; recall Figure 2. Then, the restriction of Γ to the intermediate region is given by the union of Γ_1^-, P_1^s, and Γ_1^+, as indicated in Figure 3 above.

This completes the construction of the singular heteroclinic orbit Γ.

Remark 20. The exponent p in (4.6) is given precisely by the ratio of the second and third eigenvalues of the linearization of (4.12) at P_1^s, as noted already in Section 2.1.2; cf. Remark 8.
□

4.2. Existence and asymptotics of $c(\varepsilon)$.
As in Section 2.2, the persistence of the singular heteroclinic connection Γ for a unique value $c(\varepsilon)$ of c in (4.5), with $\varepsilon \in (0, \varepsilon_0]$ sufficiently small, can be proven by considering the transition map $\Pi_1 : \Sigma_1^- \mapsto \Sigma_1^+$ between the two sections Σ_1^- and Σ_1^+ defined in (2.18). Moreover, the corresponding persistence proof will again yield the leading-order ε-asymptotics of $c(\varepsilon)$, provided Π_1 is described sufficiently accurately to the order considered here. The required analysis is carried out entirely in chart K_1, as before.

4.2.1. *Preparatory analysis.* We begin by introducing the new variables $\Delta c = c - c_0$ and $z = v_1 + \frac{1}{2}c_0$ in (4.12):

$$(4.13\text{a}) \qquad\qquad r_1' = -(\tfrac{1}{2}c_0 - z)r_1,$$

$$(4.13\text{b}) \qquad\qquad z' = (\tfrac{1}{2}c_0 - z)\Delta c - z^2 + \tfrac{1}{4}c_0^2 - g(0) - [g(r_1) - g(0)],$$

$$(4.13\text{c}) \qquad\qquad \varepsilon_1' = (\tfrac{1}{2}c_0 - z)\varepsilon_1.$$

Reparametrizing the independent variable in (4.13) by dividing out the (positive) factor of $\frac{1}{2}c_0 - z$, we find

$$(4.14\text{a}) \qquad\qquad r_1' = -r_1,$$

$$(4.14\text{b}) \qquad\qquad z' = \Delta c - \frac{z^2 - [\tfrac{1}{4}c_0^2 - g(0)]}{\tfrac{1}{2}c_0 - z} + \frac{g(0) - g(r_1)}{\tfrac{1}{2}c_0 - z},$$

$$(4.14\text{c}) \qquad\qquad \varepsilon_1' = \varepsilon_1,$$

cf. (2.20), where the prime again denotes differentiation with respect to the new variable ζ.

The normal form equations corresponding to (4.14) can now be obtained as in the proof of Proposition 2.1:

Proposition 4.1. *Let* $\mathcal{V} := \big\{(r_1, z, \varepsilon_1) \,\big|\, (r_1, z, \varepsilon_1) \in [0, \rho] \times [-z_0, 0] \times [0, 1]\big\}$, *where* $z_0 = v_0 + \frac{c_0}{2}$, *with* v_0 *as in* (2.18). *Then, there exists a* $\mathcal{C}^\infty$-*smooth coordinate transformation*

$$\psi : \begin{cases} \mathcal{V} \to \psi(\mathcal{V}), \\ (r_1, z, \varepsilon_1) \mapsto (r_1, \hat{z}, \varepsilon_1), \end{cases}$$

with $\hat{z}(z, r_1) = z + O(r_1)$, *such that* (4.13) *can be written as*

$$(4.15\text{a}) \qquad\qquad r_1' = -r_1,$$

$$(4.15\text{b}) \qquad\qquad \hat{z}' = \Delta c - \frac{\hat{z}^2 - [\tfrac{1}{4}c_0^2 - g(0)]}{\tfrac{1}{2}c_0 - \hat{z}},$$

$$(4.15\text{c}) \qquad\qquad \varepsilon_1' = \varepsilon_1.$$

As in Section 2, we write P^- and P_2^+ for the points of intersection of $\mathcal{W}^u(Q_\varepsilon^-)$ and $\mathcal{W}_2^s(Q_2^+)$ with Σ^- and Σ_2^+, respectively, where $\varepsilon \in (0, \varepsilon_0]$. Let P_1^- and P_1^+ denote the corresponding respective points in $(r_1, v_1, \varepsilon_1)$-coordinates. Given Proposition 4.1, we can then derive the following estimates for the $\hat{z}$-coordinates of the points $\widehat{P}_1^\mp$ that are obtained from $P_1^\mp$ after application of the normal form transformation ψ defined in Proposition 4.1:

Lemma 4.2. *For any $\rho \in (\varepsilon, 1)$, with $\varepsilon \in (0, \varepsilon_0]$ and Δc sufficiently small, the points $\widehat{P}_1^- = (\rho, \hat{z}^-, \varepsilon \rho^{-1})$ and $\widehat{P}_1^+ = (\varepsilon, \hat{z}^+, 1)$ satisfy*

(4.16)
$$\hat{z}^- = \hat{z}^-(\rho, \Delta c) = -\tfrac{1}{2}\sqrt{c_0^2 - 4g(0)} + \nu(\rho, \Delta c)\Delta c, \quad with \quad \nu(\rho, 0) = \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)[1 + \nu_1(\rho)] > 0,$$

and

(4.17)
$$\hat{z}^+ = \hat{z}^+(\Delta c, \varepsilon) = -\left(\frac{c_0}{2} + \Delta c\right) + \omega(\Delta c, \varepsilon)\varepsilon.$$

Here, $\nu(\rho, \Delta c)$ is a $\mathcal{C}^\infty$-smooth function in ρ and Δc, while ν_1 is $\mathcal{C}^\infty$-smooth down to $\rho = 0$, with $\nu_1(0) = 0$. Finally, $\omega(\Delta c, \varepsilon)$ is $\mathcal{C}^\infty$-smooth in Δc and ε, including in a neighborhood of $(0, 0)$.

Proof. The proof is analogous to that of Lemma 2.2: making use of the expansion for $v(u, c)$ in (4.10), we find (2.24), as before. Next, we recall that $v(\rho, c_0) = -\lambda_-^+ \rho + O(\rho^2)$; see Section 4.1.1. Given that $v^- = \rho v_1^-$ as well as that $z^- = v_1^- + \frac{c_0}{2}$, we obtain

$$z^- = -\tfrac{1}{2}\sqrt{c_0^2 - 4g(0)} + O(\rho) + \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)\Delta c + O[(\Delta c)^2];$$

cf. the proof of Lemma 2.2. Applying the normal form (near-identity) transformation ψ to z^- and noting that $\hat{z}^- = \pm\tfrac{1}{2}\sqrt{c_0^2 - 4g(0)}$ is invariant for $\Delta c = 0$ in (4.15b), we have

$$\hat{z}^- = -\tfrac{1}{2}\sqrt{c_0^2 - 4g(0)} + \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)[1 + \nu_1(\rho)]\Delta c + \nu_2(\rho, \Delta c)(\Delta c)^2$$

$$\equiv -\tfrac{1}{2}\sqrt{c_0^2 - 4g(0)} + \nu(\rho, \Delta c)\Delta c,$$

where ν_1 and ν_2 are $\mathcal{C}^\infty$-smooth in ρ and $(\rho, \Delta c)$, respectively; moreover, ν_1 is smooth down to $\rho = 0$, as before, with $\nu_1(0) = 0$. Finally, $\nu(\rho, 0)$ is positive for $\rho \in (0, 1)$ sufficiently small, by Lemma 4.1, which yields (4.16), as claimed.

The corresponding expression for $\hat{z}^+$ is obtained by noting that the v_1-coordinate of P_1^+ must necessarily satisfy $v_1^+ = -c_0 - \Delta c$ and, hence, that $z^+ = -\frac{c_0}{2} - \Delta c$ must hold. Recalling that $\hat{z} = z + O(r_1)$ and $r_1 = \varepsilon$ in Σ_1^+, we find (4.17), which completes the argument. $\qquad\square$

4.2.2. *Uniqueness of Δc.* Let $\hat{z}_-$ denote the solution to (4.15b) with initial value $\hat{z}^-(\rho, \Delta c)$, and let $\hat{z}_-^+ \equiv \hat{z}_-(\zeta^+)$, with $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$, as before. In analogy to Lemma 2.3, it then follows that the singular heteroclinic connection Γ can persist in the transition through the intermediate region for at most one value of Δc:

Lemma 4.3. *For $\hat{z}_-^+$ defined as above, there holds $\frac{\partial \hat{z}_-^+}{\partial c}(\rho, \Delta c) > 0$. Moreover, there can exist at most one value of Δc such that $\hat{z}_-^+(\rho, \Delta c) = \hat{z}^+(\Delta c, \varepsilon)$, where $\hat{z}^+$ is as in (4.17).*

4.2.3. *Existence and asymptotics of Δc.* Both the existence of $c(\varepsilon)$ and its leading-order ε-asymptotics can now be obtained from the following analogue of Proposition 2.2:

Proposition 4.2. *Let $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small. Then, there exists a function $c(\varepsilon) = c_0 + \Delta c(\varepsilon)$, with $\Delta c(0) = 0$, such that the singular orbit Γ persists if and only if $c = c(\varepsilon)$ in (4.1). Moreover, Δc is positive, and $\mathcal{C}^1$-smooth in ε (including at $\varepsilon = 0$) and p, where p is defined as*

(4.18)
$$p = \frac{2\sqrt{c_0^2 - 4g(0)}}{c_0 + \sqrt{c_0^2 - 4g(0)}}.$$

Proof. As in the proof of Proposition 2.2, we first integrate (4.15b) by separating variables to find

$$(4.19) \quad \zeta^+ - \zeta^- - \frac{1}{2}\ln\left|2\hat{z}^2 + 2\Delta c\hat{z} - c_0\Delta c - \tfrac{1}{2}c_0^2 + 2g(0)\right|\Big|_{\hat{z}^-}^{\hat{z}^+}$$

$$- \frac{c_0 + \Delta c}{\sqrt{-4g(0) + c_0^2 + 2c_0\Delta c + (\Delta c)^2}}\operatorname{arctanh}\left(\frac{2\hat{z} + \Delta c}{\sqrt{-4g(0) + c_0^2 + 2c_0\Delta c + (\Delta c)^2}}\right)\Big|_{\hat{z}^-}^{\hat{z}^+} = 0.$$

Recalling that $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$ and $\zeta^- = 0$, substituting in the expressions for $\hat{z}^+$ and $\hat{z}^-$ from (4.17) and (4.16), respectively, and making use of the identity $\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$, we obtain

$$(4.20) \quad -\ln\frac{\varepsilon}{\rho} - \frac{1}{2}\ln\left|2g(0) - 2c_0\omega(\Delta c, \varepsilon)\varepsilon + O(2)\right|$$

$$+\frac{1}{2}\ln\left|-\left\{c_0 + \sqrt{c_0^2 - 4g(0)}[1 + 2\nu(\rho, 0)] + O(1)\right\}\Delta c\right| - \frac{1}{2}\left\{\frac{c_0}{\sqrt{c_0^2 - 4g(0)}} - \frac{4g(0)}{(c_0^2 - 4g(0))^{\frac{3}{2}}}\Delta c + O(2)\right\}$$

$$\times\left\{\ln\left|\frac{-c_0 + \sqrt{c_0^2 - 4g(0)}}{c_0 + \sqrt{c_0^2 - 4g(0)}} + \frac{8g(0)}{\sqrt{c_0^2 - 4g(0)}[c_0 + \sqrt{c_0^2 - 4g(0)}]^2}\Delta c\right.\right.$$

$$+\frac{4\sqrt{c_0^2 - 4g(0)}}{[c_0^2 + \sqrt{c_0^2 - 4g(0)}]^2}\omega(\Delta c, \varepsilon)\varepsilon + O(2)\Bigg| - \ln\left|\frac{c_0 + \sqrt{c_0^2 - 4g(0)}[1 + 2\nu(\rho, 0)] + O(1)}{2[c_0^2 - 4g(0)]}\Delta c\right|\Bigg\} = 0.$$

Here, $O(1)$ and $O(2)$ denote first-order terms in Δc and second-order terms in $(\Delta c, \varepsilon)$, respectively, that are $\mathcal{C}^\infty$-smooth and uniform as long as ρ is restricted to compact subsets of $(0, 1)$; cf. the proof of Proposition 2.2.

Given that (4.20) is solvable for at most one value of Δc, as shown in Lemma 4.3, we may attempt to find a solution for positive Δc first. Since $\nu(\rho, 0) > 0$, by Lemma 4.1, we obtain

$$(4.21) \quad \left(\frac{\varepsilon}{\rho}\right)^{2\sqrt{c_0^2 - 4g(0)}} = [2|g(0)|]^{-\sqrt{c_0^2 - 4g(0)}}\left\{\left[c_0 + \sqrt{c_0^2 - 4g(0)}[1 + 2\nu(\rho, 0)]\right]\Delta c\right\}^{c_0 + \sqrt{c_0^2 - 4g(0)}}$$

$$\times\left[\frac{c_0 + \sqrt{c_0^2 - 4g(0)}}{-c_0 + \sqrt{c_0^2 - 4g(0)}}\right]^{c_0}\left[\frac{c_0 + \sqrt{c_0^2 - 4g(0)}[1 + 2\nu(\rho, 0)]}{2[c_0^2 - 4g(0)]}\right]^{c_0}[1 + O(1)],$$

where $O(1)$ denotes terms that are $\mathcal{C}^\infty$-smooth in Δc, $\Delta c\ln(\Delta c)$, and ε. As in the proof of Proposition 2.2, the Implicit Function Theorem now implies that (4.21) has a solution $\Delta c(\varepsilon, p, \rho)$, for any $\rho \in (0, 1)$ and p as defined in (4.18), which is $\mathcal{C}^1$-smooth in ε (down to $\varepsilon = 0$), p, and ρ. Moreover, Δc is necessarily unique, and independent of ρ, as it again yields the unique value of c for which Γ persists, as a heteroclinic connection between Q_ε^- and Q_ε^+ in (4.3), irrespective of the definition of Σ^-.

Finally, solving (4.21) for $\Delta c(\varepsilon) \equiv \Delta c(\varepsilon, p)$ and taking into account that $g(0) = f'(0) < 0$, by (4.2), we find $\Delta c(\varepsilon) = K\varepsilon^p + o(\varepsilon^p)$ for the leading-order ε-asymptotics of Δc. Here, the constant K is given by

$$(4.22) \quad K = 2|f'(0)|^{\frac{p}{2}}\left[\frac{-c_0 + \sqrt{c_0^2 - 4f'(0)}}{c_0 + \sqrt{c_0^2 - 4f'(0)}}\right]^{1-\frac{p}{2}}\frac{[c_0^2 - 4f'(0)]^{1-\frac{p}{2}}}{c_0 + \sqrt{c_0^2 - 4f'(0)}[1 + 2\nu(\rho, 0)]}\frac{1}{\rho^p}$$

$$\equiv |f'(0)|^{\frac{p}{2}}\left[\frac{-c_0 + \sqrt{c_0^2 - 4f'(0)}}{c_0 + \sqrt{c_0^2 - 4f'(0)}}\right]^{1-\frac{p}{2}}\frac{[c_0^2 - 4f'(0)]^{\frac{1-p}{2}}}{\delta(p)},$$

with

$$(4.23) \quad \delta(p) = \left[\frac{1}{2} + \frac{c_0}{2\sqrt{c_0^2 - 4f'(0)}} + \nu(\rho, 0)\right]\rho^p$$

a strictly positive, $\mathcal{C}^\infty$-smooth function, which concludes the argument. $\qquad\square$

This completes the proof of Theorem 4.1.

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***Remark* 21.** The numerical value of the leading-order coefficient K in the ε-asymptotics of Δc will, in general, depend on the choice of Θ in (1.5); recall Remark 12. In other words, while the exponent p in that asymptotics is universal within the class of cut-off functions defined in (1.4), cf. (4.6), the value of K given in (4.7) is specific to the Heaviside cut-off H. $\qquad\square$

4.3. Computability of Δc. We conclude this section by discussing the computability of the correction Δc that is induced by the cut-off in (4.1), for general f. We begin by noting that the exponent p in the ε-asymptotics of Δc will always be computable if the front propagation speed c_0 in the absence of a cut-off is known; cf. Equation (4.6). However, to determine the value of the corresponding coefficient K, as defined in (4.7), in closed form, we would need to evaluate $\delta(p)$. Since the definition of δ must be independent of ρ, we may take the limit as $\rho \to 0^+$ in (4.23),

$$(4.24) \qquad \delta(p) = \lim_{\rho \to 0^+} \big\{ \rho^p \nu(\rho, 0) \big\} = \lim_{\rho \to 0^+} \Big\{ \rho^{p-1} \frac{\partial v}{\partial c}(\rho, c_0) \Big\},$$

see Section 2.2, where $\frac{\partial v}{\partial c}$ denotes the solution of the variational equation in (4.11), which we restate for convenience here:

$$(4.25) \qquad \frac{\partial}{\partial u}\Big(\frac{\partial v}{\partial c}(u, c_0) \Big) = -1 + \frac{ug(u)}{[v(u, c_0)]^2} \frac{\partial v}{\partial c}(u, c_0).$$

Equation (4.25) can be solved by variation of constants; however, an exact solution for $\frac{\partial v}{\partial c}(u, c_0)$ can only be found in cases where a solution to the corresponding traveling front problem without cut-off is known explicitly.

To clarify this point further, we recall the linear approximation for $v(u, c_0)$ from Section 4.1.1: $v(u, c_0) = \lambda_-^+ u + O(u^2)$, with $\lambda_-^+ = -\frac{c_0}{2} - \frac{1}{2}\sqrt{c_0^2 - 4g(0)}$. Substituting into (4.25) and solving the resulting approximate equation

$$\frac{\partial}{\partial u}\Big(\frac{\partial v}{\partial c}(u, c_0) \Big) = -1 + \frac{g(0)}{(\lambda_-^+)^2} \frac{\partial v}{\partial c}(u, c_0),$$

we find the leading-order solution

$$(4.26) \qquad \frac{\partial v}{\partial c}(u, c_0) = -\frac{(\lambda_-^+)^2}{-g(0) + (\lambda_-^+)^2} u + C u^{\frac{g(0)}{(\lambda_-^+)^2}}$$

for $\frac{\partial v}{\partial c}(u, c_0)$. Since, however, that solution is only valid locally, in a neighborhood of $u = 0$, the constant of integration C has to remain undetermined, as the boundary condition on $\frac{\partial v}{\partial c}$ is prescribed at $u = \phi^-$. Knowledge of C, on the other hand, is necessary for evaluating (4.24): since $\frac{g(0)}{(\lambda_-^+)^2} < 0$, the dominant term in the u-asymptotics of $\frac{\partial v}{\partial c}$ is precisely that second term in (4.26). (In fact, recalling our discussion of the Nagumo and Schlögl equations in Sections 2 and 3, respectively, one finds that the order (in u) in the leading-order asymptotics of $\frac{\partial v}{\partial c}$, which was found as -2γ and $\sigma - 1$ in (2.10) and (3.9), respectively, equals $1 - p = \frac{g(0)}{(\lambda_-^+)^2}$.)

Similarly, approximating $v(u, c_0)$ locally in a neighborhood of ϕ^-, one would obtain a leading-order expression for $\frac{\partial v}{\partial c}$ that satisfies the boundary condition $\frac{\partial v}{\partial c}(\phi^-, c_0) = 0$. However, that approximation will not be valid up to and including the equilibrium state at zero. In other words, the two expansions cannot be equivalent unless $v(u, c_0)$ is known explicitly and in closed form. (Here, we only consider equivalence from an analytical point of view, as Equation (4.25) can, in principle, be integrated numerically, and the resulting approximation for $\frac{\partial v}{\partial c}$ evaluated in Σ^-, to obtain an approximate value for K.)

Hence, we conclude that, while the exact form of g does not play a role in determining the solution asymptotics of (4.25), knowledge of an explicit solution to the traveling front problem in the absence of a cut-off – and, consequently, of $v(u, c_0)$ – is crucial for evaluating the leading-order coefficient K in the ε-asymptotics of Δc in closed form. Finally, we remark that it might not be possible to evaluate K even then: while explicit knowledge of $v(u, c_0)$ certainly is a

necessary condition for the closed-form computability of K, the integrals that arise in solving (4.25) may not be computable in closed form even when $v(u, c_0)$ is known explicitly.

Remark 22. The observation that an exact solution to the traveling front problem without cut-off has to be available for the coefficient K defined in (4.7) to be computable was also made in [4]: there, it was shown that K can always be evaluated in closed form if the function that maximizes the functional underlying their variational approach is known explicitly. We conjecture that this requirement is in fact equivalent to the condition that the variational equation in (4.25) can be solved exactly. However, a proof is beyond the scope of this article. $\qquad\square$

Appendix A. Proof of Lemma 2.1

In this appendix, we give the proof of Lemma 2.1, which we restate for reference here:

Lemma A.1. *For $u \in (0, 1]$, the unique solution $\frac{\partial v}{\partial c}(u, c_0)$ to*

$$(A.1) \qquad \frac{\partial}{\partial u}\left(\frac{\partial v}{\partial c}(u, c_0)\right) = -1 + 2\frac{u - \gamma}{u(1 - u)}\frac{\partial v}{\partial c}(u, c_0)$$

that satisfies $\frac{\partial v}{\partial c}(1, c_0) = 0$ is given by

$$\frac{\partial v}{\partial c}(u, c_0) = \frac{1}{3 - 2\gamma}u^{-2\gamma}(1 - u)F(3 - 2\gamma, -2\gamma; 4 - 2\gamma; 1 - u),$$

where $F(\cdot, \cdot; \cdot; \cdot)$ denotes the hypergeometric function [1, Section 15]. In particular, $\frac{\partial v}{\partial c}(u, c_0)$ is strictly positive for any $u \in (0, 1)$.

Proof. We first note that the variational equation in (A.1) can be solved by variation of constants, which gives the general solution

$$(A.2) \qquad \frac{\partial v}{\partial c}(u, c_0) = Cu^{-2\gamma}(u - 1)^{-2(1-\gamma)} - \int_1^u s^{2\gamma}(s - 1)^{2(1-\gamma)}\,ds \cdot u^{-2\gamma}(u - 1)^{-2(1-\gamma)}.$$

To fix the constant of integration C in (A.2), we apply the boundary condition that $\frac{\partial v}{\partial c}(u, c_0)$ has to satisfy, with $\frac{\partial v}{\partial c} \to 0$ as $u \to 1^-$. Since the second term in the solution goes to zero, by l'Hôspital's Rule, we must set $C = 0$ for (A.2) to remain bounded (and, indeed, vanish) in that limit.

Next, we make the substitution $s \mapsto 1 - s$ in (A.2), which yields

$$(A.3) \qquad \begin{aligned} \frac{\partial v}{\partial c}(u, c_0) &= (-1)^{-2\gamma}\int_0^{1-u} s^{2(1-\gamma)}(1 - s)^{2\gamma}\,ds \cdot u^{-2\gamma}(u - 1)^{-2(1-\gamma)} \\ &= B_{1-u}(3 - 2\gamma, 1 + 2\gamma)u^{-2\gamma}(1 - u)^{-2(1-\gamma)}; \end{aligned}$$

here, B_x denotes the incomplete Beta function [1, Section 6.6], with

$$B_x(a, b) = \int_0^x t^{a-1}(1 - t)^{b-1}\,dt.$$

Finally, we apply the relation $B_x(a, b) = a^{-1}x^a F(a, 1 - b; a + 1; x)$, with F the hypergeometric function, see again [1, Sections 6.6 and 15], to rewrite (A.3) as

$$\frac{\partial v}{\partial c}(u, c_0) = \frac{1}{3 - 2\gamma}F(3 - 2\gamma, -2\gamma; 4 - 2\gamma; 1 - u)u^{-2\gamma}(1 - u).$$

The strict positivity of $\frac{\partial v}{\partial c}(u, c_0)$ on $(0, 1)$ now follows from the fact that F is strictly positive on that interval, which completes the proof. $\qquad\square$

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