The Burgers-FKPP advection-reaction-diffusion equation with cut-off

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Abstract

We investigate the effect of a Heaviside cut-off on the front propagation dynamics of the so-called Burgers-Fisher-Kolmogoroff-Petrowskii-Piscounov (Burgers-FKPP) advection-reaction-diffusion equation. We prove the existence and uniqueness of a "critical" travelling front solution in the presence of a cut-off in the reaction kinetics and the advection term, and we derive the leading-order asymptotics for the speed of propagation of the front in dependence on the advection strength and the cut-off parameter. Our analysis relies on geometric techniques from dynamical systems theory and specifically, on geometric desingularisation, which is also known as "blow-up".

1 Introduction

Partial differential equations (PDEs) of reaction-diffusion type are frequently derived from discrete many-particle systems in the large-scale limit as the number N of particles becomes infinite. However, discrepancies are observed between the propagation speeds of front solutions that are found numerically in the underlying many-particle systems and the corresponding speeds in the reaction-diffusion equations derived in the limit as $N \to \infty$ [1, 2, 3]. To remedy these discrepancies, Brunet and Derrida [4] introduced a cut-off in the resulting reaction kinetics; for a general reaction-diffusion equation with reaction kinetics $f(u)$, their modification takes the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)\psi(u,\varepsilon), \tag{1.1}$$

where the cut-off function ψ *a priori* only has to satisfy $\psi(u,\varepsilon) \equiv 1$ when $u > \varepsilon$ and $\psi(u,\varepsilon) < 1$ for $u < \varepsilon$. Here, $u = u(x,t)$, with $x \in \mathbb{R}$ and $t \geq 0$. The motivation in [4] was that, in N-particle systems, no reaction can take place if the particle density is below some threshold value $\frac{1}{N} = \varepsilon \ll 1$. Applying the method of matched asymptotics, they showed that for Fisher reaction kinetics $f(u) = u(1-u)$ [5] in (1.1) and a Heaviside cut-off $H(u-\varepsilon)$, with $H \equiv 0$ when $u < \varepsilon$, the shift in the propagation speed c of the front connecting the homogeneous rest states $u = 1$ and $u = 0$ that is due to a cut-off is, to leading order, given by $\Delta c = 2 - c = \frac{\pi^2}{(\ln\varepsilon)^2} + \mathscr{O}[(\ln\varepsilon)^{-3}]$.

The above asymptotics has been derived rigorously by Dumortier *et al.* [6] for a more general class of scalar reaction-diffusion equations and a broad family of cut-off functions. They applied geometric desingularisation, or "blow-up" [7, 8], to construct propagating front solutions to Equation (1.1) as heteroclinic connections in the corresponding first-order system of ordinary differential equations (ODEs) after transformation to a co-moving frame.

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Geometric desingularisation has since been applied successfully in the study of numerous other reaction-diffusion equations with a cut-off [6, 9, 10, 11, 12]. A well-developed alternative approach for determining the leading-order shift in the propagation speed due to a cut-off relies on a variational principle [13, 14, 15, 16].

In this article, we extend the results of [6, 9, 10, 11, 12] to a family of advection-reaction-diffusion equations of the form

$$\frac{\partial u}{\partial t} + g(u)\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.2}$$

with the advection term $g(u)\frac{\partial u}{\partial x}$ describing the directed transport of u. As far as we are aware, the impact of a cut-off on front propagation in advection-reaction-diffusion equations of the type in (1.2) has not been studied before. Specifically, we consider the following equation,

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \tag{1.3}$$

which is known as Burgers-Fisher-Kolmogorov-Petrowskii-Piscounov (Burgers-FKPP) equation [17, 18, 19, 20]; here, ku is the transport velocity, with $k > 0$ a real parameter. Equation (1.3) is a special case of the generalised FKPP equation, see [21], which models competing genotypes in a population and which has found applications in a variety of fields that include fluid dynamics, population modelling, and chemical kinetics. It hence serves as a prototypical model that illustrates the interaction between advection, reaction, and diffusion mechanisms in more general advection-reaction-diffusion equations of the type in (1.2). In particular, it realises a pushed front propagation regime which is not present in standard reaction-diffusion with Fisher reaction kinetics.

To study travelling wave solutions to (1.3), we introduce the travelling wave variable $\xi = x - ct$, where c denotes the propagation speed. Setting $U(\xi) = u(x,t)$, we obtain the travelling wave equation

$$-cU' + kUU' = U'' + U(1-U), \tag{1.4}$$

subject to the boundary conditions $U(-\infty) = 1$ and $U(\infty) = 0$. The corresponding solution defines a front for the Burgers-FKPP equation, (1.3), which connects the two rest states $u = 1$ and $u = 0$.

Defining $V = U'$ in (1.4), we obtain the first-order system

$$\begin{aligned} U' &= V, \\ V' &= -cV + kUV - U(1-U), \end{aligned} \tag{1.5}$$

which has equilibria at $Q^- := (1,0)$ and $Q^+ := (0,0)$. Clearly, heteroclinic orbits for (1.5) and front solutions to (1.3) are equivalent. The following result can be found in [22], where the existence of travelling front solutions to (1.3) is shown rigorously.

Theorem 1. [22, Theorem 4.1] Equation (1.5) admits a heteroclinic connection between Q^- and Q^+ for $c \geq c_{\text{crit}}$, where

$$c_{\text{crit}} = \begin{cases} 2 & \text{if } k \leq 2, \\ \frac{k}{2} + \frac{2}{k} & \text{if } k > 2. \end{cases} \tag{1.6}$$

Moreover, the corresponding front solution to (1.3) is pulled when $k \leq 2$ and pushed when $k > 2$. For $k \geq 2$ and $c_{\text{crit}} = \frac{k}{2} + \frac{2}{k}$, the heteroclinic orbit for (1.5) is given explicitly by $V(U) = -\frac{k}{2}U(1-U)$.

As is the case for standard reaction-diffusion, Equation (1.1), advection should give no contribution when $u < \varepsilon\left(= \frac{1}{N}\right)$, where N is the total number of particles in the underlying many-particle system [23]. Hence, it seems plausible that a cut-off should multiply both the reaction kinetics $f(u)$ and the advection term $g(u)\frac{\partial u}{\partial x}$ in (1.2). Our aim in this article is hence to prove an analogue of Theorem 1 for the Burgers-FKPP equation with a Heaviside cut-off $H(u-\varepsilon)$,

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x}H(u-\varepsilon) = \frac{\partial^2 u}{\partial x^2} + u(1-u)H(u-\varepsilon), \tag{1.7}$$

where $k>0$ and $\varepsilon>0$ is the cut-off parameter, as before. Our focus on the Burgers-FKPP equation is further motivated by the fact that, even in the simple Burgers-type advection-diffusion equation [24]

$$\frac{\partial u}{\partial t}+ku\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}, \tag{1.8}$$

a cut-off impacts on front propagation and the corresponding speed: it is well-known that Equation (1.8) admits a travelling front solution connecting the rest states $u=1$ and $u=0$ which propagates with the unique speed $c=\frac{k}{2}$. One can then show that introduction of a Heaviside cut-off function $H(u-\varepsilon)$ in (1.8), whence

$$\frac{\partial u}{\partial t}+ku\frac{\partial u}{\partial x}H(u-\varepsilon)=\frac{\partial^2 u}{\partial x^2}, \tag{1.9}$$

induces the shift $\Delta c=\frac{k}{2}\varepsilon^2$ in the front propagation speed which can be derived either by matched asymptotics or via an adaptation of the approach developed in this article.

Our main result can be formulated as follows.

Theorem 2. Let $\varepsilon\in[0,\varepsilon_0)$, with $\varepsilon_0>0$ sufficiently small, and let $k>0$. Then, there exists a unique, k-dependent propagation speed $c(\varepsilon)$ such that Equation (1.7) admits a unique critical front solution connecting the rest states $u=1$ and $u=0$. Moreover, $c(\varepsilon)=c(0)-\Delta c(\varepsilon)$, where $c(0)=\lim_{\varepsilon\to 0^+}c(\varepsilon)=c_{\text{crit}}$ is the critical speed in the absence of a cut-off, see Theorem 1, with

$$\Delta c(\varepsilon)=\begin{cases}\frac{\pi^2}{(\ln\varepsilon)^2} & \text{if } k\le 2,\\ \frac{2}{k^{1+8/k^2}}\frac{(k^2-4)^{4/k^2}}{\Gamma(1+4/k^2)\Gamma(1-4/k^2)}\varepsilon^{1-4/k^2} & \text{if } k>2,\end{cases} \tag{1.10}$$

to leading order in ε.

Here and in the following, $\Gamma(\cdot)$ denotes the standard Gamma function [25, Section 6.1].

In particular, Theorem 2 hence implies that the front propagation speed in (1.3) is reduced by inclusion of a (Heaviside) cut-off.

Remark 3. The above result is similar to that for the Nagumo equation with cut-off obtained in [10], which also realises pulled and pushed front propagation regimes in dependence on a control parameter. Correspondingly, the correction to the front propagation speed found in the pulled regime is again of the order $\mathscr{O}[(\ln\varepsilon)^{-2}]$, whereas in the pushed regime, it is proportional to a fractional power of ε.

Remark 4. While our choice of Heaviside cut-off in Equation (1.7) is mostly made for analytical tractability, we indicate in Section 4 below how Theorem 2 can be extended to more general choices of cut-off function $\psi(u,\varepsilon)$.

Remark 5. We note that (1.7) does not conserve mass due to the reaction kinetics $u(1-u)H(u-\varepsilon)$. However, the advection and diffusion terms can be written in mass conservation form as follows,

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}F_\varepsilon(u)=u(1-u)H(u-\varepsilon),$$

where

$$F_\varepsilon(u)=k\int_0^u\sigma H(\sigma-\varepsilon)d\sigma-\frac{\partial u}{\partial x}=\begin{cases}\frac{k}{2}(u^2-\varepsilon^2)-\frac{\partial u}{\partial x} & \text{if } u>\varepsilon,\\ -\frac{\partial u}{\partial x} & \text{if } u<\varepsilon.\end{cases}$$

The article is organised as follows: in Section 2, we apply geometric desingularisation (blow-up) to construct a singular heteroclinic orbit Γ for Equation (1.7). In Section 3, we show that Γ persists for ε sufficiently small, therefore establishing Theorem 2, and we provide numerical verification of our results. Finally, in Section 4, we discuss our findings and outline future related work.

2 Geometric desingularisation

Introducing the travelling wave variable $\xi = x - ct$ and writing $u(x,t) = U(\xi)$ in (1.7), we obtain the system of equations

$$\begin{aligned} U' &= V, \\ V' &= -cV + kUVH(U-\varepsilon) - U(1-U)H(U-\varepsilon), \\ \varepsilon' &= 0 \end{aligned} \tag{2.1}$$

in analogy to (1.5), where the (U,V)-subsystem has been extended by the trivial equation for the cut-off parameter ε. We introduce the following blow-up transformation (geometric desingularisation) at the origin in (2.1), which serves to desingularise the non-smooth transition between the outer and inner regions in $\{U = \varepsilon\}$:

$$U = \bar{r}\bar{u}, \quad V = \bar{r}\bar{v}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon}. \tag{2.2}$$

Here, $(\bar{u},\bar{v},\bar{\varepsilon}) \in \mathbb{S}^2_+ := \{(\bar{u},\bar{v},\bar{\varepsilon}) \mid \bar{u}^2+\bar{v}^2+\bar{\varepsilon}^2 = 1\} \cap \{\bar{\varepsilon} \geq 0\}$, with $\bar{r} \in [0,r_0]$ for $r_0 > 0$ sufficiently small.

As in [6, 8, 10, 26], we will analyse (2.1) in two coordinate charts, K_1 and K_2, which are obtained by setting $\bar{u} = 1$ and $\bar{\varepsilon} = 1$ in (2.2), respectively. The rescaling chart K_2 will cover the "inner region" where $U < \varepsilon$, while the phase-directional chart K_1 will allow us to describe the dynamics in the "outer" region, with $U > \varepsilon$. The transition between the two regions, at $\{U = \varepsilon\}$, will be realised in the overlap domain between these coordinate charts. We will construct the singular (in ε) heteroclinic orbit Γ for (2.1) by combining appropriate portions thereof in the two charts. As will become apparent, the uniqueness of the "critical" propagating front in Theorem 2 is a consequence of the fact that a unique choice of c yields a persistent heteroclinic connection between $Q^- = (1,0)$ and $Q^+ = (0,0)$.

Remark 6. For any object $\square$ in (U,V,ε)-space, we denote the corresponding blown-up object by $\overline{\square}$. Moreover, in chart K_i, with $i = 1,2$, that object will be denoted by $\square_i$.

2.1 Dynamics in chart K_2 ("Inner region")

In this subsection, we construct the portion Γ_2 of the singular heteroclinic orbit Γ in chart K_2. Setting $\bar{\varepsilon} = 1$ in (2.2), we have the transformation

$$U = r_2u_2, \quad V = r_2v_2, \quad \text{and} \quad \varepsilon = r_2, \tag{2.3}$$

which we apply to (2.1) to obtain the system of equations

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -cv_2 + kr_2u_2v_2H(u_2-1) - u_2(1-r_2u_2)H(u_2-1), \\ r_2' &= 0. \end{aligned} \tag{2.4}$$

We consider (2.4) in the inner region where $U < \varepsilon$, which is equivalent to $u_2 < 1$, by (2.3). Therefore, $H(u_2-1) \equiv 0$, which implies that (2.4) reduces to

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -cv_2, \\ r_2' &= 0. \end{aligned} \tag{2.5}$$

We define the line of equilibria $\ell_2^+ = \{(0,0,r_2) \mid r_2 \in [0,r_0]\}$ for (2.5). Here, we are particularly interested in the point $Q_2^+ = (0,0,0) \in \ell_2^+$, which is found by taking the singular limit as $r_2 \to 0^+$ on ℓ_2^+. (While Equation (2.5) admits equilibria for any $u_2 \in (0,1)$, we only consider $u_2 = 0$, which corresponds to the point Q^+ before blow-up.) The eigenvalues of the linearisation of (2.5) about Q_2^+ are $-c$ and 0 (double), where the second zero eigenvalue is due to the trivial r_2-equation. Taking $r_2 \to 0^+$ in (2.5), we find the system

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -c(0)v_2, \end{aligned} \tag{2.6}$$

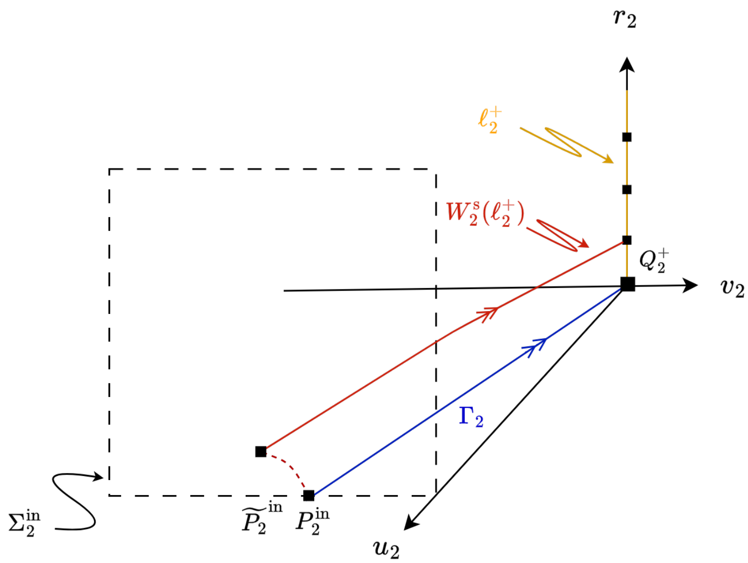

Figure 1: Geometry and dynamics in chart K_2.

where we have used that $c \to c(0) = c_{\text{crit}}$ as $r_2 (= \varepsilon) \to 0^+$ and c_{crit} is as defined in Theorem 1.

We can solve for the stable manifold $W_2^s(Q_2^+)$ of Q_2^+ by writing

$$\frac{dv_2}{du_2} = -c(0),$$

which we can integrate under the condition that $v_2(0) = 0$; the unique solution is given by

$$\Gamma_2 : \ v_2(u_2) = -c(0)u_2.$$

Therefore, invoking again Theorem 1, we can write

$$\Gamma_2 : \ v_2(u_2) = \begin{cases} -2u_2 & \text{if } k \leq 2, \\ -\left(\frac{k}{2} + \frac{2}{k}\right) u_2 & \text{if } k > 2. \end{cases} \tag{2.7}$$

We emphasise that Γ_2 is linear in u_2 for both $k \leq 2$ and $k > 2$; while the corresponding slope is k-dependent in the latter case, by (1.6), that difference is immaterial for the dynamics.

Next, we introduce the entry section

$$\Sigma_2^{\text{in}} = \{(1, v_2, r_2) \mid (v_2, r_2) \in [-v_0, 0] \times [0, r_0]\}, \tag{2.8}$$

which is the equivalent of the hyperplane $\{U = \varepsilon\}$ in chart K_2, to describe the transition of the orbit Γ_2 between the inner and outer regions; here, $v_0 > 0$ is an appropriately chosen constant. We define the entry point into K_2 as $P_2^{\text{in}} = \Gamma_2 \cap \Sigma_2^{\text{in}} = (1, -c(0), 0)$, where $P_2^{\text{in}} = (1, -2, 0)$ for $k \leq 2$ and $P_2^{\text{in}} = \left(1, -\left(\frac{k}{2} + \frac{2}{k}\right), 0\right)$ for $k > 2$, by (2.7). The geometry in chart K_2 is illustrated in Figure 1. The singular orbit Γ_2 (in blue), which corresponds to the stable manifold $W_2^s(Q_2^+)$ of the equilibrium at the origin, intersects the entry section Σ_2^{in} in P_2^{in}. For $r_2 \in (0, r_0]$, Γ_2 will perturb to the stable manifold $W_2^s(\ell_2^+)$ (in red) of the line of equilibria ℓ_2^+ (in yellow).

2.2 Dynamics in chart K_1 ("Outer region")

In this subsection, we will analyse the dynamics of Equation (2.1) in the phase-directional chart K_1. Our aim is to construct the singular orbit Γ_1, which is the continuous extension of Γ_2, as defined in (2.7), to K_1.

Setting $\bar{u} = 1$ in (2.2), we have

$$U = r_1, \quad V = r_1 v_1, \quad \text{and} \quad \varepsilon = r_1 \varepsilon_1, \tag{2.9}$$

which we apply to (2.1) in the outer region where $U > \varepsilon$ to find

$$\begin{aligned} r_1' &= r_1 v_1, \\ v_1' &= -c v_1 + k r_1 v_1 H(1-\varepsilon_1) - (1-r_1) H(1-\varepsilon_1) - v_1^2, \\ \varepsilon_1' &= -\varepsilon_1 v_1. \end{aligned} \tag{2.10}$$

Here, $H(1-\varepsilon_1) \equiv 1$ due to $\varepsilon_1 < 1$ in chart K_1. The system of equations in (2.10) has a line of equilibria at $\ell_1^- = \{(1,0,\varepsilon_1) \mid \varepsilon_1 \in [0,\varepsilon_0]\}$ which corresponds to the steady state at Q^- before blow-up. As other equilibria of (2.10) depend on k, we will discuss them systematically in Sections 2.2.1 and 2.2.2 below. The point $Q_1^- = (1,0,0) \in \ell_1^-$ is obtained in the limit as $\varepsilon_1 \to 0^+$.

Since $\varepsilon = r_1 \varepsilon_1$, we will have to consider both $r_1 \to 0$ and $\varepsilon_1 \to 0$ in the singular limit of $\varepsilon = 0$. We will denote the corresponding portions of the singular orbit Γ_1 in the invariant planes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$ by Γ_1^+ and Γ_1^-, respectively.

We introduce the exit section

$$\Sigma_1^{\text{out}} = \{(r_1, v_1, 1) \mid (r_1, v_1) \in [0, r_0] \times [-v_0, 0]\} \tag{2.11}$$

to track Γ_1^+ as it leaves chart K_1; here, $v_0 > 0$ is defined as in (2.8). Clearly, Σ_1^{out} is equivalent to the entry section Σ_2^{in} in chart K_2 after transformation to K_1: as the change of coordinates $\kappa_{21} : K_2 \to K_1$ between the two charts is given by

$$\kappa_{21}: \ r_1 = r_2 u_2, \quad v_1 = v_2 u_2^{-1}, \quad \text{and} \quad \varepsilon_1 = u_2^{-1}, \tag{2.12}$$

we have $\kappa_{21}(\Sigma_2^{\text{in}}) = \Sigma_1^{\text{out}}$. Correspondingly, we can write $P_1^{\text{out}} = (0, -c(0), 1) = \kappa_{21}(P_2^{\text{in}})$ for the exit point in Σ_1^{out}, where $P_2^{\text{in}} = (1, -c(0), 0)$, as before.

In contrast to chart K_2, the singular geometry and dynamics in K_1 are qualitatively different for $k \leq 2$ and $k > 2$, in that the corresponding phase portraits will not be topologically equivalent. Therefore, we consider these regimes in (2.1) separately.

2.2.1 Pulled front propagation: $k \leq 2$

We note that, when $k \leq 2$, the propagation speed c reduces to $c(0) = 2 = c_{\text{crit}}$ when either $r_1 \to 0^+$ or $\varepsilon_1 \to 0^+$, recall Theorem 1. In addition to the line of equilibria ℓ_1^-, we have an equilibrium at $P_1 = (0,-1,0)$. A simple calculation shows the following result.

Lemma 7. The eigenvalues of the linearisation of (2.10) at P_1 are given by -1, 0, and 1, with corresponding eigenvectors $(1, k-1, 0)^T, (0,1,0)^T$, and $(0,0,1)^T$, respectively.

We first outline the construction of Γ_1^+. Taking $r_1 \to 0^+$ in (2.10), we obtain

$$\begin{aligned} v_1' &= -2v_1 - 1 - v_1^2, \\ \varepsilon_1' &= -\varepsilon_1 v_1, \end{aligned} \tag{2.13}$$

which we can write as

$$\frac{dv_1}{d\varepsilon_1} = \frac{(v_1+1)^2}{\varepsilon_1 v_1}. \tag{2.14}$$

To find a solution to Equation (2.14) so that the orbit Γ_1^+ connects to Γ_2 in the section $\Sigma_1^{\text{out}} = \kappa_{21}(\Sigma_2^{\text{in}})$, we require $v_1(1) = -c(0) = -2(= v_2(1))$, by (2.12). The corresponding (unique) solution is given by

$$\Gamma_1^+ : \ v_1(\varepsilon_1) = -\frac{1 + W_0\left(\frac{e}{\varepsilon_1}\right)}{W_0\left(\frac{e}{\varepsilon_1}\right)}, \tag{2.15}$$

where W_0 denotes the Lambert W function [27], which is defined as the solution to $W_0(z)e^{W_0(z)} = z$. We note that $\Gamma_1^+ \to P_1 = (0,-1,0)$ as $\varepsilon_1 \to 0^+$, which completes the construction.

To construct Γ_1^-, we take $\varepsilon_1 \to 0^+$ in (2.10), which yields

$$\begin{aligned} r_1' &= r_1 v_1, \\ v_1' &= -2v_1 + k r_1 v_1 - (1 - r_1) - v_1^2. \end{aligned} \tag{2.16}$$

Clearly, (2.16) is equivalent to the unmodified first-order system in (2.1) with $c = c(0)$ after blow-down, i.e., after transformation to the original (U,V,ε)-space before the blow-up:

$$\begin{aligned} U' &= V, \\ V' &= -2V + kUV - U(1-U). \end{aligned} \tag{2.17}$$

While we cannot explicitly solve (2.17) for $k < 2$, the following two results imply the existence of the orbit Γ_1^-. The first of these is obtained by simple linearisation.

Lemma 8. The origin Q^+ in (2.17) is a degenerate stable node with eigenvalue -1 (double) and eigenvector $(-1,1)^T$, while the equilibrium at $Q^- = (1,0)$ is a saddle point with eigenvalues $\frac{k-2\pm\sqrt{k^2-4k+8}}{2}$ and corresponding eigenvectors $\left(\frac{1}{2}\left(2-k\pm\sqrt{8-4k+k^2}\right),1\right)^T$.

Next, we show that (2.17) admits a trapping region for $k < 2$; the proof is inspired by [22, Theorem 2.1].

Proposition 9. The curves $\{V = 0\}$ and $\{V = -U(1-U)\}$ form a trapping region $\mathscr{T}$ for the flow of Equation (2.17) when $k < 2$. Moreover, the curve $\{V = -U(1-U)\}$ is invariant under the flow of (2.17) when $k = 2$.

Proof. Substitution of $V = 0$ into (2.17) gives

$$\begin{aligned} U' &= 0, \\ V' &= -U(1-U), \end{aligned} \tag{2.18}$$

which implies $(0,1)\cdot(0,-U(1-U))^T = -U(1-U) < 0$ due to $0 < U < 1$. Similarly, substituting $V = V(U) = -U(1-U)$ into (2.17), we obtain

$$\begin{aligned} U' &= -U(1-U), \\ V' &= U(1-U)(1-kU) \end{aligned} \tag{2.19}$$

and $(-V'(U),1)\cdot(U',V')^T = -(k-2)U^2(1-U) > 0$ when $k < 2$, which also implies that $V(U) = -U(1-U)$ is invariant under the flow of (2.17) when $k = 2$. ☐

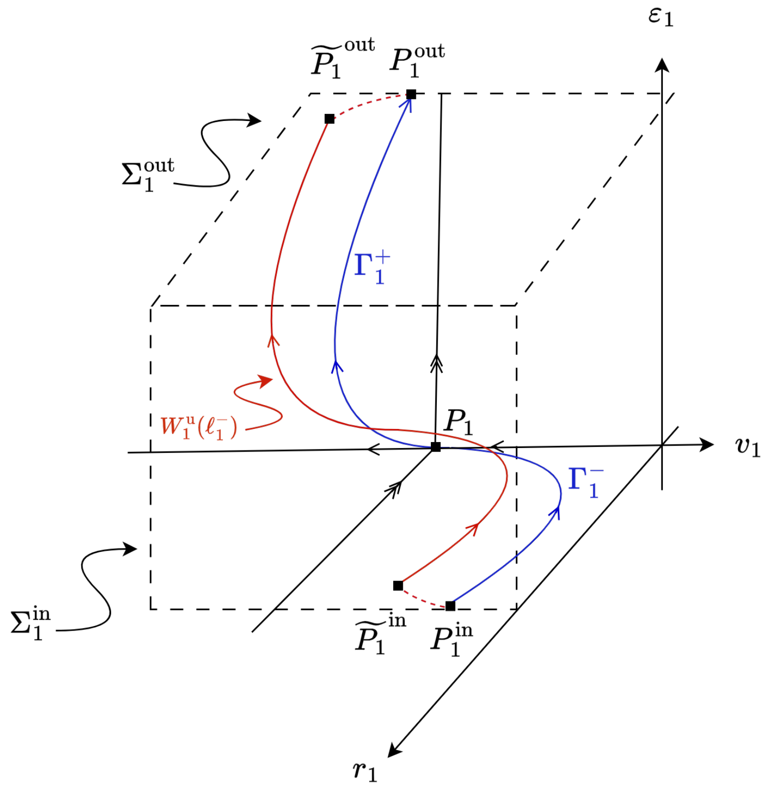

Figure 2: Geometry and dynamics in chart K_1 for $k \leq 2$.

Since the trapping region $\mathscr{T}$ contains only the two equilibria Q^- and Q^+, and since the divergence of the vector field in (2.17) is negative for $U \in [0,1)$ and all V, there are no periodic orbits in $\mathscr{T}$. Hence, there must exist a heteroclinic connection between Q^- and Q^+. That connection must pass through the negative V-plane and is consistent with the stability properties of $Q^\mp$ stated in Lemma 8. It follows that the flow of (2.16) must enter an equivalent trapping region $\mathscr{T}_1$ in chart K_1. The closed region $\mathscr{T}_1$ is bounded by the lines $\{r_1 = 0\}$, $\{v_1 = 0\}$, and $\{v_1 = r_1 - 1\}$ in the plane $\{\varepsilon_1 = 0\}$. Therefore, we can conclude that the orbit Γ_1^- exists and is forward asymptotic to P_1. Defining the section

$$\Sigma_1^{\text{in}} = \{(r_0, v_1, \varepsilon_1) \mid (v_1, \varepsilon_1) \in [-v_0, 0] \times [0,1]\}, \tag{2.20}$$

with $v_0 > 0$ as in (2.8), we see that the point of intersection $P_1^{\text{in}} = \Gamma_1^- \cap \Sigma_1^{\text{in}}$ is given by $P_1^{\text{in}} = (r_0, v_1^{\text{in}}, 0)$, where $v_1^{\text{in}} > -1$, as $v_1^{\text{in}} \in [r_0 - 1, 0]$ by the proof of Proposition 9.

Hence, the construction of $\Gamma_1 = \Gamma_1^- \cup P_1 \cup \Gamma_1^+$ is complete in the case where $k \leq 2$; see Figure 2 for an illustration of the geometry in chart K_1 in that case. The portions Γ_1^- and Γ_1^+ (in blue) of Γ_1 are forward and backward asymptotic, respectively, to the equilibrium at P_1 and intersect the sections Σ_1^{in} and Σ_1^{out} in P_1^{in} and P_1^{out}, respectively. For $\varepsilon_1 \in (0, \varepsilon_0]$, Γ_1 will perturb to the unstable manifold $W_1^{\text{u}}(\ell_1^-)$ (in red) of the line of equilibria ℓ_1^-.

2.2.2 Pushed front propagation: $k > 2$

We now consider the singular dynamics in chart K_1 in the pushed propagation regime where $k > 2$. In analogy to the pulled regime, ℓ_1^- is still a line of equilibria for (2.10). Since, however, $c \to c(0) = \frac{k}{2} + \frac{2}{k}$ as $r_1 \to 0^+$, the point P_1 is no longer an equilibrium for (2.10). Instead, we have two equilibria, at $\hat{P}_1 = \left(0, -\frac{k}{2}, 0\right)$ and $\check{P}_1 = \left(0, -\frac{2}{k}, 0\right)$, which undergo a saddle-node bifurcation as $k \to 2^+$. We are interested in the strong stable eigendirection of the linearisation about the origin in (1.5), i.e., in the absence of a cut-off. Since the heteroclinic orbit $V(U) = -\frac{k}{2}U(1-U)$ is the union of the unstable manifold $W^{\mathrm{u}}(Q^-)$ of Q^- and the strong stable manifold $W^{\mathrm{ss}}(Q^+)$ of Q^+, we restrict our attention to $\hat{P}_1$. The point $\check{P}_1$ corresponds to the weak stable eigendirection at the origin in (1.5), which is not relevant here.

The following lemma summarises the stability properties of $\hat{P}_1$.

Lemma 10. The eigenvalues of the linearisation of (2.10) at $\hat{P}_1 = \left(0, -\frac{k}{2}, 0\right)$ are given by $-\frac{k}{2}$, $\frac{k}{2} - \frac{2}{k}$, and $\frac{k}{2}$, with corresponding eigenvectors $\left(\frac{2}{k}, 1, 0\right)^T$, $(0,1,0)^T$, and $(0,0,1)^T$, respectively.

We again first construct the portion Γ_1^+ of Γ_1. Taking $r_1 \to 0^+$ in (2.10), we obtain

$$\begin{aligned} v_1' &= -\left(\frac{k}{2} + \frac{2}{k}\right)v_1 - 1 - v_1^2, \\ \varepsilon_1' &= -\varepsilon_1 v_1, \end{aligned} \tag{2.21}$$

which we rewrite as

$$\frac{dv_1}{d\varepsilon_1} = \frac{1 + \left(\frac{k}{2} + \frac{2}{k}\right)v_1 + v_1^2}{\varepsilon_1 v_1}.$$

Solving by separation of variables, we find

$$\ln \frac{|k + 2v_1|^{k^2}}{|kv_1 + 2|^4} = (\ln \varepsilon_1 + \alpha)(k^2 - 4),$$

where α is a constant of integration. Exponentiating both sides in the above equation, we have

$$\frac{(k + 2v_1)^{k^2}}{(kv_1 + 2)^4} = \alpha' \varepsilon_1^{k^2 - 4}, \tag{2.22}$$

with $\alpha' = \pm e^{\alpha(k^2-4)}$.

We choose α' so that the orbit Γ_1^+ connects to Γ_2 in the section $\Sigma_1^{\mathrm{out}} = \kappa_{21}(\Sigma_2^{\mathrm{in}})$. Thus, we require $v_1(1) = -c(0) = -\left(\frac{k}{2} + \frac{2}{k}\right)(= v_2(1))$, which is satisfied for $\alpha' = \frac{(-4/k)^{k^2}}{(k^2/2)^4}$.

Finally, we note that Γ_1^+ is backward asymptotic to $\hat{P}_1$. Taking $\varepsilon_1 \to 0^+$, we conclude that

$$\frac{(k + 2v_1)^{k^2}}{(kv_1 + 2)^4} \to 0 \tag{2.23}$$

must hold, which is only true when $v_1 \to -\frac{k}{2}$. Hence, the construction of Γ_1^+ is complete.

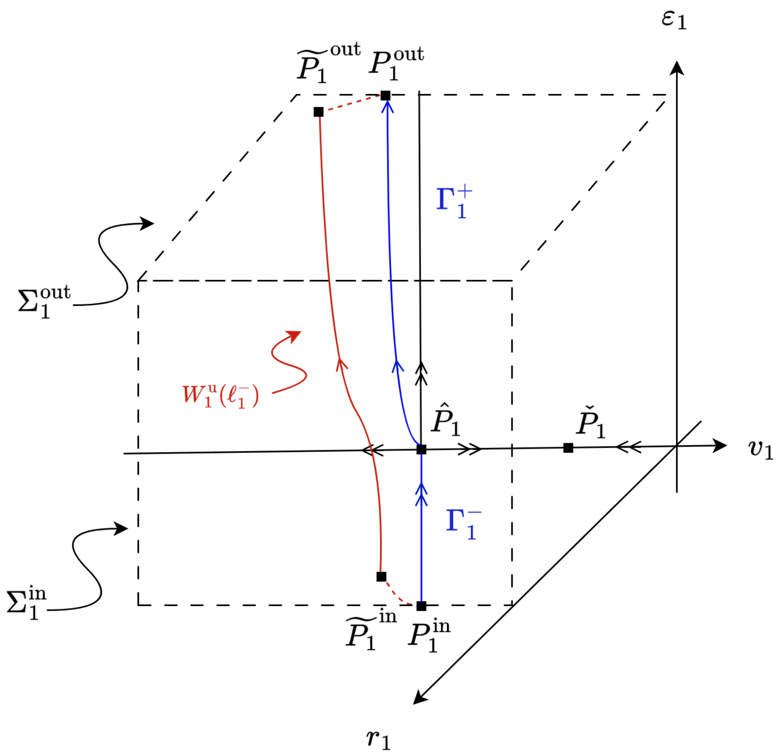

Figure 3: Geometry and dynamics in chart K_1 for $k > 2$.

Next, we consider the portion Γ_1^- of Γ_1. The limit as $\varepsilon_1 \to 0^+$ in (2.10) gives the system of equations

$$\begin{aligned} r_1' &= r_1 v_1, \\ v_1' &= -\left(\frac{k}{2} + \frac{2}{k}\right) v_1 + k r_1 v_1 - (1 - r_1) - v_1^2, \end{aligned} \tag{2.24}$$

which is equivalent to (2.1) after blow-down.

Lemma 11. For $r_1 \in (0, 1]$ the system of equations in (2.24) admits an explicit orbit that is given by

$$\Gamma_1^- : \; v_1(r_1) = -\frac{k}{2}(1 - r_1), \tag{2.25}$$

with $v_1(1) = 0$.

Proof. Recall that when $k \geq 2$ and $c(0) = \frac{2}{k} + \frac{k}{2} = c_{\text{crit}}$, we have an explicit orbit for (2.1) that is given by $V(U) = -\frac{k}{2}U(1 - U)$, see Theorem 1. Transformation to chart K_1, with $v = r_1 v_1$ and $u = r_1$, yields the result. □

We conclude that Γ_1^- is forward asymptotic to $\hat{P}_1$ and backward asymptotic to Q_1^-. Moreover, $P_1^{\text{in}} = \Gamma_1^- \cap \Sigma_1^{\text{in}} = \big(r_0, \frac{k}{2}(r_0 - 1), 0\big)$, which completes the construction of the singular orbit Γ_1 in the case where $k > 2$. The geometry in chart K_1 is illustrated in Figure 3 in that case.

2.3 Singular orbit $\overline{\Gamma}$

We now combine the results of the previous two subsections to define the singular orbit $\overline{\Gamma}$ in $(\bar{u}, \bar{v}, \bar{\varepsilon})$-space.

Proposition 12. For any $k > 0$, there exists a singular heteroclinic orbit $\overline{\Gamma}$ for Equations (2.4) and (2.10) that connects Q_1^- to Q_2^+.

Proof. We first consider the case where $k \leq 2$, i.e., the pulled front propagation regime, which is analogous to the standard FKPP equation with a cut-off [6]. The orbit Γ_2 given by $v_2(u_2) = -2u_2$, see (2.7), connects to Q_2^+ and intersects Σ_2^{in} in $P_2^{\text{in}} = (1, -2, 0)$. Next, we apply the change of coordinates in (2.12) to find $\kappa_{21}\big(P_2^{\text{in}}\big) = P_1^{\text{out}} = (0, -2, 1)$. By construction, Γ_1^+ passes through P_1^{out} and is backward asymptotic to P_1, see (2.15). Similarly, Γ_1^- is forward asymptotic to P_1 and backward asymptotic to Q_1^-, by Proposition 9. Therefore, we can now write $\overline{\Gamma}$ as the union of Γ_1^-, Γ_1^+, and Γ_2 with Q_1^-, P_1, and Q_2^+ in blown-up phase space, which proves the result for $k \leq 2$.

Next, we consider the case where $k > 2$, corresponding to the pushed propagation regime. Here, Γ_2 is given by $v_2(u_2) = -\big(\frac{k}{2} + \frac{2}{k}\big)u_2$, see (2.7), which again connects to Q_2^+ and intersects Σ_2^{in} in $P_2^{\text{in}} = \big(1, -\big(\frac{k}{2} + \frac{2}{k}\big), 0\big)$. Applying the change of coordinates in (2.12), we find $\kappa_{21}\big(P_2^{\text{in}}\big) = P_1^{\text{out}} = \big(0, -\big(\frac{k}{2} + \frac{2}{k}\big), 1\big)$. We know that Γ_1^+, constructed in (2.22), passes through P_1^{out} and is backward asymptotic to $\hat{P}_1 = \big(0, -\frac{k}{2}, 0\big)$. Similarly, Γ_1^-, defined in (2.25), is forward asymptotic to $\hat{P}_1$ and backward asymptotic to Q_1^-. Therefore, we can write $\overline{\Gamma}$ as the union of the orbits Γ_1^-, Γ_1^+, and Γ_2 with Q_1^-, $\hat{P}_1$, and Q_2^+ in blown-up space, which shows the result for $k > 2$. □

3 Proof of Theorem 2

In this section, we prove our main result, Theorem 2. We first show that the singular orbit Γ, which is obtained from the orbit $\overline{\Gamma}$ constructed in Section 2 after blow-down, persists for ε sufficiently small in Equation (2.1). Then, we derive the leading-order asymptotics of the correction $\Delta c(\varepsilon)$ to the critical speed $c(0) = c_{\text{crit}}$ that is due to the cut-off. Finally, we illustrate our results numerically.

3.1 Persistence of Γ

The following result implies the existence of a unique front propagation speed $c(\varepsilon)$ for which there exists a critical heteroclinic orbit in (2.1). While the proof is similar to that of [6, Proposition 3.1], we give it here for completeness.

Proposition 13. For $\varepsilon \in (0, \varepsilon_0)$, with ε_0 sufficiently small, $k > 0$, and c close to $c(0)$, there exists a critical heteroclinic connection between Q^- and Q^+ in Equation (2.1) for a unique speed $c(\varepsilon)$ which depends on k. Furthermore, there holds $c(\varepsilon) \leq c(0)$.

Proof. We first analyse (2.4) in the inner region, where $U < \varepsilon$. In particular, we are interested in the stable manifold $W_2^{\text{s}}(\ell_2^+)$, which is given explicitly by $v_2(u_2) = -c(r_2)u_2$ when $r_2 (= \varepsilon) > 0$, for general values of c. For r_2 fixed, $W_2^{\text{s}}(\ell_2^+)$ intersects Σ_2^{in} in the point $(1, v_2^{\text{in}}, r_2)$, where $v_2^{\text{in}} = -c(\varepsilon)$. From the definition of the blow-up transformation in (2.2), we have that $V^{\text{in}} = v_2^{\text{in}}\varepsilon = -c(\varepsilon)\varepsilon \lesssim 0$, which implies $\frac{\partial V^{\text{in}}}{\partial c} = -\varepsilon$.

We now consider the outer region, where $U > \varepsilon$. For general c, the dynamics in that region are governed by

$$\begin{aligned} U' &= V, \\ V' &= -cV + kUV - U(1-U), \end{aligned} \tag{3.1}$$

recall (1.5). The intersection of the unstable manifold $W^{\text{u}}(Q^-)$ of Q^- with $\{U = \varepsilon\}$ can be written as the graph of an analytic function $V^{\text{out}}(c, \varepsilon)$, with $\frac{\partial V^{\text{out}}}{\partial c} > 0$. A standard phase plane argument shows that $V^{\text{out}}(c, \varepsilon)$ must be $\mathscr{O}(1)$ and negative for $c \lesssim c(0)$, which implies $V^{\text{in}} > V^{\text{out}}$.

Finally, we consider the case where $c = c(0)$ and $\varepsilon > 0$. First, we take $k < 2$, in which case $c(0) = 2$. The trapping region argument in Proposition 9 then shows that V is bounded between the curves $\{V = 0\}$ and $\{V = -U(1-U)\}$

and, hence, that $V^{\mathrm{out}} \geq -\varepsilon(1-\varepsilon)$ in $\{U = \varepsilon\}$. Therefore, we can conclude that $V^{\mathrm{in}} = -2\varepsilon < -\varepsilon(1-\varepsilon) \leq V^{\mathrm{out}}$ for any $\varepsilon > 0$. Next, we take $k \geq 2$, in which case the singular heteroclinic orbit is known explicitly as $V(U) = -\frac{k}{2}U(1-U)$. Therefore, we can write $V^{\mathrm{out}}(c(0),\varepsilon) = -\frac{k}{2}\varepsilon(1-\varepsilon)$, which again implies $V^{\mathrm{in}} = -\left(\frac{k}{2}+\frac{2}{k}\right)\varepsilon < -\frac{k}{2}\varepsilon(1-\varepsilon) = V^{\mathrm{out}}$ for any $\varepsilon > 0$.

We conclude by observing that $W^{\mathrm{s}}(Q^+)$ and $W^{\mathrm{u}}(Q^-)$ must intersect in $\{U = \varepsilon\}$ for a unique value of $c(\varepsilon) \lesssim c(0)$, which follows from the implicit function theorem and the fact that $\frac{\partial V^{\mathrm{out}}}{\partial c} - \frac{\partial V^{\mathrm{in}}}{\partial c} > 0$. □

It follows from Proposition 13 that a Heaviside cut-off reduces the critical front propagation speed in Equation (1.3); correspondingly, $\Delta c(\varepsilon) = c(0) - c(\varepsilon)$ must be positive for ε sufficiently small.

3.2 Leading-order asymptotics of Δc

In this subsection, we derive the asymptotics of the correction Δc to $c(0)$ to leading order in ε. Again, we distinguish between the pulled and pushed front propagation regimes in (2.1).

3.2.1 Pulled front propagation: $k \leq 2$

We first consider the case where $k \leq 2$. Recall that the dynamics in chart K_1 are governed by the system of equations in (2.10). Our aim is to approximate the transition map $\Pi_1 : \Sigma_1^{\mathrm{in}} \to \Sigma_1^{\mathrm{out}}$ under the flow of (2.10) for $\varepsilon \in (0,\varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small.

To that end, we first shift the equilibrium at $P_1 = (0,-1,0)$ to the origin via the transformation $V_1 = v_1 + 1$, and we set $c = c(0) - \Delta c = 2 - \eta^2$. With these transformations, we can write (2.10) as

$$
\begin{aligned}
r_1' &= -r_1(1-V_1),\\
V_1' &= (2-\eta^2)(1-V_1) - kr_1(1-V_1) - 1 + r_1 - (1-V_1)^2,\\
\varepsilon_1' &= \varepsilon_1(1-V_1).
\end{aligned}
\tag{3.2}
$$

Rescaling "time" by dividing out a positive factor of $1-V_1$ from the right-hand sides in (3.2), noting that the ε_1-equation decouples, and appending the trivial equation for η, we obtain

$$
\begin{aligned}
\dot{r}_1 &= -r_1,\\
\dot{V}_1 &= -\eta^2 + \frac{(1-k)r_1 + kr_1V_1 - V_1^2}{1-V_1},\\
\dot{\eta} &= 0,
\end{aligned}
\tag{3.3}
$$

where the overdot denotes differentiation with respect to the new independent variable ζ.

Remark 14. The rescaling of "time" in (3.2) is implicitly defined via $(1-V_1(\xi))\frac{d}{d\zeta} = \frac{d}{d\xi}$, and merely affects the parametrisation of solutions while leaving the phase portrait unchanged.

We have the following result, which can be shown in close analogy to [6, Proposition 3.2].

Lemma 15. There exists a normal form transformation $(r_1,V_1,\eta) \to (S(r_1,V_1,\eta),W(r_1,V_1,\eta),\eta)$ that transforms Equation (3.3) to

$$
\begin{aligned}
\dot{S} &= -S,\\
\dot{W} &= -\eta^2 - \frac{W^2}{1-W},\\
\dot{\eta} &= 0.
\end{aligned}
\tag{3.4}
$$

That transformation respects the invariance of $\{r_1 = 0\}$ and $\{\eta = \eta_0\}$, for any $\eta_0 \in \mathbb{R}$.

Proof. The statement follows from [28, Theorem 1]. □

We note that the only resonant terms in (3.3) are of the form V_1^n, for $n \geq 2$. Therefore, all other terms can be removed via a sequence of smooth near-identity transformations.

The normal form in (3.4) is identical to the one stated in [6, Equation (34)]. Moreover, the analysis in [6] shows that the correction Δc to $c(0)$ is given by $\eta^2 = \frac{\pi^2}{(\ln \varepsilon)^2} + \mathscr{O}[(\ln \varepsilon)^{-3}]$ to leading order, as well as that it is independent of the transformed coordinates W^{in} and W^{out} of the entry and exit points P_1^{in} and P_1^{out}, respectively, following the normal form transformation in Lemma 15. We note that both W^{in} and W^{out} are well-defined by Proposition 9 and our analysis in chart K_2, in which the point P_2^{in} and, therefore, also the point P_1^{out}, is known explicitly.

In summary, we find the same correction Δc as in [6] for the pulled propagation regime, i.e., when $k \leq 2$, which completes the proof of Theorem 2 in that case.

Remark 16. Setting $k = 0$ in Theorem 2, we recover the main result from [6, Theorem 1.1], as is to be expected.

3.2.2 Pushed front propagation: $k > 2$

The pushed propagation regime where $k > 2$ is significantly more involved algebraically than the pulled regime discussed in the previous subsection.

Our aim is again to approximate the transition map $\Pi_1 : \Sigma_1^{\text{in}} \to \Sigma_1^{\text{out}}$ under the flow of (2.10). Now, the point $\hat{P}_1 = \left(0, -\frac{k}{2}, 0\right)$ is shifted to the origin via the transformation $V_1 = v_1 + \frac{k}{2}$; moreover, we write $c = c(0) - \Delta c = \frac{k}{2} + \frac{2}{k} - \Delta c$. The resulting system of equations is given by

$$\begin{aligned}
r_1' &= -r_1\left(\frac{k}{2} - V_1\right), \\
V_1' &= -\Delta c\left(\frac{k}{2} - V_1\right) + r_1\left(1 - \frac{k^2}{2} + kV_1\right) + \left(\frac{k}{2} - \frac{2}{k}\right)V_1 - V_1^2, \\
\varepsilon_1' &= \varepsilon_1\left(\frac{k}{2} - V_1\right).
\end{aligned} \tag{3.5}$$

Next, we rescale "time" by a (positive) factor of $\frac{k}{2} - V_1$, with $\left(\frac{k}{2} - V_1(\xi)\right)\frac{d}{d\zeta} = \frac{d}{d\xi}$, which yields

$$\begin{aligned}
\dot{r}_1 &= -r_1, \\
\dot{V}_1 &= -\Delta c + \frac{r_1\left(1 - \frac{k^2}{2} + kV_1\right) + \left(\frac{k}{2} - \frac{2}{k}\right)V_1 - V_1^2}{\frac{k}{2} - V_1}, \\
\dot{\varepsilon}_1 &= \varepsilon_1.
\end{aligned} \tag{3.6}$$

We note that the equation for ε_1 in (3.6) again decouples. Finally, we separate the r_1-dependent terms in the V_1-equation in (3.6), and we append the trivial equation for Δc:

$$\begin{aligned}
\dot{r}_1 &= -r_1, \\
\dot{V}_1 &= -\Delta c + r_1\frac{1 - \frac{k^2}{2} + kV_1}{\frac{k}{2} - V_1} + \frac{\left(\frac{k}{2} - \frac{2}{k}\right)V_1 - V_1^2}{\frac{k}{2} - V_1}, \\
\dot{\Delta c} &= 0.
\end{aligned} \tag{3.7}$$

For the linearisation of (3.7) at the origin, we obtain the eigenvalues -1, $1 - \frac{4}{k^2}$, and 0. It is straightforward to show that the monomial $r_1V_1^j$ in (3.7) can only be resonant for integer-valued $j = \frac{2-4/k^2}{1-4/k^2}$. In particular, the lowest-order resonance is realised at order 4, since $1(-1) + 3\left(1 - \frac{4}{k^2}\right) = 1 - \frac{4}{k^2}$ when $k = 2\sqrt{2}$, corresponding to the fourth-order monomial $r_1V_1^3$. To approximate Π_1, we hence first derive a normal form for (3.7) by eliminating all non-resonant r_1-dependent terms via a sequence of near-identity transformations.

Lemma 17. There exists a sequence of smooth transformations that transforms Equation (3.7) to

$$\begin{aligned}\dot{r}_1 &= -r_1,\\ \dot{W} &= -\Delta c + \frac{\left(\frac{k}{2}-\frac{2}{k}\right)W - W^2}{\frac{k}{2}-W} + \mathcal{O}(r_1 W^j),\\ \dot{\Delta c} &= 0,\end{aligned} \tag{3.8}$$

with $j \geq 3$. Specifically, that sequence is composed of the transformation $V_1 = \frac{k}{2}r_1 + Z$ in (3.7), followed by the near-identity transformation $Z = W + \frac{4}{k^2}r_1 W$ and, finally, a sequence of smooth near-identity transformations.

Proof. The existence of such a transformation follows from standard normal form theory [29], as the lowest-order potentially resonant monomial in (3.7) is of the form $r_1 V_1^3$ for $k = 2\sqrt{2}$. All higher-order non-resonant terms can be removed by a sequence of smooth near-identity transformations. □

Next, we approximate $\widetilde{P}_1^{\text{in}}$ and $\widetilde{P}_1^{\text{out}}$, which are the entry and exit points in Σ_1^{in} and Σ_1^{out}, respectively, under Π_1, to a sufficiently high order in Δc, ε, and r_0. We first show the following preparatory result.

Lemma 18. For U and V defined as in (1.5), $U \in [0, U_0]$ with $U_0 > 0$ sufficiently small, and any $k \geq 2$, there holds

$$\frac{\partial V}{\partial c}(U, c(0)) = \begin{cases} \frac{U(1-U+\ln U)}{U-1} & \text{if } k = 2,\\ \frac{k^2}{k^2+4} U^{\frac{4}{k^2}} (1-U)\,{}_2F_1\left(1+\frac{4}{k^2}, \frac{4}{k^2}, 2+\frac{4}{k^2}, 1-U\right) & \text{if } k > 2,\end{cases} \tag{3.9}$$

where ${}_2F_1$ is the hypergeometric function, see, e.g., [25, Section 15].

Proof. We rewrite (1.5) with U as the independent variable,

$$V\frac{dV}{dU} = -cV + kUV - U(1-U). \tag{3.10}$$

Differentiation with respect to c gives

$$\frac{\partial V}{\partial c}\frac{\partial V}{\partial U} + V\frac{\partial}{\partial c}\frac{\partial V}{\partial U} = -V - c\frac{\partial V}{\partial c} + kU\frac{\partial V}{\partial c}. \tag{3.11}$$

Evaluating at $V(U, c(0)) = -\frac{k}{2}U(1-U)$ and making use of $\frac{\partial V}{\partial U}(U, c(0)) = -\frac{k}{2}(1-2U)$, we find

$$\frac{d}{dU}\left(\frac{\partial V}{\partial c}(U, c(0))\right) = -1 + \frac{4}{k^2}\frac{1}{U(1-U)}\frac{\partial V}{\partial c}(U, c(0)). \tag{3.12}$$

We note that (3.12) is an ordinary differential equation for $\frac{\partial V}{\partial c}(U, c(0))$ in the variable U. For $k = 2$, the unique solution that remains bounded as $U \to 1^-$ is given by

$$\frac{\partial V}{\partial c}(U, c(0)) = \frac{U(1-U+\ln U)}{U-1}.$$

For $k > 2$, we can solve (3.12) by variation of constants, which gives

$$\frac{\partial V}{\partial c}(U, c(0)) = \beta(1-U)^{-\frac{4}{k^2}}U^{\frac{4}{k^2}} + (1-U)^{-\frac{4}{k^2}}U^{\frac{4}{k^2}}\left[-\int_1^U (1-s)^{\frac{4}{k^2}} s^{-\frac{4}{k^2}}\, ds\right],$$

for some constant of integration β that is to be determined. We require that $\frac{\partial V}{\partial c}(U, c(0)) \to 0$ when $U \to 1^-$. Therefore, $\beta = 0$, since the second term goes to zero by L'Hôpital's Rule. Next, we make the substitution $s = 1 - \sigma$, which gives

$$\frac{\partial V}{\partial c}(U, c(0)) = (1-U)^{-\frac{4}{k^2}}U^{\frac{4}{k^2}}\int_0^{1-U} \sigma^{\frac{4}{k^2}}(1-\sigma)^{-\frac{4}{k^2}}\, d\sigma. \tag{3.13}$$

The integral in (3.13) is of the form of an Incomplete Beta function, see [25, Section 6.6], which is defined by the expression

$$B_x(a,b) := \int_0^x \sigma^{a-1}(1-\sigma)^{b-1}d\sigma.$$

Setting $x = 1-U$, $a = 1+\frac{4}{k^2}$, and $b = 1-\frac{4}{k^2}$, we can write $\frac{\partial V}{\partial c}(U,c(0))$ in terms of $B_{1-U}\left(1+\frac{4}{k^2}, 1-\frac{4}{k^2}\right)$. Finally, the relation [25, Equation 6.6.8 and Section 15]

$$B_x(a,b) = a^{-1}x^a {}_2F_1(a,1-b,a+1,x)$$

implies

$$\frac{\partial V}{\partial c}(U,c(0)) = \frac{k^2}{k^2+4}U^{4/k^2}(1-U){}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1-U\right), \tag{3.14}$$

which completes the proof. □

Lemma 19. For $k>2$ and Δc and ε sufficiently small, the points $\widetilde{P}_1^{\text{in}} = \left(r_0, W^{\text{in}}, \frac{\varepsilon}{r_0}\right)$ and $\widetilde{P}_1^{\text{out}} = (\varepsilon, W^{\text{out}}, 1)$ satisfy

$$W^{\text{in}} = \nu(r_0)\Delta c + \mathcal{O}(\Delta c^2, r_0^{4/k^2}\Delta c) \quad \text{and} \quad W^{\text{out}} = -\frac{2}{k} + \Delta c - \frac{k}{2}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{3.15}$$

where

$$\nu(r_0) = -\frac{k^2}{k^2+4}r_0^{4/k^2-1}(1-r_0){}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1-r_0\right). \tag{3.16}$$

Proof. Recall that, by (2.2), $r_1v_1 = r_2v_2$, which implies $v_1^{\text{out}} = v_2^{\text{in}} = -c(\varepsilon) = -\left(\frac{k}{2}+\frac{2}{k}\right)+\Delta c$. As the point P_1 was shifted to the origin via the transformation $V_1 = v_1 + \frac{k}{2}$, we have $V_1^{\text{out}} = -\frac{2}{k}+\Delta c$. The normal form transformation given by Lemma 17 implies that $W = V_1 - \frac{k}{2}r_1 + \mathcal{O}(r_1V_1)$. Moreover, $r_1 = \varepsilon$ in Σ_1^{out}; therefore, $W^{\text{out}} = -\frac{2}{k}+\Delta c - \frac{k}{2}\varepsilon + \mathcal{O}(\varepsilon^2)$.

We now consider W^{in}. As $W^{\text{u}}(Q^-)$ is analytic in U and c, we can write

$$\begin{aligned}
V(U,c) &= \sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j V}{\partial c^j}(U,c(0))(-\Delta c)^j \\
&= -\frac{k}{2}U(1-U) - \frac{k^2}{k^2+4}U^{4/k^2}(1-U){}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1-U\right)\Delta c + \mathcal{O}(\Delta c^2),
\end{aligned}$$

by Lemma 18. Next, we make use of $U = r_1$, $V = r_1v_1$, and the fact that $r_1 = r_0$ in Σ_1^{in}, as well as of the transformation $V_1 = v_1 + \frac{k}{2}$, to obtain

$$V_1^{\text{in}} = \frac{k}{2}r_0 - \frac{k^2}{k^2+4}r_0^{4/k^2-1}(1-r_0){}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1-r_0\right)\Delta c + \mathcal{O}(\Delta c^2).$$

Finally, since $W = V_1 - \frac{k}{2}r_1 + \mathcal{O}(r_1V_1)$, we have

$$\begin{aligned}
W^{\text{in}} &= -\frac{k^2}{k^2+4}r_0^{4/k^2-1}(1-r_0){}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1-r_0\right)\Delta c + \mathcal{O}(\Delta c^2, r_0^{4/k^2}\Delta c) \\
&= \nu(r_0)\Delta c + \mathcal{O}(\Delta c^2, r_0^{4/k^2}\Delta c),
\end{aligned} \tag{3.17}$$

where $\nu(r_0)$ is as defined in (3.16). In particular, the invariance of $\{W=0\}$ for $\Delta c = 0$ in the normal form, Equation (3.8), implies that the error term in (3.17) has to be proportional to Δc, as stated. □

Remark 20. Lemma 19 implies that W^{in} and W^{out} are both negative for Δc and r_0 sufficiently small. In particular, $v(r_0)$, as defined in (3.16), is negative, which follows from the identity ${}_2F_1(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$; see, e.g., [25, Equation 15.1.20]. Incidentally, that identity is valid for $\Re(c-a-b) > 0$ which, for $a = 1+4/k^2$, $b = 4/k^2$, and $c = 2+4/k^2$, is equivalent to requiring $k > 2$.

Instead of integrating the "full" normal form in (3.8) to determine the leading-order asymptotics of $\Delta c(\varepsilon)$, we will consider the simplified equations that are obtained by omitting the higher-order $\mathscr{O}(r_1 W^j)$-terms with $j \geq 3$ therein:

$$\dot{\widehat{W}} = -\Delta c + \frac{\left(\frac{k}{2} - \frac{2}{k}\right)\widehat{W} - \widehat{W}^2}{\frac{k}{2} - \widehat{W}}. \tag{3.18}$$

We now show that, to leading order, the asymptotics of $\Delta c(\varepsilon)$ is not affected by the omission of the $\mathscr{O}(r_1 W^j)$-terms in (3.8). In our proof, we make the *a priori* assumption that $\Delta c = \mathscr{O}\left(\varepsilon^{1-4/k^2}\right)$, which we then show to be consistent in Proposition 23.

Lemma 21. Let $\zeta \in [0, \zeta^{\text{out}}]$, with W^{in} and W^{out} defined as in Lemma 19, let $k > 2$, and let $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small. Then, for $W^{\text{in}} = W(0) = \widehat{W}^{\text{in}}$, we have

$$\left|W^{\text{out}} - \widehat{W}^{\text{out}}\right| = \mathscr{O}(\varepsilon^{\kappa}),$$

where $\kappa > \frac{1}{2}$.

Proof. Considering the difference between the equations for W and $\widehat{W}$ in (3.8) and (3.18), respectively, and multiplying the result with $W - \widehat{W}$, we find

$$\left|W(\zeta) - \widehat{W}(\zeta)\right|^2 \leq \left|W^{\text{in}} - \widehat{W}^{\text{in}}\right|^2 + 2\int_0^{\zeta} \left[\left|\frac{(k-4/k)W - 2W^2}{k-2W} - \frac{(k-4/k)\widehat{W} - 2\widehat{W}^2}{k-2\widehat{W}}\right| + C|r_1 W^j|\right]\left|W - \widehat{W}\right| ds,$$

where C is a generic constant. For $\zeta \in [0, \zeta^{\text{out}}]$ and $k > 2$, the integral term $\mathscr{I}$ in the above inequality is estimated as

$$\mathscr{I} \leq 2\int_0^{\zeta} \left|W - \widehat{W}\right|^2 \left|\frac{5}{4} - \frac{4}{(k-2W)(k-2\widehat{W})}\right| ds + C\int_0^{\zeta} |r_1 W^j|^2 ds,$$

where we have used Young's inequality. Since $\left|\frac{5}{4} - \frac{4}{(k-2W)(k-2\widehat{W})}\right|$ is monotonic for $W, \widehat{W} \in [W^{\text{out}}, W^{\text{in}}]$, and since $W^{\text{out}} = -\frac{2}{k} + \Delta c - \frac{k}{2}\varepsilon + \mathscr{O}(\varepsilon^2)$ by Lemma 19, with $\Delta c = \mathscr{O}(\varepsilon^{1-4/k^2})$ positive, there exists $\varepsilon_0 > 0$ sufficiently small such that $-\frac{2}{k} \leq W^{\text{out}}$ for $\varepsilon \in (0, \varepsilon_0)$. Therefore, we can estimate $\left|\frac{5}{4} - \frac{4}{(k-2W)(k-2\widehat{W})}\right| \leq \frac{5}{4} - \frac{4}{(k+4/k)^2} \leq \frac{5}{4} - \frac{1}{k^2}$.

Thus, taking $W^{\text{in}} = W(0) = \widehat{W}^{\text{in}}$, we have

$$\left|W(\zeta) - \widehat{W}(\zeta)\right|^2 \leq 2\left(\frac{5}{4} - \frac{1}{k^2}\right)\int_0^{\zeta} \left|W - \widehat{W}\right|^2 ds + C\int_0^{\zeta} |r_1 W^j|^2 ds.$$

An application of the Grønwall inequality then yields

$$\left|W(\zeta) - \widehat{W}(\zeta)\right|^2 \leq C e^{2\left(\frac{5}{4} - \frac{1}{k^2}\right)\zeta} \int_0^{\zeta} |r_1 W^j|^2 ds. \tag{3.19}$$

Next, we write $|r_1 W^j| = \frac{|(r_1 W)^j|}{r_1^{j-1}} = \frac{|r_1 W|^j}{r_0^{j-1}} e^{(j-1)\zeta}$. To estimate $r_1 W$, we consider

$$(r_1 W)' = -r_1 W + r_1 W' = -r_1 \Delta c - r_1 \frac{\frac{2}{k} W}{\frac{k}{2} - W}[1 + \mathscr{O}(r_1 W^j)].$$

Setting $y := r_1 W$, making use of $r_1(\zeta) = r_0 e^{-\zeta}$, and denoting $F(\zeta) \equiv F(r_1(\zeta), W(\zeta)) = \frac{2/k}{k/2 - W(\zeta)}\big[1 + \mathscr{O}(r_1(\zeta) W(\zeta)^j)\big]$, we can write the above as $y' = -r_0 e^{-\zeta}\Delta c - yF(\zeta)$. Solving by variation of constants, with $y(0) = r_0 W^{\text{in}} = r_0 \Delta c\, \omega(\Delta c, r_0)$ for $\omega(\Delta c, r_0) = \nu(r_0) + \mathscr{O}(\Delta c, r_0^{4/k^2})$, recall Lemma 19, we find

$$\begin{aligned} y(\zeta) &= r_0 \Delta c e^{-\int_0^\zeta F(s)\,ds}\Big[\omega(\Delta c, r_0) - \int_0^\zeta \exp\Big(-s + \int_0^s F(\sigma)\,d\sigma\Big) ds\Big] \\ &= r_0 \Delta c e^{-\int_0^\zeta F(s)\,ds}\Big[\omega(\Delta c, r_0) - 1 + e^{-\zeta}\exp\Big(\int_0^\zeta F(\sigma)\,d\sigma\Big) - \int_0^\zeta e^{-s}\exp\Big(\int_0^s F(\sigma)\,d\sigma\Big) F(s)\,ds\Big]. \end{aligned}$$

Here, the second line follows from integration by parts. Since $-\frac{2}{k} \leq W^{\text{out}}$, and since $W \in [W^{\text{out}}, W^{\text{in}}]$, we can estimate $\frac{1}{1+k^2/4} \leq F(\zeta)$ for $\zeta \in [0, \zeta^{\text{out}}]$ and r_0 sufficiently small. Similarly, for every fixed $k > 2$, there exists μ such that $k^2 > \mu > 4$, which implies $F(\zeta) \leq \frac{\mu}{k^2}$ for ε and r_0 sufficiently small.

Hence, and since $\omega(\Delta c, r_0)$ is negative for Δc and r_0 sufficiently small, by Remark 20, we find

$$|y(\zeta)| \leq r_0 \Delta c e^{-\frac{1}{1+k^2/4}\zeta}\Big[1 - \omega(\Delta c, r_0) + e^{-(1-\mu/k^2)\zeta} + \frac{\mu}{k^2}\int_0^\zeta e^{-(1-\mu/k^2)s}\,ds\Big]. \tag{3.20}$$

Since the term in square brackets in (3.20) is bounded for $\zeta \in [0, \infty)$, we find that

$$|(r_1 W)(\zeta^{\text{out}})| \leq C \Delta c \varepsilon^{\frac{1}{1+k^2/4}},$$

where $\zeta^{\text{out}} = -\ln\frac{\varepsilon}{r_0}$, as before.

Therefore, we can estimate

$$\int_0^{\zeta^{\text{out}}} |r_1 W^j|^2\,ds \leq C\int_0^{\zeta^{\text{out}}} \Big(\Delta c \varepsilon^{\frac{1}{1+k^2/4}}\Big)^{2j} \frac{e^{(j-1)s}}{r_0^{j-1}}\,ds \leq C\Big(\Delta c \varepsilon^{\frac{1}{1+k^2/4}}\Big)^{2j}\varepsilon^{-(j-1)} = \mathscr{O}\Big(\varepsilon^{2j(1-4/k^2) + \frac{2j}{1+k^2/4} - j + 1}\Big), \tag{3.21}$$

where we have made use of $\Delta c = \mathscr{O}\big(\varepsilon^{1-4/k^2}\big)$. Finally, we recall that the $r_1 W^j$-terms in (3.7) can only be resonant for integer-valued $j = \frac{2-4/k^2}{1-4/k^2}$, which implies that (3.21) is of the order $\mathscr{O}\Big(\varepsilon^{\frac{3k^4+16k^2-32}{k^2(k^2+4)}}\Big)$. Furthermore, the exponential term in (3.19) satisfies $e^{2\left(\frac{5}{4} - \frac{1}{k^2}\right)\zeta^{\text{out}}} = \mathscr{O}\big(\varepsilon^{2/k^2 - 5/2}\big)$. Combining the above, we conclude that $|W^{\text{out}} - \widehat{W}^{\text{out}}| = \mathscr{O}(\varepsilon^{\kappa(k)})$, where $\kappa(k) = \frac{k^4+16k^2-48}{2k^2(k^2+4)}$. We note that $\kappa(k) > \frac{1}{2}$ for $k \in (2, \infty)$, with $\lim_{k\to\infty}\kappa(k) = \frac{1}{2}$. Hence, it follows that $\big|W^{\text{out}} - \widehat{W}^{\text{out}}\big| = \mathscr{O}(\varepsilon^\kappa) \to 0$ as $\varepsilon \to 0$ with $\kappa > \frac{1}{2}$, as stated. □

We can now solve (3.18) by separation of variables,

$$\frac{-2kW + k^2}{-2kW^2 + W(k^2 + 2\Delta ck - 4) - \Delta ck^2}\,dW = d\zeta, \tag{3.22}$$

where we have omitted overhats from $\widehat{W}$ for simplicity of notation. Integration of (3.22) gives

$$\begin{aligned} &\zeta^{\text{out}} - \zeta^{\text{in}} - \frac{1}{2}\ln\big|-2kW^2 + (k^2 + 2k\Delta c - 4)W - k^2\Delta c\big|\Big|_{W^{\text{in}}}^{W^{\text{out}}} \\ &- \frac{\frac{k^2}{2} + 2 - k\Delta c}{\sqrt{(k^2 + 2k\Delta c - 4)^2 - 8k^3\Delta c}} \\ &\times \ln\left|\frac{-4kW + k^2 + 2k\Delta c - 4 - \sqrt{(k^2 + 2k\Delta c - 4)^2 - 8k^3\Delta c}}{-4kW + k^2 + 2k\Delta c - 4 + \sqrt{(k^2 + 2k\Delta c - 4)^2 - 8k^3\Delta c}}\right|\Bigg|_{W^{\text{in}}}^{W^{\text{out}}} = 0. \end{aligned} \tag{3.23}$$

Recall that, as $\varepsilon_1(\zeta) = \frac{\varepsilon}{r_0}e^{\zeta}$, we have $\zeta^{\text{in}} = 0$ and $\zeta^{\text{out}} = -\ln\frac{\varepsilon}{r_0}$ in (3.23). Moreover, by Lemma 19,

$$W^{\text{in}} = \nu(r_0)\Delta c + \mathscr{O}(\Delta c^2, r_0^{4/k^2}\Delta c) \quad \text{and} \quad W^{\text{out}} = -\frac{2}{k} + \Delta c - \frac{k}{2}\varepsilon + \mathscr{O}(\varepsilon^2).$$

We now proceed as follows: given (3.23), we derive a necessary condition on Δc which will determine the leading-order asymptotics thereof in ε.

We begin by substituting our estimates for W^{in} and W^{out} into the first logarithmic term in (3.23), which gives

$$\begin{aligned}&\frac{1}{2}\ln\left|-2k(W^{\text{in}})^2 + (k^2 + 2k\Delta c - 4)W^{\text{in}} - k^2\Delta c\right| \\ &\quad = \frac{1}{2}\ln\left|(k^2-4)\nu(r_0)\Delta c - k^2\Delta c + \mathscr{O}(\Delta c^2, r_0^{4/k^2}\Delta c)\right|\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}&\frac{1}{2}\ln\left|-2k(W^{\text{out}})^2 + (k^2 + 2k\Delta c - 4)W^{\text{out}} - k^2\Delta c\right| \\ &\quad = \frac{1}{2}\ln\left|-2k + \mathscr{O}(\Delta c, \varepsilon)\right|,\end{aligned} \tag{3.25}$$

respectively.

Now, we expand the rational function multiplying the second logarithmic term in (3.23) as

$$-\frac{\frac{k^2}{2} + 2 - k\Delta c}{\sqrt{(k^2+2k\Delta c-4)^2 - 8k^3\Delta c}} = -\frac{k^2+4}{2(k^2-4)} - \frac{16k^3}{(k^2-4)^3}\Delta c + \mathscr{O}(\Delta c^3), \tag{3.26}$$

and we write the argument of the logarithm therein as

$$\begin{aligned}&\frac{-4kW + k^2 + 2k\Delta c - 4 - \sqrt{(k^2+2k\Delta c-4)^2 - 8k^3\Delta c}}{-4kW + k^2 + 2k\Delta c - 4 + \sqrt{(k^2+2k\Delta c-4)^2 - 8k^3\Delta c}} \\ &\quad = -1 + 2\frac{-4kW + k^2 + 2k\Delta c - 4}{-4kW + k^2 + 2k\Delta c - 4 + \sqrt{(k^2+2k\Delta c-4)^2 - 8k^3\Delta c}}.\end{aligned} \tag{3.27}$$

Substituting the estimate for W^{in} into (3.27), we have

$$\begin{aligned}&-1 + 2\frac{(-4k\nu(r_0) + 2k)\Delta c + k^2 - 4}{-4k\nu(r_0)\Delta c + k^2 + 2k\Delta c - 4 + \sqrt{(k^2+2k\Delta c-4)^2 - 8k^3\Delta c}} + \mathscr{O}(\Delta c^2, r_0^{4/k^2}\Delta c) \\ &\quad = \left[\frac{2k^3}{(k^2-4)^2} - \frac{2k\nu(r_0)}{k^2-4}\right]\Delta c + \mathscr{O}(\Delta c^2, r_0^{4/k^2}\Delta c).\end{aligned} \tag{3.28}$$

Similarly, we can use our estimate for W^{out} in (3.27) to obtain

$$-1 + 2\frac{4+k^2}{2k^2} + \mathscr{O}(\Delta c, \varepsilon) = \frac{4}{k^2} + \mathscr{O}(\Delta c, \varepsilon). \tag{3.29}$$

Summarising the above calculations, we can write (3.23) as

$$\begin{aligned}&-\ln\frac{\varepsilon}{r_0} + \frac{1}{2}\ln\left|(k^2-4)\nu(r_0)\Delta c - k^2\Delta c + \mathscr{O}(\Delta c^2, r_0^{4/k^2}\Delta c)\right| - \frac{1}{2}\ln\left|-2k + \mathscr{O}(\Delta c, \varepsilon)\right| \\ &\quad - \left[\frac{k^2+4}{2(k^2-4)} + \mathscr{O}(\Delta c)\right]\left[-\ln\left|\left[\frac{2k^3}{(k^2-4)^2} - \frac{2k\nu(r_0)}{k^2-4}\right]\Delta c + \mathscr{O}(\Delta c^2, r_0^{4/k^2}\Delta c)\right|\right. \\ &\quad \left. + \ln\left|\frac{4}{k^2} + \mathscr{O}(\Delta c, \varepsilon)\right|\right] = 0.\end{aligned} \tag{3.30}$$

Now, we exponentiate (3.30) to obtain

$$\left(\frac{\varepsilon}{r_0}\right)^2 = \frac{[k^2-(k^2-4)\nu(r_0)]\Delta c + \mathscr{O}(2)}{2k+\mathscr{O}(1)} \times \left(\frac{\left[\frac{2k^3}{(k^2-4)^2} - \frac{2k\nu(r_0)}{k^2-4}\right]\Delta c + \mathscr{O}(2)}{\frac{4}{k^2}+\mathscr{O}(1)}\right)^{\frac{k^2+4}{k^2-4}}, \tag{3.31}$$

where $\mathscr{O}(1)$ denotes terms that are of at least order 1 in Δc and ε, while $\mathscr{O}(2)$ stands for terms of at least order 2 in Δc and r_0^{4/k^2}. Solving for Δc in (3.31), we obtain

$$\Delta c = \alpha(k)\varepsilon^{1-\frac{4}{k^2}}[1+o(1)], \tag{3.32}$$

where

$$\alpha(k) = \frac{1}{r_0^{1-4/k^2}\left[k^2-(k^2-4)\nu(r_0)\right]} \frac{(2k)^{1/2(1-4/k^2)}\left[2(k^2-4)^2\right]^{1/2(1+4/k^2)}}{k^{3/2(1+4/k^2)}}. \tag{3.33}$$

For future reference, we label the r_0-dependent contribution to $\alpha(k)$ as

$$\delta(r_0) = r_0^{1-4/k^2}\left[k^2-(k^2-4)\nu(r_0)\right]. \tag{3.34}$$

In spite of the function $\nu(r_0)$, as defined in Lemma 19, being dependent on r_0, that dependence must cancel, as the choice of r_0 in the definition of Σ_1^{in} is arbitrary. Therefore, we can take the limit as $r_0 \to 0^+$ in (3.34).

Lemma 22. The function δ defined in Equation (3.34) satisfies

$$\lim_{r_0\to 0^+} \delta(r_0) = (k^2-4)\Gamma(1+4/k^2)\Gamma(1-4/k^2), \tag{3.35}$$

where $k > 2$.

Proof. We begin by writing $\delta(r_0)$ as

$$\delta(r_0) = r_0^{1-4/k^2}k^2 + \frac{k^2}{k^2+4}(k^2-4)(1-r_0)\,{}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1-r_0\right), \tag{3.36}$$

using the definition of $\nu(r_0)$ from Lemma 19. Taking $r_0 \to 0^+$, we find

$$\begin{aligned}\lim_{r_0\to 0^+}\delta(r_0) &= \frac{k^2}{k^2+4}(k^2-4)\,{}_2F_1\left(1+4/k^2, 4/k^2, 2+4/k^2, 1\right)\\ &= \frac{k^2-4}{1+4/k^2}\frac{\Gamma(2+4/k^2)\Gamma(1-4/k^2)}{\Gamma(1)\Gamma(2)}\\ &= (k^2-4)\Gamma(1+4/k^2)\Gamma(1-4/k^2).\end{aligned} \tag{3.37}$$

Here, we have used the identities ${}_2F_1(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ [25, Equation 15.1.20] and $\Gamma(2+4/k^2) = (1+4/k^2)\Gamma(1+4/k^2)$, as well as the fact that $\Gamma(1) = 1 = \Gamma(2)$, which completes the proof. □

Proposition 23. Let $\varepsilon \in (0,\varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small, and let $k > 2$. Then, the function Δc defined in Theorem 2 satisfies

$$\Delta c(\varepsilon) = \frac{2}{k^{1+8/k^2}} \frac{(k^2-4)^{4/k^2}}{\Gamma(1+4/k^2)\Gamma(1-4/k^2)} \varepsilon^{1-4/k^2}[1+o(1)]. \tag{3.38}$$

Proof. The statement follows directly from Lemma 22 and Equations (3.32) and (3.33). □

Hence, the proof of Theorem 2 is complete in the pushed front propagation regime, which is realised for $k > 2$ in (2.1).

Remark 24. We note that $\varepsilon^{1-4/k^2} \to \varepsilon^0 = 1$ as $k \to 2^+$ in (3.38), i.e., as we approach the pulled propagation regime. L'Hôpital's Rule shows that the corresponding coefficient tends to 0 in that limit, which is consistent with Theorem 2, as the correction Δc is logarithmic in ε for $k \leq 2$.

Remark 25. A simplification of the general expression for Δc in Equation (3.38) is achieved for specific values of k in (2.1); e.g., $k = 2\sqrt{2}$ gives $c(\varepsilon) = c(0) - \Delta c(\varepsilon) = \sqrt{2} + \frac{1}{\sqrt{2}} - \frac{1}{\pi}\varepsilon^{1/2}[1 + o(1)]$. Similarly, for $k = 4$, we have $c(\varepsilon) = \frac{5}{2} - \frac{\sqrt[4]{3}}{\pi}\varepsilon^{3/4}[1 + o(1)]$.

3.3 Numerical verification

In this subsection, we verify the asymptotics in Theorem 2 by calculating numerically the error incurred by approximating $c(\varepsilon)$ with the corresponding first-order expansion (in ε), which we denote by $\hat{c}(\varepsilon)$; for $k = 4$, e.g., we have $\hat{c}(\varepsilon) = \frac{5}{2} - \frac{\sqrt[4]{3}}{\pi}\varepsilon^{3/4}$. The numerical value of $c(\varepsilon)$ is obtained by integrating Equation (2.1) and storing the final value of $U = U_{\text{final}}(c)$ obtained after a sufficiently large number of time steps. We then minimise $|U_{\text{final}}(c)|$, taking $\hat{c}(\varepsilon)$ as our initial value of c. Our findings are illustrated in Figure 4 for $k \in \{1, \frac{3}{2}, 2\sqrt{2}, 4\}$, where we have used a double logarithmic scale, with $\varepsilon \in [10^{-4}, 10^{-2}]$. Figure 4 suggests that the next-order correction to $c(\varepsilon)$ will be of the order $\mathscr{O}[(\ln\varepsilon)^{-3}]$ for $k = 1$ and $k = \frac{3}{2}$, whereas it will be $\mathscr{O}(\varepsilon)$ for $k = 2\sqrt{2}$ and $\mathscr{O}(\varepsilon^{3/2})$ for $k = 4$.

4 Discussion

In this article, we have proven the existence of "critical" travelling front solutions to the Burgers-FKPP equation with a Heaviside cut-off multiplying both the reaction kinetics and the advection term, recall Equation (1.7). Moreover, we have rigorously derived the leading order ε-asymptotics of the unique front propagation speed $c(\varepsilon)$. To the best of our knowledge, the effects of a cut-off on advection-reaction-diffusion equations of the type in (1.2) have not been studied before.

For $k \leq 2$, the front is pulled and behaves as the pulled fronts with a cut-off considered, e.g., in [6, 10], with the correction to the front propagation speed being negative and of the order $\mathscr{O}[(\ln\varepsilon)^{-2}]$. When $k > 2$, the front is pushed, and the correction to the speed of propagation is also negative, and proportional to a fractional power of ε, again in analogy to the pushed fronts in reaction-diffusion equations with a cut-off analysed in [10]. While the proof of Theorem 2 in the pulled propagation regime where $k \leq 2$ closely follows the proof of [6, Theorem 1], we have included it for completeness. The analysis of the pushed regime, with $k > 2$, is significantly more involved algebraically and relies on a modification of the approach developed in [9, 10, 11]. Our main analytical contribution in this article can be found in Section 3, where we adapt techniques from both [6] and [9, 10, 11] to derive the asymptotics in (1.10).

It is important to emphasise that the blow-up technique is applied in the present context to remedy a discontinuity in the governing equations, rather than a loss of hyperbolicity, as is typically the case in applications of blow-up [8, 26]. Given that the regularisation of piecewise smooth systems via the alternative methodology developed in [30] typically results in a singular perturbation problem, it may be feasible to adapt that well-established methodology to our setting; see [31] for a specific application.

It may be possible to calculate higher-order terms in ε in the expansion of $c(\varepsilon)$ in the pushed regime; however, to do so, one must solve for $\frac{\partial^j V}{\partial c^j}(U, c(0))$ $(j \geq 2)$ via the procedure outlined in Lemma 18. We note that the procedure will fail for general pulled fronts with $k < 2$, as the front is not explicitly known in those cases, preventing us from calculating W^{in} to a sufficiently high order to determine higher-order terms in $c(\varepsilon)$. However, at the boundary between the pushed and pulled regimes, when $k = 2$, the front is known explicitly, as is $\frac{\partial V}{\partial c}(U, c(0))$, see Lemma 18. Hence, one could approximate W^{in} to a sufficiently high order, which may make it possible to determine the next term in the expansion of $c(\varepsilon)$ when $k = 2$.

By retracing the proof of Theorem 2, one can show that the leading-order correction to the front speed is independent of whether or not the advection term in Equation (1.7) is multiplied with the cut-off function $H(u - \varepsilon)$. That observation

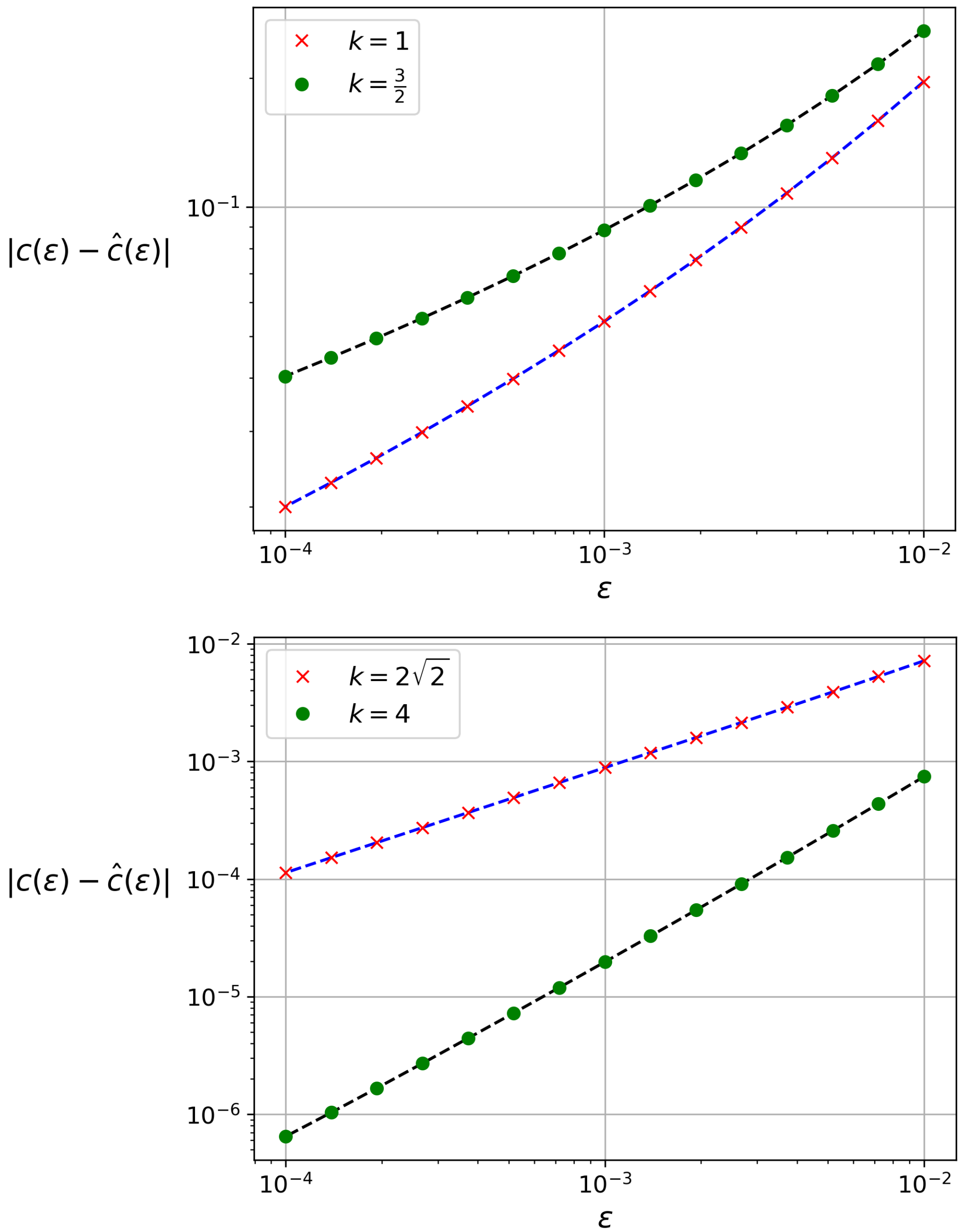

Figure 4: Error of the approximation of $c(\varepsilon)$ by $\hat{c}(\varepsilon)$ for $k \in \{1, \frac{3}{2}, 2\sqrt{2}, 4\}$ and $\varepsilon \in [10^{-4}, 10^{-2}]$.

is supported by the motivating example of the Burgers equation with a cut-off, Equation (1.9), where the correction to the front speed is given by $\Delta c(\varepsilon) = \frac{k}{2}\varepsilon^2$ to leading order. As both $(\ln\varepsilon)^{-2}$ and ε^{1-4/k^2} are of lower order compared to ε^2, it is to be expected that a cut-off in the advection term will not affect the leading-order asymptotics of $\Delta c(\varepsilon)$. Since the requisite argument is very similar to the proof of Theorem 2, we outline it in Appendix A.

In Theorem 2, we have restricted to a Heaviside cut-off function $H(u-\varepsilon)$ in Equation (1.7); that restriction appears reasonable, as the Heaviside cut-off cancels the reaction kinetics and the advection term exactly when no particles are present in the underlying N-particle system. One can instead introduce a general cut-off function $\psi(u,\varepsilon)$ in (1.7) which satisfies $\psi(u,\varepsilon) \equiv 1$ when $u > \varepsilon$ and $\psi(u,\varepsilon) < 1$ for $u < \varepsilon$. Equation (1.1) with Fisher reaction kinetics and a general cut-off has been considered in [6], while a linear cut-off function was studied explicitly in [11]. In the context of Equation (1.7), one can show that to leading order, $\Delta c = \frac{\pi^2}{\ln(\varepsilon)^2}$ in the pulled propagation regime for a wide range of cut-off functions which includes the Heaviside cut-off [6]. Hence, the leading-order asymptotics of Δc is then universal, as was also the case in [6]. In the pushed regime, one again obtains $\Delta c = \mathscr{O}(\varepsilon^{1-4/k^2})$; however, it is not possible to calculate explicitly the corresponding leading-order coefficient for a general cut-off function ψ, since explicit knowledge of the entry point in chart K_2 is required. In particular, that coefficient will be cut-off-dependent then, in contrast to the pulled propagation regime, as is also the case in [6, 10, 32].

Finally, we note that Theorem 2 can be extended to advection-reaction-diffusion equations with a more general advection term,

$$\frac{\partial u}{\partial t} + ku^n \frac{\partial u}{\partial x} H(u-\varepsilon) = \frac{\partial^2 u}{\partial x^2} + u(1-u^n)H(u-\varepsilon), \tag{4.1}$$

where $n \geq 2$ is integer-valued. For $\varepsilon = 0$, Equation (4.1) admits a pulled front for $k \leq n+1$ and $c \geq 2$, whereas for $k > n+1$, there exists a pushed front solution for $c \geq \frac{k}{n+1} + \frac{n+1}{k}$, which can be shown in analogy to the proof of Theorem 1 via the approach outlined in [22]. For $c = \frac{k}{n+1} + \frac{n+1}{k}$, the sought-after front corresponds, in a co-moving frame, to the heteroclinic orbit $V(U) = -\frac{k}{n+1}U(1-U^n)$. Due to the increased algebraic complexity, we leave the study of the impact of a cut-off on Equation (4.1) for the future. However, we note that, as the front is explicitly known in the pushed regime, it is likely that the leading-order correction to the propagation speed $c(\varepsilon)$ can be calculated explicitly for all $k > 0$.

A Proof of Theorem 2 without cut-off in advection

Here, we briefly show that Theorem 2 remains equally valid for the advection-reaction-diffusion-equation

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u)H(u-\varepsilon), \tag{A.1}$$

in which the advection term $ku\frac{\partial u}{\partial x}$ is not affected by the cut-off. The corresponding first-order system then reads

$$\begin{aligned} U' &= V, \\ V' &= -\gamma V + kUV - U(1-U)H(U-\varepsilon), \\ \varepsilon' &= 0, \end{aligned} \tag{A.2}$$

where the travelling wave variable is now defined by $\xi = x - \gamma t$, with γ the front propagation speed. The analysis of (A.2) again relies on the blow-up transformation in (2.2). We observe that (1.7) and (A.1) are identical for $u > \varepsilon$; therefore, it suffices to study (A.2) in the rescaling chart K_2 only, which is again defined by (2.3):

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -\gamma v_2 + kr_2u_2v_2, \\ r_2' &= 0. \end{aligned} \tag{A.3}$$

By taking $\varepsilon = r_2 \to 0^+$ and solving for $v_2(u_2)$, we obtain (2.7), i.e., the singular orbit Γ_2 in K_2 is given as before. By the above, we can conclude that Proposition 12 holds for (A.1).

It remains to consider the persistence of Γ for (A.1). We can solve (A.3) explicitly for general γ and $\varepsilon(= r_2) > 0$ to obtain

$$W_2^{\mathrm{s}}(\ell_2^+):\ v_2(u_2) = -\gamma u_2 + \frac{k}{2} r_2 u_2^2. \tag{A.4}$$

The point of intersection of $W_2^{\mathrm{s}}(\ell_2^+)$ with Σ_2^{in} is given by $P_2^{\mathrm{in}} = (1, v_2^{\mathrm{in}}, \varepsilon)$, where $v_2^{\mathrm{in}} = -\gamma + \frac{k}{2}\varepsilon$. From the definition of (2.2), it follows that $V^{\mathrm{in}} = v_2^{\mathrm{in}}\varepsilon = -\gamma\varepsilon + \frac{k}{2}\varepsilon^2$. One can now prove a similar result to Proposition 13 for (A.2), where $c(0) = c_{\mathrm{crit}}$ is again defined as in Theorem 1.

Proposition 26. For $\varepsilon \in (0, \varepsilon_0)$, with ε_0 sufficiently small, $k > 0$, and γ close to $c(0)$, there exists a critical heteroclinic connection between Q^- and Q^+ in Equation (A.2) for a unique speed $\gamma(\varepsilon)$ which depends on k. Furthermore, there holds $\gamma(\varepsilon) \leq c(0)$.

The proof is similar to that of Proposition 13, the only difference being that $V^{\mathrm{in}} = -c\varepsilon$ is replaced by $V^{\mathrm{in}} = v_2^{\mathrm{in}}\varepsilon = -\gamma\varepsilon + \frac{k}{2}\varepsilon^2$. In spite of that difference, the argument from the proof of Proposition 13 carries over verbatim.

The remainder of the analysis in Section 3 equally translates to Equation (A.1). The sole difference concerns the point $P_1^{\mathrm{out}} = (\varepsilon, W^{\mathrm{out}}, 1)$, where W^{out} is derived from v_2^{in}. We have the following result on the leading-order asymptotics of W^{out}.

Lemma 27. For $k > 2$ and ε and $\Delta\gamma$ sufficiently small, the point $P_1^{\mathrm{out}} = (\varepsilon, W^{\mathrm{out}}, 1)$ satisfies

$$W^{\mathrm{out}} = -\frac{2}{k} + \mathcal{O}(\Delta\gamma, \varepsilon^2), \tag{A.5}$$

where $\gamma(\varepsilon) = c(0) - \Delta\gamma$.

Proof. Equation (A.5) follows from the definition of V^{in} for (A.2) and the sequence of transformations defined in Lemma 17. □

In Section 3.2.1, i.e., in the pulled regime, we found that the leading-order asymptotics of Δc is independent of W^{in} and W^{out}; therefore, we can conclude that Theorem 2 holds for Equation (A.1) when $k \leq 2$, i.e., that $\Delta\gamma = \Delta c$ to leading order.

As the asymptotics of W^{out} for (A.1) in Lemma 27 differs from that in Lemma 19 at $\mathcal{O}(\varepsilon)$, and as only the constant term $-\frac{2}{k}$ is required to derive the leading-order asymptotics of Δc, we can conclude that Theorem 2 holds for (A.1) in the pushed front propagation regime where $k > 2$.

In Figure 5, we compare the propagation speeds $c(\varepsilon)$ and $\gamma(\varepsilon)$ for (1.7) and (4.1), respectively. We find that $|c(\varepsilon) - \gamma(\varepsilon)|$ is of higher order than $|c(\varepsilon) - \hat{c}(\varepsilon)|$, which we plot in red for comparison. For example, for $k = 4$, the propagation speeds $c(\varepsilon)$ and $\gamma(\varepsilon)$ differ approximately at order $\mathcal{O}(\varepsilon^{9/5})$, whereas the difference between the propagation speed $c(\varepsilon)$ for (1.7) and its leading-order approximation $\hat{c}(\varepsilon)$, given in Theorem 2, is of the order $\mathcal{O}(\varepsilon^{3/2})$.

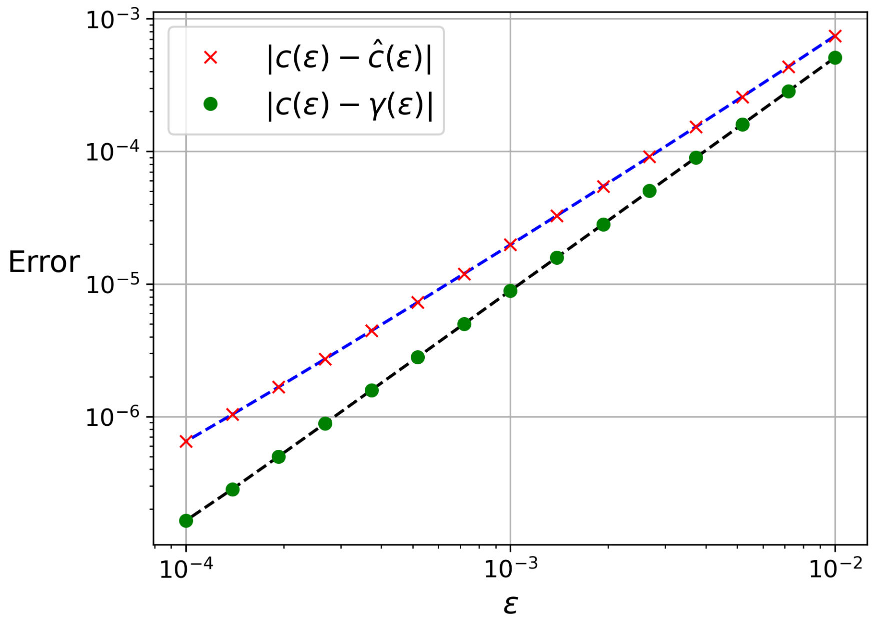

Figure 5: Numerical difference between $c(\varepsilon)$ and $\gamma(\varepsilon)$ (green) for $k = 4$ and $\varepsilon \in [10^{-4}, 10^{-2}]$; the error $|c(\varepsilon) - \hat{c}(\varepsilon)|$ is plotted for comparison (red).

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