Cut-offs in a degenerate advection-reaction-diffusion equation – a case study

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Abstract

We investigate the effect of a Heaviside cut-off on the front propagation dynamics of a degenerate advection-reaction -diffusion equation. In particular, we consider two formulations of the equation, one with the cut-off function multiplying the reaction kinetics alone and one in which the cut-off is also applied to the advection term. We prove the existence and uniqueness of a "critical" front solution in both cases, and we derive the leading-order correction to the front propagation speed in dependence on the advection strength and the cut-off parameter. We show that, while the asymptotics of the correction in the cut-off parameter remains unchanged to leading order when the advection term is cut off, the corresponding coefficient is different. Finally, we consider a generalised family of advection-reaction-diffusion equations, and we identify scenarios in which the application of a cut-off to the advection term substantially affects the front propagation speed. Our analysis relies on geometric techniques from dynamical systems theory and, specifically, on geometric desingularisation, also known as "blow-up".

1 Introduction

The Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation with a Heaviside cut-off function multiplying the reaction kinetics was first studied by Brunet and Derrida [3]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)H(u-\varepsilon), \tag{1.1}$$

with $u = u(x,t)$ for $x \in \mathbb{R}$ and $t \geq 0$. Here, the cut-off models the situation where no reaction is possible for low particle densities or concentrations, i.e., when $u \leq \frac{1}{N} =: \varepsilon$ for N sufficiently large, with N being the number of particles. Specifically, the Heaviside cut-off in Equation (1.1) produces an $\mathscr{O}[(\ln\varepsilon)^{-2}]$-shift in the speed of the front connecting the rest states $u = 1$ and $u = 0$, and hence addresses the discrepancy observed between the front propagation speed found for the FKPP equation without cut-off and the speed in the corresponding, discrete N-particle system.

In this article, we aim to show that a cut-off in the nonlinear advection term can also induce a shift in the front propagation speed in a family of advection-reaction-diffusion equations. Nonlinear advection terms naturally appear in the continuum approximation of discrete systems; examples include models for traffic flow [11, 12] or density-dependent migration of populations [8, 7]. In both of these, nonlinear advection may produce a discrepancy in front speed between the advection-reaction-diffusion equation and the corresponding discrete system.

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As a motivating example, we consider the (viscous) Burgers equation

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \sigma \frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

where the term $ku\frac{\partial u}{\partial x}$ models the directed transport of u, with $k > 0$ the strength of the advection, while $\sigma > 0$ is the diffusion coefficient. It is straightforward to show that Equation (1.2) admits a front solution connecting $u = 1$ and $u = 0$ that propagates with speed $c = \frac{k}{2}$. As is the case for the reaction kinetics in (1.1), the advection term in (1.2) should be zero when $u \leq \varepsilon\big(= \frac{1}{N}\big)$. Therefore, it seems plausible to multiply that term with a Heaviside cut-off function, which yields

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x}H(u-\varepsilon) = \sigma \frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

In analogy to (1.2), Equation (1.3) admits a front connecting $u = 1$ and $u = 0$ that propagates with speed $c = \frac{k}{2} - \Delta c(\varepsilon)$, where $\Delta c(\varepsilon) = \frac{k}{2}\varepsilon^2$. In particular, since $\Delta c(\varepsilon)$ is positive, the speed of that front is therefore reduced by a cut-off. This reduction in propagation speed is substantiated by simulations of a corresponding N-particle system for the Burgers equation which is introduced in [2]. We provide the computational details in Appendix A. In Figure 1, we plot the tails of the fronts computed for the N-particle system, as well as the exact solution $u(x,t) = \big(e^{5(x-t/2)} + 1\big)^{-1}$ to (1.2), for $k = 1$ and $\sigma = \frac{1}{10}$; here, $T = 3$, with step size $\Delta_t = \frac{1}{1000}$ [2]. It is evident that the simulated, discrete fronts lag behind the true solution, and that they approach it with increasing N. As the shift in the speed of the front induced by a cut-off in (1.3) is $\mathscr{O}(\varepsilon^2)$, we expect the optimal rate of convergence for the front speed in the underlying N-particle system to be of the order $\mathscr{O}(N^{-2})$.

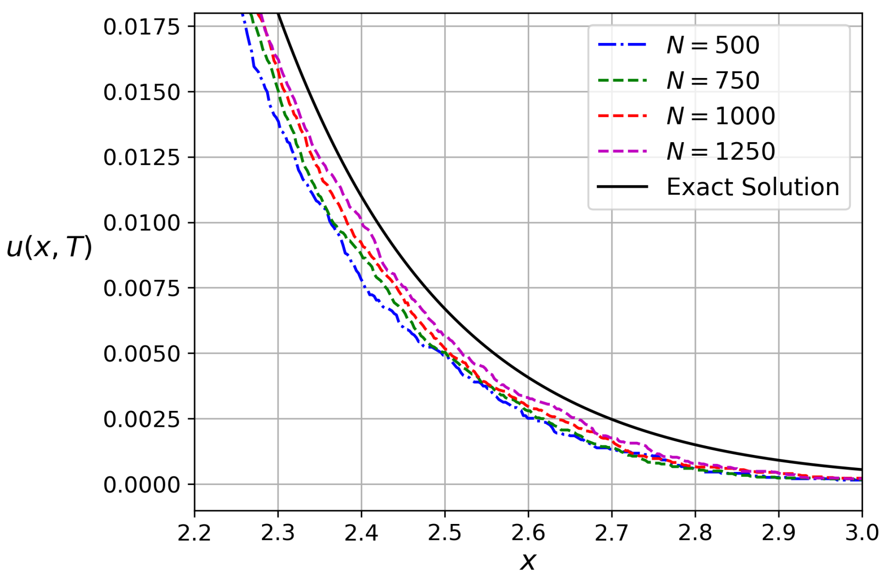

Figure 1: Particle simulation, averaged over 40 simulation runs, for Equation (1.3) with $k = 1$ and $\sigma = \frac{1}{10}$.

Front propagation in the Burgers-FKPP advection-reaction-diffusion equation with cut-off,

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x}H(u-\varepsilon) = \frac{\partial^2 u}{\partial x^2} + u(1-u)H(u-\varepsilon), \tag{1.4}$$

has been studied in [10], where it was observed that the leading-order correction to the propagation speed of the front is independent of a Heaviside cut-off multiplying the advection term. That is not too surprising, as in both the pulled and pushed front propagation regimes, the leading-order ε-correction in the expansion for the speed $c(\varepsilon)$ is of lower order than the correction to the speed of propagation induced by the advection term alone, as seen in the Burgers equation with cut-off in (1.3). Hence, from a modelling perspective, it is often sufficient to consider merely a cut-off in the reaction kinetics.

However, we do not expect that observation to be true in general. For certain advection-reaction-diffusion equations where advection is of comparable order (in u) to the reaction kinetics, application of a cut-off to the kinetics only should produce a significantly different front propagation speed than when both the reaction kinetics and the advection term are cut off. Moreover, the speed of the front should differ substantially in the underlying N-particle system in these cases. The purpose of this article is to consider such a case. Recall that the speed of the front solution in the Burgers equation with cut-off in (1.3) is reduced by $\Delta c(\varepsilon) = \frac{k}{2}\varepsilon^2$. For the Zeldovich equation with cut-off,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u)H(u-\varepsilon), \tag{1.5}$$

on the other hand, it has been shown in [9, Theorem 1.1] that the shift in the propagation speed of the front due to a Heaviside cut-off is also of the order $\mathscr{O}(\varepsilon^2)$. Hence, one may expect that a combination of these two equations will produce an example where the leading-order ε-asymptotics of the correction to the front propagation speed is driven by both advection and the reaction kinetics.

Correspondingly, in this article, we consider the degenerate advection-reaction-diffusion equation

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u), \tag{1.6}$$

which is a realisation of the so-called generalised Burgers-Fisher equation [7, Equation (1.1)] from population dynamics. Here and in the following, we assume without loss of generality that the diffusion coefficient has been rescaled to $\sigma = 1$.

As we are interested in travelling wave solutions to (1.6), we set $\xi = x - ct$ for $c > 0$. With $U(\xi) = u(x,t)$ and $U' = V$, we then obtain the first-order system of equations

$$\begin{aligned} U' &= V, \\ V' &= -cV + kUV - U^2(1-U). \end{aligned} \tag{1.7}$$

We note that heteroclinic orbits connecting the equilibria of (1.7) are equivalent to travelling wave solutions of (1.6). We have the following basic result; as the proof is straightforward, we omit it here.

Proposition 1. The first-order system in (1.7) admits a heteroclinic orbit connecting the equilibria $Q^- = (1,0)$ and $Q^+ = (0,0)$ for $c \geq c_{\text{crit}} = \frac{1}{4}(k + \sqrt{k^2+8})$. Moreover, for $c = c_{\text{crit}}$, that heteroclinic orbit is given explicitly by $V = -c_{\text{crit}}U(1-U)$.

We note that the orbit obtained for the "critical" speed c_{crit} corresponds to the front solution of (1.6) with the fastest decay rate (in ξ), whereas fronts propagating with $c > c_{\text{crit}}$ decay at slower rates [1].

After multiplication of either both the reaction kinetics and the advection term in (1.6), or of the kinetics only, with a Heaviside cut-off, we obtain the two degenerate advection-reaction-diffusion equations

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x}H(u-\varepsilon) = \frac{\partial^2 u}{\partial x^2} + u^2(1-u)H(u-\varepsilon) \quad \text{and} \tag{1.8}$$

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u)H(u-\varepsilon), \tag{1.9}$$

which will be considered in the following.

Our main result can be stated as follows.

Theorem 2. Let $\varepsilon \in [0, \varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small, and let $k > 0$. Then, there exist unique propagation speeds $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$, with $\lim_{\varepsilon \to 0^+} c^{(i)}(\varepsilon) = c_{\text{crit}}$ $(i = 1, 2)$ the critical speed in the absence of a cut-off, such that

Equations (1.8) and (1.9), respectively, admit unique critical front solutions connecting the rest states $u = 1$ and $u = 0$. Moreover, $c^{(i)}(\varepsilon) = c_{\text{crit}} - \Delta c^{(i)}(\varepsilon)$, where

$$\Delta c^{(1)}(\varepsilon) = \frac{1}{4}\big(-k + 3\sqrt{k^2+8}\big)\varepsilon^2 \quad \text{and} \tag{1.10}$$

$$\Delta c^{(2)}(\varepsilon) = 2\frac{-k+3\sqrt{k^2+8}}{\big(k+\sqrt{k^2+8}\big)^2}\varepsilon^2, \tag{1.11}$$

to leading order in ε.

Remark 3. We note that Theorem 2 only considers front solutions to (1.8) and (1.9) which are perturbations of the critical fronts obtained for $\varepsilon = 0$ and $c(0) := \lim_{\varepsilon\to 0^+} c^{(i)}(\varepsilon) = c_{\text{crit}}$, with $i = 1, 2$.

Theorem 2 implies, in particular, that the front propagation speed in (1.8) and (1.9) is reduced by a Heaviside cut-off, as $\Delta c^{(i)} > 0$ for $i = 1, 2$ when $k > 0$.

This article is organised as follows: in Section 2, we apply the blow-up technique (geometric desingularisation) to construct a singular heteroclinic orbit. The proof of Theorem 2 is completed in Section 3, where we prove persistence of the singular orbit and derive Equations (1.10) and (1.11). Furthermore, we provide numerical verification of our results. In Section 4, we extend our findings to a generalised family of advection-reaction-diffusion equations. Finally, in Section 5, we discuss our results and possible topics for future work.

2 Geometric desingularisation

We introduce the travelling wave variable $\xi^{(i)} = x - c^{(i)}t$ $(i = 1, 2)$ in (1.8) and (1.9), respectively. Writing $u(x,t) = U(\xi^{(i)})$ and appending the trivial equation for the cut-off parameter ε, we obtain

$$\begin{aligned} U' &= V, \\ V' &= -c^{(1)}V + kUVH(U-\varepsilon) - U^2(1-U)H(U-\varepsilon), \\ \varepsilon' &= 0 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} U' &= V, \\ V' &= -c^{(2)}V + kUV - U^2(1-U)H(U-\varepsilon), \\ \varepsilon' &= 0 \end{aligned} \tag{2.2}$$

from (1.8) and (1.9), respectively. Next, we define the following blow-up transformation at the origin in (2.1) and (2.2),

$$U = \bar{r}\bar{u}, \quad V = \bar{r}\bar{v}, \quad \text{and} \quad \varepsilon = \bar{r}\bar{\varepsilon}, \tag{2.3}$$

which serves to desingularise the non-smooth transition between the inner and outer regions that are defined by $\{U < \varepsilon\}$ and $\{U > \varepsilon\}$, respectively. Here, $(\bar{u}, \bar{v}, \bar{\varepsilon}) \in \mathbb{S}^2_+ := \{(\bar{u}, \bar{v}, \bar{\varepsilon}) \,|\, \bar{u}^2 + \bar{v}^2 + \bar{\varepsilon}^2 = 1\}$, with $\bar{r} \in [0, r_0]$ for $r_0 > 0$ sufficiently small and $\bar{\varepsilon} \geq 0$.

We will analyse the dynamics of (2.1) and (2.2) in two coordinate charts: we will consider the rescaling chart K_2 and the phase-directional chart K_1 which are defined by $\bar{\varepsilon} = 1$ and $\bar{u} = 1$, respectively. By construction, these charts correspond to the inner and outer regions where $U < \varepsilon$ and $U > \varepsilon$, respectively. By combining the dynamics in these two charts, we will construct a singular (in ε) heteroclinic orbit Γ. In particular, we will show that Γ represents the singular orbit for both (2.1) and (2.2).

2.1 Dynamics in the rescaling chart K_2

In this subsection, we construct Γ_2, which is the segment of the singular heteroclinic orbit Γ in chart K_2. We will find that Γ_2 is equal for (2.1) and (2.2), but that its perturbation for ε positive and small differs for the two systems.

Setting $\bar{\varepsilon} = 1$ in (2.3), we find the transformation

$$U = r_2 u_2, \quad V = r_2 v_2, \quad \text{and} \quad \varepsilon = r_2 \tag{2.4}$$

in K_2. We first describe the singular geometry of (2.1); then, we show that the geometry of (2.2) is identical in the singular limit.

2.1.1 Dynamics of (2.1) in chart K_2

Applying the transformation in (2.4) to (2.1), we obtain

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -c^{(1)} v_2 + k r_2 u_2 v_2 H(u_2 - 1) - r_2 u_2^2 (1 - r_2 u_2) H(u_2 - 1), \\ r_2' &= 0. \end{aligned} \tag{2.5}$$

In the inner region, we have $U < \varepsilon$, which is equivalent to $u_2 < 1$. Hence, $H(u_2 - 1) \equiv 0$, and we can reduce (2.5) to

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -c^{(1)} v_2, \\ r_2' &= 0. \end{aligned} \tag{2.6}$$

Equation (2.6) admits a line of equilibria at $\ell_2^+ = \{(0,0,r_2) \,|\, r_2 \in [0, r_0]\}$. (While equilibria are found for any $u_2 \in (0,1)$, only those on ℓ_2^+ correspond to Q^+ after blow-down.) We are particularly interested in the point $Q_2^+ = (0,0,0)$ which is obtained by taking the singular limit as $r_2 \to 0^+$ on ℓ_2^+.

Lemma 4. The eigenvalues of the linearisation of (2.6) about Q_2^+ are given by $-c(0)$ and 0 (double), where the second zero eigenvalue is due to the trivial r_2-equation. The corresponding eigenspaces are spanned by $(1, -c(0), 0)^T$ and $\{(1,0,0)^T, (0,0,1)^T\}$, respectively.

Taking $r_2 (= \varepsilon) \to 0^+$, we have $c^{(1)} \to c(0)$, which further reduces (2.6) to

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -c(0) v_2. \end{aligned} \tag{2.7}$$

Solving the equivalent equation $\frac{dv_2}{du_2} = -c(0)$ under the condition that $v_2(0) = 0$, we find the singular orbit

$$\Gamma_2 : \; v_2(u_2) = -c(0) u_2. \tag{2.8}$$

The section

$$\Sigma_2^{\text{in}} = \{(1, v_2, r_2) \,|\, (v_2, r_2) \in [-v_0, 0] \times [0, r_0]\}$$

is introduced to track Γ_2 as it leaves the rescaling chart K_2 in backward "time". (Here, $v_0 > 0$ is an appropriately chosen constant.) We define the entry point into K_2 as $P_2^{\text{in}} = \Gamma_2 \cap \Sigma_2^{\text{in}} = (1, -c(0), 0)$. For $r_2 \in (0, r_0]$, with $r_0 > 0$ sufficiently small, the stable manifold $W_2^{\text{s}}(\ell_2^+)$ of ℓ_2^+ can be obtained directly from (2.6) by solving $\frac{dv_2}{du_2} = -c^{(1)}$ under the condition that $v_2(0) = 0$, which gives

$$W_2^{\text{s}}(\ell_2^+) : \; v_2(u_2) = -c^{(1)} u_2. \tag{2.9}$$

2.1.2 Dynamics of (2.2) in chart K_2

We now consider the dynamics of (2.2) in chart K_2. Applying (2.4) to (2.2) and recalling that $H(u_2-1)\equiv 0$, we obtain

$$\begin{aligned} u_2' &= v_2, \\ v_2' &= -c^{(2)}v_2 + kr_2u_2v_2, \\ r_2' &= 0. \end{aligned} \tag{2.10}$$

Taking $r_2 \to 0^+$ in (2.10), we again find (2.7). Hence, the singular orbit Γ_2 for (2.2) is given by (2.8), as before. Correspondingly, the entry point $P_2^{\text{in}} = \Gamma_2 \cap \Sigma_2^{\text{in}}$ into chart K_2 also equals $(1, -c(0), 0)$.

While the singular geometries of (2.6) and (2.10) are identical, differences arise when considering a perturbation to Γ_2 for $r_2 \in (0, r_0]$, with r_0 sufficiently small: we can determine the (non-singular) stable manifold $W_2^s(\ell_2^+)$ of ℓ_2^+ by solving $\frac{dv_2}{du_2} = -c^{(2)} + kr_2u_2$ under the condition that $v_2(0) = 0$ to obtain

$$W_2^s(\ell_2^+): \; v_2(u_2) = -c^{(2)}u_2 + \frac{k}{2}r_2u_2^2, \tag{2.11}$$

which clearly differs from (2.9).

The geometry of (2.6) and (2.10) in the rescaling chart K_2 is depicted in Figure 2; recall that, when $r_2 > 0$, $W_2^s(\ell_2^+)$ is given by (2.9) for the former, whereas it is defined in (2.11) for the latter.

Remark 5. The stable manifolds $W_2^s(\ell_2^+)$ in (2.9) and (2.11), respectively, are not required for the construction of the singular orbit Γ. However, precise knowledge of those manifolds, and of their intersections with Σ_2^{in} for $\varepsilon = r_2 > 0$, is necessary for determining $\Delta c^{(i)}$ $(i = 1, 2)$ to leading order in ε, see Section 3.

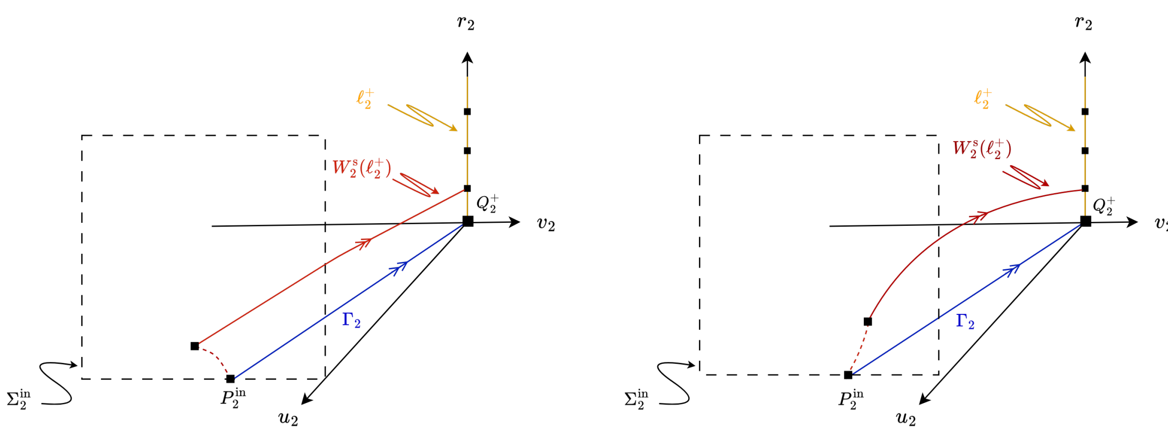

Figure 2: Geometry and dynamics of (2.6) (left) and (2.10) (right) in chart K_2.

2.2 Dynamics in the phase-directional chart K_1

In this subsection, we analyse the dynamics of (2.1) and (2.2) in the phase-directional chart K_1. We construct the singular orbit Γ_1, which is the continuous extension of Γ_2 in K_1. In the outer region, where $U > \varepsilon$, we have $H(U-\varepsilon) \equiv 1$. Hence, (2.1) and (2.2) are identical, and we can perform the analysis of both systems simultaneously.

For $\bar{u} = 1$ in (2.3), we obtain the following transformation in K_1,

$$U = r_1, \quad V = r_1v_1, \quad \text{and} \quad \varepsilon = r_1\varepsilon_1. \tag{2.12}$$

Applying (2.12) to (2.1) and (2.2) and noting that $H(1-\varepsilon_1) \equiv 1$ for $\varepsilon_1 < 1$, we find

$$\begin{aligned} r_1' &= r_1 v_1, \\ v_1' &= -c^{(i)} v_1 + k r_1 v_1 - r_1(1-r_1) - v_1^2, \\ \varepsilon_1' &= -\varepsilon_1 v_1, \end{aligned} \tag{2.13}$$

for $i = 1,2$.

The system in (2.13) admits a line of equilibria at $\ell_1^- = \{(1,0,\varepsilon_1) \,|\, \varepsilon_1 \in [0,\varepsilon_0]\}$. We are interested in the point $Q_1^- = (1,0,0) \in \ell_1^-$, which is obtained in the limit as $\varepsilon_1 \to 0^+$ and which corresponds to Q^- in the singular limit after blow-down.

A further equilibrium of (2.13) is found at $P_1 = (0,-c(0),0)$, for which we have the following result.

Lemma 6. The eigenvalues of the linearisation of (2.13) about $P_1 = (0,-c(0),0)$ are given by $-c(0)$ and $c(0)$ (double). The corresponding eigenspaces are spanned by $\big(2c(0), 1+kc(0), 0\big)^T$ and $\big\{(0,0,1)^T, (0,1,0)^T\big\}$, respectively.

Remark 7. For future reference, we note the occurrence of potential resonances in (2.13) at P_1, corresponding to monomials of the form $r_1^{j-1} v_1^j$ for integer-valued $j \geq 2$.

As $\varepsilon = r_1 \varepsilon_1$ in (2.3), we have to analyse both the limit as $r_1 \to 0^+$ and the limit of $\varepsilon_1 \to 0^+$. We will construct the corresponding segments Γ_1^+ and Γ_1^- of Γ_1 in the invariant planes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$, respectively.

The change of coordinates $\kappa_{21} : K_2 \to K_1$ between charts K_2 and K_1 is given by

$$\kappa_{21}: \; r_1 = r_2 u_2, \quad v_1 = v_2 u_2^{-1}, \quad \text{and} \quad \varepsilon_1 = u_2^{-1}. \tag{2.14}$$

We introduce the section Σ_1^{out} as

$$\Sigma_1^{\text{out}} = \{(r_1, v_1, 1) \,|\, (r_1, v_1) \in [0, r_0] \times [-v_0, 0]\}, \tag{2.15}$$

which is defined such that $\kappa_{21}(\Sigma_2^{\text{in}}) = \Sigma_1^{\text{out}}$, with $v_0 > 0$ chosen as before. Since $u_2 = 1$ in Σ_2^{in} or, equivalently, since $\varepsilon_1 = 1$ in Σ_1^{out}, we have $v_1^{\text{out}} = v_2^{\text{in}}$ and, therefore, $P_1^{\text{out}} = (0,-c(0),1)$.

To construct the segment Γ_1^+ of Γ_1, we take the limit as $r_1 \to 0^+$ in (2.13) to obtain

$$\begin{aligned} v_1' &= -c(0) v_1 - v_1^2, \\ \varepsilon_1' &= -\varepsilon_1 v_1. \end{aligned} \tag{2.16}$$

Solving $\frac{dv_1}{d\varepsilon_1} = -\frac{c(0)+v_1}{\varepsilon_1}$, with $v_1(1) = -c(0)$ due to $v_1^{\text{out}} = -c(0) = v_2^{\text{in}}$, we find

$$\Gamma_1^+ : \; v_1(\varepsilon_1) = -c(0). \tag{2.17}$$

The segment Γ_1^- is found in the limit as $\varepsilon_1 \to 0^+$ in (2.13). Hence,

$$\begin{aligned} r_1' &= r_1 v_1, \\ v_1' &= -c(0) v_1 + k r_1 v_1 - r_1(1-r_1) - v_1^2, \end{aligned} \tag{2.18}$$

which is equivalent to (1.7) after blow-down.

Lemma 8. For $r_1 \in (0,1]$, the system in (2.18) admits an explicit orbit

$$\Gamma_1^- : \; v_1(r_1) = -c(0)(1-r_1), \tag{2.19}$$

with $v_1(1) = 0$.

Proof. For $c(0) = \frac{1}{4}(k + \sqrt{k^2+8})(= c_{\text{crit}})$, we have an explicit orbit for (1.7) which is given by $V(U) = -c(0)U(1-U)$, see Proposition 1. Application of the transformation in (2.12) to that orbit gives the stated result. □

To track Γ_1^- as it enters a neighbourhood of P_1, we introduce the following section,

$$\Sigma_1^{\text{in}} = \{(r_0, v_1, \varepsilon_1) \,|\, (v_1, \varepsilon_1) \in [-v_0, 0] \times [0,1]\}, \tag{2.20}$$

which implies $P_1^{\text{in}} = \Gamma_1^- \cap \Sigma_1^{\text{in}} = (r_0, -c(0)(1-r_0), 0)$. The geometry of (2.13) in the phase-directional chart K_1 is depicted in Figure 3.

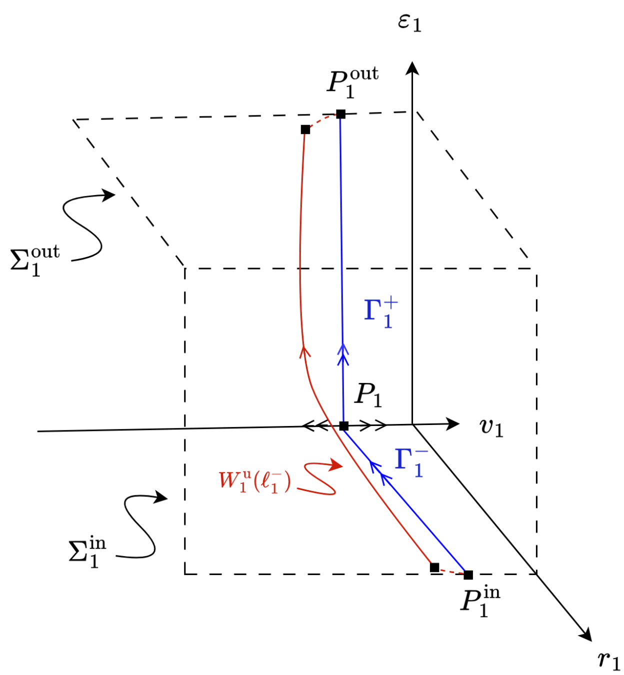

Figure 3: Geometry and dynamics of (2.13) in chart K_1.

2.3 Construction of the singular orbit $\overline{\Gamma}$

We conclude this section by constructing the singular orbit $\overline{\Gamma}$ in $(\bar{u}, \bar{v}, \bar{\varepsilon})$-space.

Proposition 9. For (2.1) and (2.2), there exists a singular heteroclinic orbit $\overline{\Gamma}$ connecting Q_1^- and Q_2^+.

Proof. The orbit $\overline{\Gamma}$ is given by the union of the orbits Γ_1^-, Γ_1^+, and Γ_2 with the equilibria Q_1^-, P_1, and Q_2^+ in blown-up phase space; see Figure 4 for an illustration. □

3 Persistence and leading-order asymptotics

In this section, we prove our main result, Theorem 2. We begin by showing that the singular orbit Γ constructed in the previous section persists for ε sufficiently small.

3.1 Persistence of the singular orbit Γ

Our first result implies the existence of a unique speed $c^{(i)}$ $(i = 1,2)$ for which Γ will persist as a heteroclinic connection between Q^- and Q^+.

Proposition 10. For $\varepsilon \in (0, \varepsilon_0)$, with ε_0 sufficiently small, $k > 0$, and $c^{(i)}$ $(i = 1,2)$ close to $c(0)$, there exists a critical heteroclinic orbit for (2.1) and (2.2), respectively, which connects Q^- and Q^+ for a unique, k-dependent speed $c^{(i)}(\varepsilon)$. Moreover, $c^{(i)}(\varepsilon) \leq c(0)$ for $i = 1,2$.

Proof. We will first prove the result for (2.1). In the inner region, where $U < \varepsilon$, we find $W_2^{\mathrm{s}}(\ell_2^+) \cap \Sigma_2^{\mathrm{in}} = (1, v_2^{\mathrm{in}}, r_2)$ in chart K_2, where $W_2^{\mathrm{s}}(\ell_2^+)$ is given by (2.9) and $v_2^{\mathrm{in}} = -c^{(1)}(r_2)$. From (2.4), we know that $V = r_2 v_2$ and $\varepsilon = r_2$; hence, $V^{\mathrm{in}} = -c^{(1)}(\varepsilon)\varepsilon$ and $\frac{\partial V^{\mathrm{in}}}{\partial c^{(1)}} = -\varepsilon$.

For general $c^{(1)}$, the dynamics of (2.1) in the outer region $\{U > \varepsilon\}$ are governed by

$$\begin{aligned} U' &= V, \\ V' &= -c^{(1)} V + kUV - U^2(1-U). \end{aligned} \tag{3.1}$$

The intersection of the unstable manifold $W^{\mathrm{u}}(Q^-)$ of Q^- with $\{U = \varepsilon\}$ can be written as the graph of an analytic function $V^{\mathrm{out}}(c^{(1)}, \varepsilon)$, with $\frac{\partial V^{\mathrm{out}}}{\partial c^{(1)}} > 0$. A standard phase plane argument shows that $V^{\mathrm{out}}(c^{(1)}, \varepsilon)$ must be $\mathscr{O}(1)$ and negative for $c^{(1)} \lesssim c(0)$, which implies $V^{\mathrm{in}} > V^{\mathrm{out}}$.

It remains to consider the case where $c^{(1)} = c(0)$. The singular heteroclinic orbit is given explicitly by $V(U) = -c(0)U(1-U)$ in that case, see Proposition 1. Therefore, $V^{\mathrm{out}}(c(0), \varepsilon) = -c(0)\varepsilon(1-\varepsilon)$ and, hence, $-c(0)\varepsilon = V^{\mathrm{in}} < V^{\mathrm{out}} = -c(0)\varepsilon(1-\varepsilon)$ for ε sufficiently small.

Next, we consider (2.2) in the inner region where $U < \varepsilon$. We find $W_2^{\mathrm{s}}(\ell_2^+) \cap \Sigma_2^{\mathrm{in}} = (1, v_2^{\mathrm{in}}, r_2)$, where $W_2^{\mathrm{s}}(\ell_2^+)$ is given by (2.11) and $v_2^{\mathrm{in}} = -c^{(2)}(r_2) + \frac{k}{2} r_2$. It follows from (2.4) that $V^{\mathrm{in}} = -c^{(2)}(\varepsilon)\varepsilon + \frac{k}{2}\varepsilon^2$ and, hence, that $\frac{\partial V^{\mathrm{in}}}{\partial c^{(2)}} = -\varepsilon$.

For general $c^{(2)}$, the dynamics of (2.2) in the outer region $\{U > \varepsilon\}$ are governed by

$$\begin{aligned} U' &= V, \\ V' &= -c^{(2)} V + kUV - U^2(1-U). \end{aligned} \tag{3.2}$$

We conclude that $V^{\mathrm{out}}(c^{(2)}, \varepsilon)$ must be $\mathscr{O}(1)$ and negative for $c^{(2)} \lesssim c(0)$, which implies $V^{\mathrm{in}} > V^{\mathrm{out}}$; moreover, $\frac{\partial V^{\mathrm{out}}}{\partial c^{(2)}} > 0$.

Finally, we again consider the case where $c^{(2)} = c(0)$. As before, the singular heteroclinic orbit is then given by $V(U) = -c(0)U(1-U)$. Therefore, $-c(0)\varepsilon + \frac{k}{2}\varepsilon^2 = V^{\mathrm{in}} < V^{\mathrm{out}} = -c(0)\varepsilon(1-\varepsilon)$, as $\frac{k}{2} < \frac{1}{4}(k + \sqrt{k^2+8}) = c(0)$ for $k \geq 0$.

Applying the implicit function theorem and noting that $\frac{\partial V^{\mathrm{out}}}{\partial c^{(i)}} - \frac{\partial V^{\mathrm{in}}}{\partial c^{(i)}} > 0$ for both (2.1) $(i = 1)$ and (2.2) $(i = 2)$, we conclude that $W^{\mathrm{s}}(Q^+)$ and $W^{\mathrm{u}}(Q^-)$ intersect at $\{U = \varepsilon\}$ for a unique value of $c^{(i)}(\varepsilon) \lesssim c(0)$, which completes the proof. □

The persistent heteroclinic connection constructed in Proposition 10 is again illustrated in Figure 4.

Remark 11. We note that $v_2^{\mathrm{in}} (= v_1^{\mathrm{out}})$ differs for (2.1) and (2.2) when $r_2 = \varepsilon > 0$, see Figure 2. Therefore, we will label the v_1-coordinates of the corresponding exit points from chart K_1 for (2.1) and (2.2) as $v_1^{\mathrm{out},(1)}$ and $v_1^{\mathrm{out},(2)}$, respectively.

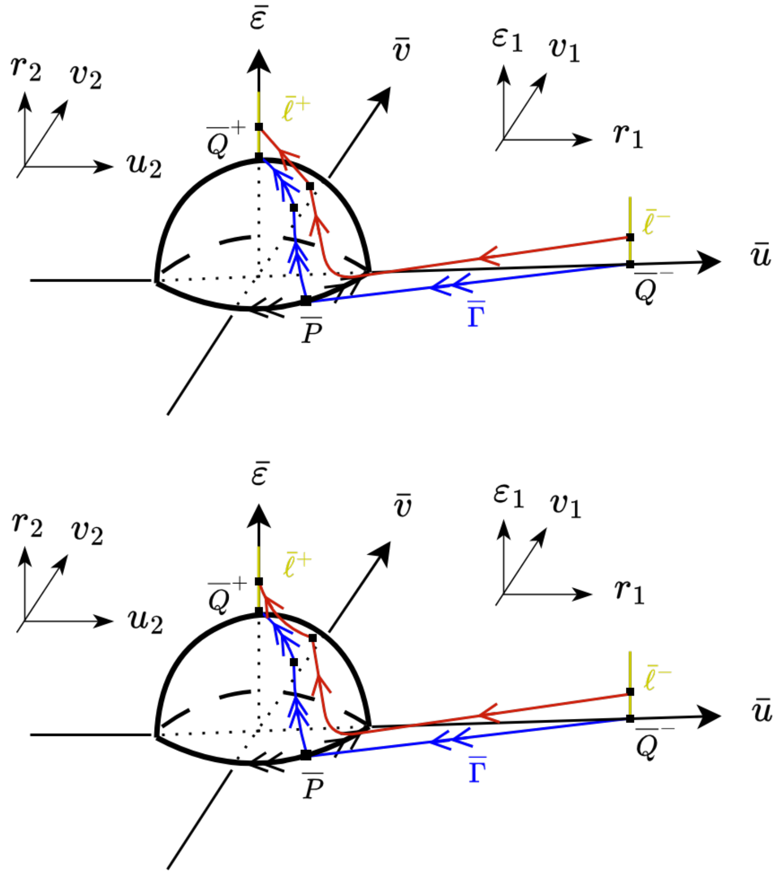

Figure 4: Global geometry of the blown-up vector field obtained from (2.1) (top) and (2.2) (bottom).

3.2 Leading-order asymptotics of $\Delta c^{(i)}$

In this subsection, we determine the leading-order asymptotics (in ε) of $\Delta c^{(i)} = c(0) - c^{(i)}(\varepsilon)$. Recall that in chart K_1, the dynamics of (2.1) and (2.2) are governed by (2.13). Therefore, we will approximate the transition map $\Pi_1 : \Sigma_1^{\text{in}} \to \Sigma_1^{\text{out}}$ under the flow of (2.13) for $\varepsilon \in (0, \varepsilon_0)$, with ε_0 sufficiently small. In preparation, we shift the point $P_1 = (0, -c(0), 0)$ to the origin via the transformation $z = v_1 + c(0)$, and we rewrite (2.13) as

$$\begin{aligned} r_1' &= -(c(0) - z) r_1, \\ z' &= (c(0) - z)(z - \Delta c^{(i)}) - k r_1 (c(0) - z) - r_1 (1 - r_1), \\ \varepsilon_1' &= (c(0) - z)\varepsilon_1. \end{aligned} \tag{3.3}$$

Rescaling "time" in (3.3) by a positive factor of $c(0) - z$, noting that the equation for ε_1 decouples, and appending the trivial equation for $\Delta c^{(i)}$, we obtain

$$\begin{aligned} \dot{r_1} &= -r_1, \\ \dot{z} &= z - \Delta c^{(i)} - k r_1 - \frac{r_1(1 - r_1)}{c(0) - z}, \\ \dot{\Delta c^{(i)}} &= 0. \end{aligned} \tag{3.4}$$

Here, the overdot denotes differentiation with respect to the new independent variable $\zeta^{(i)}$.

Lemma 12. There exists a sequence of smooth transformations that transforms (3.4) into

$$\begin{aligned}\dot{r_1} &= -r_1,\\ \dot{\hat{w}} &= \hat{w} - \frac{1}{c^{(i)}(\varepsilon)^3} r_1 \hat{w}^2[1+\mathcal{O}(r_1\hat{w})],\\ \dot{\Delta c^{(i)}} &= 0.\end{aligned} \tag{3.5}$$

Specifically, that sequence is composed of the transformation $w = z - \Delta c^{(i)}$, followed by $\tilde{w} = w - \left(\frac{k}{2} + \frac{1}{2c^{(i)}(\varepsilon)}\right) r_1$ and a sequence of smooth near-identity transformations of the form $\hat{w} = \tilde{w} + \mathcal{O}(r_1^2, r_1\tilde{w})$.

Proof. We first define the new variable $w = z - \Delta c^{(i)}$, which removes $\Delta c^{(i)}$ from the z-equation in (3.4), to obtain

$$\dot{w} = w - kr_1 - \frac{r_1(1-r_1)}{c^{(i)}(\varepsilon) - w}. \tag{3.6}$$

Expanding $\frac{1}{c^{(i)}(\varepsilon)-w} = \frac{1}{c^{(i)}(\varepsilon)}\left[1 + \frac{w}{c^{(i)}(\varepsilon)} + \left(\frac{w}{c^{(i)}(\varepsilon)}\right)^2 + \mathcal{O}(w^3)\right]$, we can write (3.6) as

$$\dot{w} = w - \left(k + \frac{1}{c^{(i)}(\varepsilon)}\right) r_1 + \frac{r_1^2}{c^{(i)}(\varepsilon)} - \frac{r_1(1-r_1)}{c^{(i)}(\varepsilon)}\left[\frac{w}{c^{(i)}(\varepsilon)} + \left(\frac{w}{c^{(i)}(\varepsilon)}\right)^2 + \mathcal{O}(w^3)\right]. \tag{3.7}$$

Similarly, the transformation $\tilde{w} = w - \left(\frac{k}{2} + \frac{1}{2c^{(i)}(\varepsilon)}\right) r_1$ removes the $\mathcal{O}(r_1)$-term from (3.7).

Finally, we remove all non-resonant terms by applying a sequence of near-identity transformations, with $\hat{w} = \tilde{w} + \mathcal{O}(r_1^2, r_1\tilde{w})$. The existence of such a transformation follows from standard normal form theory, see e.g. [4]. Since the lowest-order resonant term is of the form $r_1\tilde{w}^2$, we find

$$\dot{\hat{w}} = \hat{w} - \frac{1}{c^{(i)}(\varepsilon)^3} r_1 \hat{w}^2[1+\mathcal{O}(r_1\hat{w})], \tag{3.8}$$

as claimed. ◻

Next, we approximate $\widehat{P}_1^{\text{in}}$ and $\widehat{P}_1^{\text{out},(i)}$ $(i = 1,2)$, which are the entry and exit points in the sections Σ_1^{in} and Σ_1^{out} under the flow of (3.5), respectively, to a sufficient order in ε, r_0, and $\Delta c^{(i)}$. To that end, we first prove the following preparatory result.

Lemma 13. For (U,V) defined as in (1.7) and $U \in [0, U_0]$, with U_0 sufficiently small, there holds

$$\frac{\partial V}{\partial c}(U, c(0)) = \frac{1}{3 - \frac{k}{c(0)}}(1-U). \tag{3.9}$$

Proof. We rewrite (1.7) with U as the independent variable, which gives

$$V\frac{dV}{dU} = -cV + kUV - U^2(1-U). \tag{3.10}$$

Differentiation of (3.10) with respect to c yields

$$\frac{\partial V}{\partial c}\frac{\partial V}{\partial U} + V\frac{\partial}{\partial c}\left(\frac{\partial V}{\partial U}\right) = -V - c\frac{\partial V}{\partial c} + kU\frac{\partial V}{\partial c}. \tag{3.11}$$

Next, we evaluate (3.11) along the singular orbit $V = -c(0)U(1-U)$, as given by Proposition 1, and we make use of $\frac{\partial V}{\partial U} = -c(0)(1-2U)$, with $c = c(0)$, to obtain the ordinary differential equation

$$\frac{d}{dU}\left(\frac{\partial V}{\partial c}\right) = \left(2 - \frac{k}{c(0)}\right)\frac{1}{1-U}\frac{\partial V}{\partial c} - 1 \tag{3.12}$$

for $\frac{\partial V}{\partial c}$. Solving (3.12) under the condition that $\frac{\partial V}{\partial c}(U, c(0))$ remains bounded as $U \to 1^-$, we find (3.9). □

Lemma 14. For ε and $\Delta c^{(i)}$ $(i = 1, 2)$ sufficiently small and $k > 0$, the points $\widehat{P}_1^{\text{in}} = \left(r_0, \hat{w}^{\text{in}}, \frac{\varepsilon}{r_0}\right)$ and $\widehat{P}_1^{\text{out},(i)} = (\varepsilon, \hat{w}^{\text{out},(i)}, 1)$ satisfy

$$\hat{w}^{\text{in}} = -\frac{1}{r_0}\frac{1}{3 - \frac{k}{c(0)}}\Delta c^{(i)}\left[1 + o(1)\right], \tag{3.13}$$

as well as

$$\hat{w}^{\text{out},(1)} = -c(0)\varepsilon\left[1 + o(1)\right] \quad \text{and} \quad \hat{w}^{\text{out},(2)} = -\frac{1}{2c(0)}\varepsilon\left[1 + o(1)\right], \tag{3.14}$$

where $o(1)$ denotes higher-order terms in ε, r_0, and $\Delta c^{(i)}$. Furthermore, $\hat{w}^{\text{out},(1)}$ and $\hat{w}^{\text{out},(2)}$ correspond to the transformed exit points from chart K_1, which are derived from $v_1^{\text{out},(1)}$ and $v_1^{\text{out},(2)}$, respectively.

Proof. We begin by approximating $\hat{w}^{\text{out},(1)}$. Recall from the proof of Proposition 10 that $v_1^{\text{out},(1)} = -c^{(1)}(\varepsilon) = -c(0) + \Delta c^{(1)}$ for (2.1). Having shifted the point P_1 to the origin via the transformation $z = v_1 + c(0)$, we have $z^{\text{out},(1)} = \Delta c^{(1)}$. As we removed the $\Delta c^{(1)}$-term by defining $w = z - \Delta c^{(1)}$, it follows that $w^{\text{out},(1)} = 0$. The final two transformations are given by $\tilde{w} = w - \left(\frac{k}{2} + \frac{1}{2c^{(1)}(\varepsilon)}\right) r_1$ and the sequence of near-identity transformations $\hat{w} = \tilde{w} + \mathcal{O}(r_1^2, r_1\tilde{w})$. Recalling that $r_1 = \varepsilon$ in Σ_1^{out}, we finally have

$$\hat{w}^{\text{out},(1)} = -\left(\frac{k}{2} + \frac{1}{2c(0)}\right)\varepsilon = -c(0)\varepsilon \tag{3.15}$$

to leading order, where we have made use of $c(0) = \frac{1}{4}(k + \sqrt{k^2 + 8})$.

Next, we approximate $\hat{w}^{\text{out},(2)}$. Again, recall from the proof of Proposition 10 that $v_1^{\text{out},(2)} = -c^{(2)}(\varepsilon) + \frac{k}{2}\varepsilon = -c(0) + \Delta c^{(2)} + \frac{k}{2}\varepsilon$. Shifting P_1 to the origin via the transformation $z = v_1 + c(0)$, we find $z^{\text{out},(2)} = \Delta c^{(2)} + \frac{k}{2}\varepsilon$. The term $\Delta c^{(2)}$ is removed by defining $w = z - \Delta c^{(2)}$, which yields $w^{\text{out},(2)} = \frac{k}{2}\varepsilon$. The final two transformations are given by $\tilde{w} = w - \left(\frac{k}{2} + \frac{1}{2c^{(2)}(\varepsilon)}\right) r_1$ and the sequence of near-identity transformations $\hat{w} = \tilde{w} + \mathcal{O}(r_1^2, r_1\tilde{w})$ which, together with $r_1 = \varepsilon$ in Σ_1^{out}, shows

$$\hat{w}^{\text{out},(2)} = \frac{k}{2}\varepsilon - \left(\frac{k}{2} + \frac{1}{2c(0)}\right)\varepsilon = -\frac{1}{2c(0)}\varepsilon \tag{3.16}$$

to leading order.

Finally, we derive the approximation for $\hat{w}^{\text{in}}$. Since $W^{\text{u}}(Q^-)$ is analytic in U and $c^{(i)}$, we can write

$$\begin{aligned} V(U, c^{(i)}) &= \sum_{j=0}^{\infty} \frac{1}{j!}\frac{\partial^j V}{\partial (c^{(i)})^j}(U, c(0))\left(-\Delta c^{(i)}\right)^j \\ &= V(U, c(0)) - \frac{\partial V}{\partial c^{(i)}}(U, c(0))\Delta c^{(i)} + \mathcal{O}\left[\left(\Delta c^{(i)}\right)^2\right] \\ &= -c(0)U(1-U) - \frac{1}{3 - \frac{k}{c(0)}}(1-U)\Delta c^{(i)} + \mathcal{O}\left[\left(\Delta c^{(i)}\right)^2\right] \end{aligned} \tag{3.17}$$

for $i = 1,2$. Here, the coefficient of $\Delta c^{(i)}$ is obtained from Lemma 13, as Equations (2.1) and (2.2) are identical for $\varepsilon = 0$, and given by (1.7). Making use of $V = r_1 v_1$, $U = r_1$, and $r_1 = r_0$ in Σ_1^{in}, we find

$$v_1^{\text{in}} = c(0)(r_0 - 1) - \frac{1}{r_0}\frac{1}{3 - \frac{k}{c(0)}}(1 - r_0)\Delta c^{(i)} + \mathscr{O}\big[\big(\Delta c^{(i)}\big)^2\big]. \tag{3.18}$$

Then, considering $z = v_1 + c(0)$ and $w = z - \Delta c^{(i)}$, we have

$$w^{\text{in}} = c(0)r_0 - \frac{1}{r_0}\frac{1}{3 - \frac{k}{c(0)}}(1 - r_0)\Delta c^{(i)} - \Delta c^{(i)} + \mathscr{O}\big[\big(\Delta c^{(i)}\big)^2\big]. \tag{3.19}$$

Finally, since $\hat{w} = w - \left(\frac{k}{2} + \frac{1}{2c^{(i)}(\varepsilon)}\right) r_1 + \mathscr{O}(r_1^2, r_1 w)$ and since we can expand $\frac{1}{c^{(i)}(\varepsilon)} = \frac{1}{c(0) - \Delta c^{(i)}} = \frac{1}{c(0)} + \frac{1}{c(0)^2}\Delta c^{(i)} + \mathscr{O}[(\Delta c^{(i)})^2]$, we obtain

$$\begin{aligned}\hat{w}^{\text{in}} &= \left[c(0) - \frac{k}{2} - \frac{1}{2}\left(\frac{1}{c(0)} + \frac{1}{c(0)^2}\Delta c^{(i)} + \mathscr{O}\big[\big(\Delta c^{(i)}\big)^2\big]\right)\right] r_0 - \frac{1}{r_0}\left[\frac{1}{3 - \frac{k}{c(0)}}\Delta c^{(i)} + \mathscr{O}\big(r_0 \Delta c^{(i)}\big)\right] \\ &= -\frac{1}{r_0}\left[\frac{1}{3 - \frac{k}{c(0)}}\Delta c^{(i)}[1 + \mathscr{O}(r_0)] + \mathscr{O}\big[\big(\Delta c^{(i)}\big)^2\big]\right],\end{aligned} \tag{3.20}$$

where the constant terms cancel due to $c(0) = \frac{1}{4}(k + \sqrt{k^2 + 8})$. We note that $\Delta c^{(i)}$ must be independent of r_0, which was defined as an arbitrary small constant in (2.20). Moreover, we will find that $\hat{w}^{\text{in}}$, as given in (3.20), is sufficient for calculating $\Delta c^{(i)}$ to leading order in ε. In other words, terms of the order $\mathscr{O}[r_0 \Delta c^{(i)}, (\Delta c^{(i)})^2]$ have no effect on the leading-order asymptotics of $\Delta c^{(i)}$. Hence, the proof is complete. ☐

Consider the simplified normal form obtained by omitting the $\mathscr{O}(r_1^2 \hat{w}^3)$-terms in (3.8),

$$\dot{\check{w}} = \check{w} - \frac{1}{c^{(i)}(\varepsilon)^3} r_1 \check{w}^2. \tag{3.21}$$

The next result is required to show that Equation (3.21) is sufficient for determining $\Delta c^{(1)}$ and $\Delta c^{(2)}$ to leading order in ε.

Proposition 15. Let $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small, let $k > 0$, and let $\hat{w}$ and $\check{w}$ be defined as in (3.8) and (3.21), respectively. Then, for $\hat{w}^{\text{in}} = \check{w}^{\text{in}}$, we have

$$\big|\hat{w}^{\text{out,(i)}} - \check{w}^{\text{out,(i)}}\big| \leq \frac{|r_0 \hat{w}^{\text{in}}|}{\varepsilon}\mathscr{O}\big(|r_0 \hat{w}^{\text{in}}|^2 |\ln \varepsilon|\big), \tag{3.22}$$

where $i = 1,2$.

Proof. Let the right-hand side of (3.21) be denoted by $g(\zeta^{(i)}, \check{w})$. Then,

$$\begin{aligned}\big|g(\zeta^{(i)}, \check{w}_1) - g(\zeta^{(i)}, \check{w}_2)\big| &= \left|\check{w}_1 - \frac{1}{c^{(i)}(\varepsilon)^3} r_1 \check{w}_1^2 - \check{w}_2 + \frac{1}{c^{(i)}(\varepsilon)^3} r_1 \check{w}_2^2\right| \\ &= |\check{w}_1 - \check{w}_2|\left|1 - \frac{1}{c^{(i)}(\varepsilon)^3} r_1(\check{w}_1 + \check{w}_2)\right|.\end{aligned} \tag{3.23}$$

The equation $\frac{d}{d\zeta^{(i)}}(r_1 \check{w}) = -\frac{1}{c^{(i)}(\varepsilon)^3} r_1^2 \check{w}^2$, with $(r_1 \check{w})(0) = r_0 \check{w}^{\text{in}}$, has the unique solution

$$(r_1 \check{w})(\zeta^{(i)}) = \frac{r_0 \check{w}^{\text{in}}}{1 + \frac{1}{c^{(i)}(\varepsilon)^3} r_0 \check{w}^{\text{in}} \zeta^{(i)}} = r_0 \check{w}^{\text{in}}[1 + o(1)]$$

for $|r_0\check{w}^{\text{in}}|$ sufficiently small. Therefore, we can estimate

$$\left|1-\frac{1}{c^{(i)}(\varepsilon)^3}r_1(\check{w}_1+\check{w}_2)\right| \le 1+\frac{2}{c^{(i)}(\varepsilon)^3}|r_0\check{w}^{\text{in}}|+o(r_0\check{w}^{\text{in}}).$$

Denoting the right-hand side of (3.8) by $f(\zeta^{(i)},\hat{w})$, we obtain

$$\begin{aligned}\left|f(\zeta^{(i)},\hat{w})-g(\zeta^{(i)},\hat{w})\right| &= \left|\frac{1}{c^{(i)}(\varepsilon)^3}r_1^2\hat{w}^3[1+\mathscr{O}(r_1\hat{w})]\right| \\ &\le \frac{2}{c^{(i)}(\varepsilon)^3}\frac{|r_0\hat{w}^{\text{in}}|^3}{r_1} = \frac{2}{c^{(i)}(\varepsilon)^3}\frac{|r_0\hat{w}^{\text{in}}|^3}{r_0}e^{\zeta^{(i)}},\end{aligned} \tag{3.24}$$

where we have used that $r_1(\zeta^{(i)}) = r_0 e^{-\zeta^{(i)}}$, as is found by solving the r_1-equation in (3.5). Then, an application of Gronwall's inequality implies

$$\begin{aligned}\left|(\hat{w}-\check{w})(\zeta^{(i)})\right| &\le \exp\left[\left(1+\frac{2}{c^{(i)}(\varepsilon)^3}|r_0\hat{w}^{\text{in}}|\right)\zeta^{(i)}\right]\int_0^{\zeta^{(i)}}\exp\left[-\left(1+\frac{2}{c^{(i)}(\varepsilon)^3}|r_0\hat{w}^{\text{in}}|\right)x\right]\frac{2}{c^{(i)}(\varepsilon)^3}\frac{|r_0\hat{w}^{\text{in}}|^3}{r_0}e^x dx \\ &= \frac{|r_0\hat{w}^{\text{in}}|^2}{r_0}e^{\zeta^{(i)}}\left[\exp\left(\frac{2}{c^{(i)}(\varepsilon)^3}|r_0\hat{w}^{\text{in}}|\zeta^{(i)}\right)-1\right] \\ &= \frac{|r_0\hat{w}^{\text{in}}|^2}{r_0}e^{\zeta^{(i)}}\mathscr{O}(|r_0\hat{w}^{\text{in}}|\zeta^{(i)}).\end{aligned} \tag{3.25}$$

Evaluating the right-hand side of (3.25) at $\zeta^{\text{out},(\text{i})} = -\ln\frac{\varepsilon}{r_0}$, we obtain the stated result. ☐

We are now in a position to prove the main result of this section.

Proposition 16. For $\varepsilon \in [0,\varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small, and $k > 0$, the functions $\Delta c^{(1)}$ and $\Delta c^{(2)}$ defined in Theorem 2 satisfy

$$\Delta c^{(1)}(\varepsilon) = \frac{1}{4}\big(-k+3\sqrt{k^2+8}\big)\varepsilon^2[1+o(1)] \quad \text{and} \tag{3.26}$$

$$\Delta c^{(2)}(\varepsilon) = 2\frac{-k+3\sqrt{k^2+8}}{\big(k+\sqrt{k^2+8}\big)^2}\varepsilon^2[1+o(1)], \tag{3.27}$$

respectively.

Proof. We begin by solving the simplified normal form in (3.21). Substituting $r_1(\zeta^{(i)}) = r_0 e^{-\zeta^{(i)}}$ into (3.21), we obtain

$$\dot{\check{w}} = \check{w}-\frac{1}{c^{(i)}(\varepsilon)^3}r_0 e^{-\zeta^{(i)}}\check{w}^2, \tag{3.28}$$

which has the solution

$$\check{w}(\zeta^{(i)}) = \frac{e^{\zeta^{(i)}}}{\beta+\frac{1}{c^{(i)}(\varepsilon)^3}r_0\zeta^{(i)}}, \tag{3.29}$$

where β is a constant of integration. We require that $\check{w}(\zeta^{(i)}) = \hat{w}^{\text{in}}$ at $\zeta^{\text{in},(i)} = 0$ and, hence, $\beta = \frac{1}{\hat{w}^{\text{in}}}$. Expanding (3.29) in terms of $|r_0\hat{w}^{\text{in}}|$, we have

$$\check{w}(\zeta^{(i)}) = \hat{w}^{\text{in}}e^{\zeta^{(i)}}\left[1+\mathscr{O}\big(r_0\hat{w}^{\text{in}}\zeta^{(i)}\big)\right]. \tag{3.30}$$

Evaluating (3.30) at $\zeta^{\text{out},(i)} = -\ln\frac{\varepsilon}{r_0}$, we find

$$\hat{w}^{\text{out},(i)} = \frac{r_0\hat{w}^{\text{in}}}{\varepsilon}\left[1+\mathscr{O}\big(r_0\hat{w}^{\text{in}}|\ln\varepsilon|\big)\right], \tag{3.31}$$

for $i = 1,2$. We note that (3.31) is of lower order than (3.22), as derived in Proposition 15, which implies that the simplified normal form in (3.21) is sufficient for approximating $\Delta c^{(i)}$ to leading order. Substituting the estimate for $\hat{w}^{\text{in}}$ from Lemma 14 into (3.31), we obtain

$$\begin{aligned}\hat{w}^{\text{out},(i)} &= -\frac{r_0}{\varepsilon}\frac{1}{r_0}\frac{1}{3-\frac{k}{c(0)}}\Delta c^{(i)}[1+o(1)]\left[1+\mathscr{O}\left(\Delta c^{(i)}\ln\varepsilon\right)\right]\\ &= -\frac{1}{\varepsilon}\frac{1}{3-\frac{k}{c(0)}}\Delta c^{(i)}[1+o(1)].\end{aligned} \tag{3.32}$$

Since $\hat{w}^{\text{out},(1)} = -c(0)\varepsilon[1+o(1)]$, by Lemma 14, we conclude that

$$\Delta c^{(1)} = [3c(0)-k]\varepsilon^2[1+o(1)]. \tag{3.33}$$

Similarly, as $\hat{w}^{\text{out},(2)} = -\frac{1}{2c(0)}\varepsilon[1+o(1)]$, recall again Lemma 14, we have

$$\Delta c^{(2)} = \frac{3-\frac{k}{c(0)}}{2c(0)}\varepsilon^2[1+o(1)]. \tag{3.34}$$

Finally, substitution of $c(0) = \frac{1}{4}(k+\sqrt{k^2+8})$ into (3.33) and (3.34), respectively, yields the stated result. ☐

Proposition 16 implies, in particular, that $\Delta c^{(i)} > 0$ for $i = 1,2$, in agreement also with Proposition 10, which completes the proof of Theorem 2.

Remark 17. We note that Proposition 16 remains true for $k = 0$, which corresponds to the Zeldovich equation with cut-off [9]: both (3.26) and (3.27) reduce to $\Delta c(\varepsilon) = \frac{3}{\sqrt{2}}\varepsilon^2[1+o(1)]$ then, which agrees with [9, Theorem 1.2].

To compare the leading-order corrections $\Delta c^{(1)}$ and $\Delta c^{(2)}$ to $c(0)$ for Equations (1.8) and (1.9), we evaluate their difference

$$\left|c^{(1)}(\varepsilon)-c^{(2)}(\varepsilon)\right| = \left|\Delta c^{(1)}(\varepsilon)-\Delta c^{(2)}(\varepsilon)\right| = \left|\frac{1}{4}k\left(6+k^2-k\sqrt{k^2+8}\right)\varepsilon^2[1+o(1)]\right|. \tag{3.35}$$

We observe that Equation (3.35) has a unique root at $k = 0$, as is to be expected, since (1.8) and (1.9) are identical for $k = 0$. In addition, the right-hand side in (3.35) is monotonically increasing in k and approaches $\frac{k}{2}\varepsilon^2[1+\mathscr{O}(k^{-1})]$ as $k\to\infty$. These observations are consistent with the corresponding result for the motivating example of the Burgers equation in (1.2), where multiplication of the advection term with a Heaviside cut-off produced a shift $\Delta c = \frac{k}{2}\varepsilon^2$ in the front propagation speed for (1.3), and is related to the fact that, for large k, the advection terms in Equations (1.8) and (1.9) are dominant. In other words, the reaction kinetics becomes negligible for k large, which implies that (1.8) and (1.9) are well approximated by (1.3) and (1.2), respectively, in that limit.

Finally, we provide numerical verification of Proposition 16. To that end, we calculate the error incurred by approximating $c^{(i)}(\varepsilon)$ with the corresponding first-order expansion $\hat{c}^{(i)}(\varepsilon)$ for $i = 1,2$ and $k = 1$, which yields $\hat{c}^{(1)}(\varepsilon) = 1-2\varepsilon^2$ and $\hat{c}^{(2)}(\varepsilon) = 1-\varepsilon^2$. The numerical values of $c^{(i)}(\varepsilon)$ $(i = 1,2)$ are determined by integrating Equations (2.1) and (2.2) and returning final values of $U = U_{\text{final}}(c)$ that are obtained after a sufficiently large number of time steps. We then minimise $|U_{\text{final}}(c)|$ by applying a Nelder–Mead method [6]. Our findings are illustrated in Figure 5, where we have used a double logarithmic scale, with $\varepsilon \in [10^{-4}, 10^{-2}]$. Figure 5 suggests that $|c^{(i)}(\varepsilon)-\hat{c}^{(i)}(\varepsilon)| = \mathscr{O}(\varepsilon^3)$ for $i = 1,2$, which is consistent with [9, Theorem 1.2] for $k = 0$; specifically, the slopes of the lines $|c^{(i)}(\varepsilon)-\hat{c}^{(i)}(\varepsilon)|$ are given by 3.005078 ($i = 1$) and 3.002047 ($i = 2$), respectively.

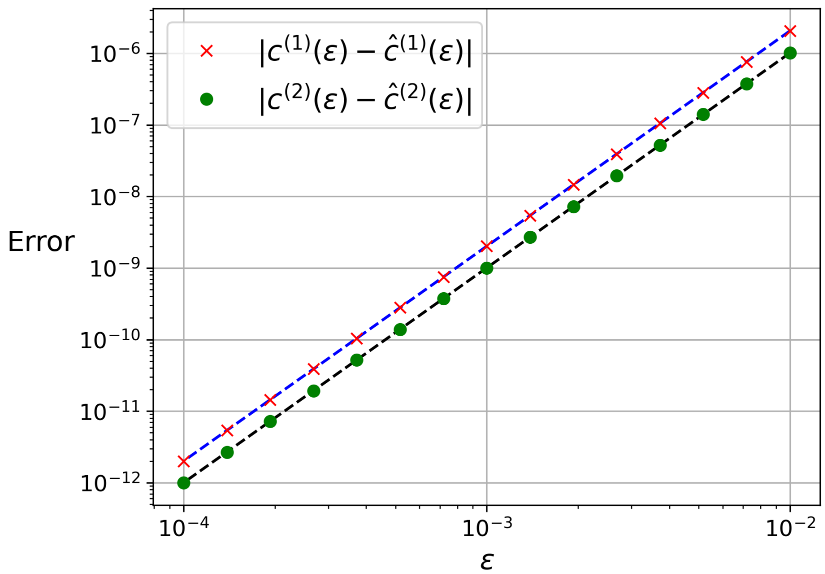

Figure 5: Error of the approximation for $c^{(i)}(\varepsilon)$ by $\hat{c}^{(i)}(\varepsilon)$ for Equations (1.8) and (1.9), where $i = 1,2$, $k = 1$, and $\varepsilon \in [10^{-4}, 10^{-2}]$.

4 Generalisation

In this section, we briefly consider generalisations of Equations (1.8) and (1.9) where the reaction kinetics in both equations is replaced with $u^n(1-u)H(u-\varepsilon)$ for $n = 1,2,3$, which gives

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x}H(u-\varepsilon) = \frac{\partial^2 u}{\partial x^2} + u^n(1-u)H(u-\varepsilon) \quad \text{and} \tag{4.1}$$

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u^n(1-u)H(u-\varepsilon), \tag{4.2}$$

respectively, where $k > 0$, as before. For $n = 1$, the corresponding equations without a cut-off are known as the Burgers-FKPP equation [5, Chapter 11]. The change in front propagation speed due to the Heaviside cut-off in (4.1) and (4.2) was studied in [10], where it was shown that for $k \leq 2$, the speeds of the critical front solutions to Equations (4.1) and (4.2) converge to the singular speed $c(0)$ logarithmically in ε, whereas for $k > 2$, convergence is sub-linear in ε. As in the statement of Theorem 2, let $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$ denote the unique propagation speeds of critical front solutions to (4.1) and (4.2), respectively; then, it follows from [10, Theorem 2 and Appendix A] that $c^{(1)}(\varepsilon) = c^{(2)}(\varepsilon)$ to leading order, for all $k > 0$.

That result contrasts with our main result in this article, Theorem 2, which applies precisely in the case where $n = 2$ in (4.1) and (4.2): here, the orders (in ε) of $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$ are equal, while the corresponding coefficients differ.

Finally, we consider Equations (4.1) and (4.2) with $n = 3$. The analysis of these equations is complicated by the fact that the singular critical speed $c(0) = c_{\text{crit}}$ is not known explicitly. However, an adaptation of the approach in [9] still allows us to determine the leading-order asymptotics of $c^{(i)}(\varepsilon)$. In Figure 6, we show similar error estimates as are displayed in Figure 5 for Equations (1.8) and (1.9). We first compute numerical values for the singular propagation speed $c(0)$ for Equations (4.1) and (4.2) and then apply the same approach as at the end of Section 3 to approximate $c^{(i)}(\varepsilon)$. In Figure 6, we plot $c(0) - c^{(i)}(\varepsilon) = \Delta c^{(i)}(\varepsilon)$ for $i = 1,2$, which implies that $\Delta c^{(1)}(\varepsilon) = \mathscr{O}(\varepsilon^2)$ and $\Delta c^{(2)}(\varepsilon) = \mathscr{O}(\varepsilon^3)$ for $n = 3$: the slopes of the lines $\Delta c^{(i)}(\varepsilon)$ are given by 2.001391 ($i = 1$) and 2.999524 ($i = 2$), respectively.

Our findings are summarised in Table 1, where $\alpha_k^{(i)}$ denotes the leading-order (ε-independent) coefficient in the expansion of $\Delta c^{(i)}(\varepsilon)$ for $i = 1, 2$.

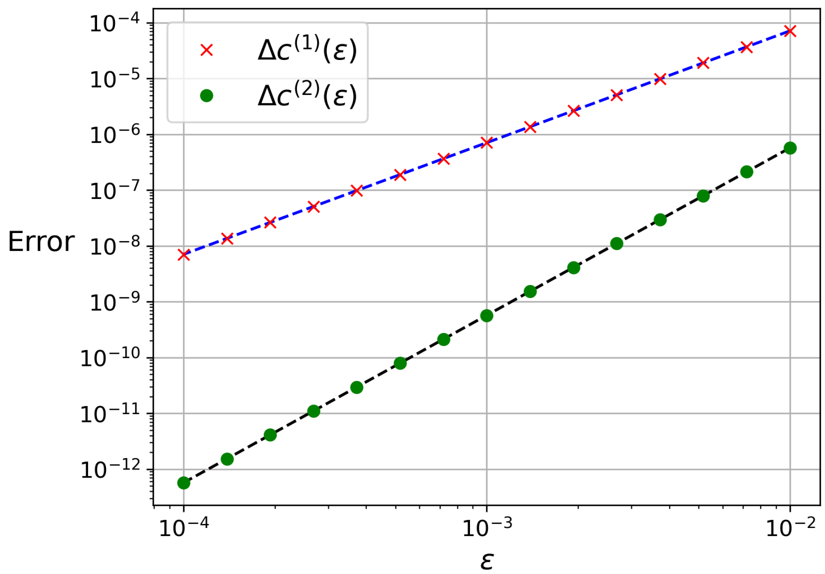

Figure 6: Approximation of $\Delta c^{(i)}(\varepsilon)$ for Equations (4.1) and (4.2) with $n = 3$, where $i = 1, 2$, $k = 1$, and $\varepsilon \in [10^{-4}, 10^{-2}]$.

n	$\alpha_k^{(1)} = \alpha_k^{(2)}$	$\Delta c^{(1)}$	$\Delta c^{(2)}$
1 ($k \leq 2$)	✔	$\mathscr{O}[(\ln \varepsilon)^{-2}]$	$\mathscr{O}[(\ln \varepsilon)^{-2}]$
1 ($k > 2$)	✔	$\mathscr{O}(\varepsilon^{1-4/k^2})$	$\mathscr{O}(\varepsilon^{1-4/k^2})$
2	✘	$\mathscr{O}(\varepsilon^2)$	$\mathscr{O}(\varepsilon^2)$
3	✘	$\mathscr{O}(\varepsilon^2)$	$\mathscr{O}(\varepsilon^3)$

Table 1: Correction to the propagation speed of front solutions to Equations (4.1) and (4.2) for $n = 1, 2, 3$.

5 Discussion

In this article, we have proven the existence and uniqueness of "critical" propagating fronts for two degenerate advection -reaction-diffusion equations in the presence of a cut-off. In particular, in Equation (1.8), a Heaviside cut-off function multiplies both the reaction kinetics and the advection term, whereas in Equation (1.9), only the kinetics is affected by the cut-off. In both cases, we have derived the leading-order asymptotics (in ε) of the corresponding unique front propagation speeds $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$, respectively.

It is possible to determine higher-order terms in the expansions of $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$ via the approach outlined in [9], since the singular front is known explicitly and since the calculation of $\frac{\partial V}{\partial c}(U, c(0))$ in Lemma 13 can be generalised to higher orders, which hence allows one to approximate $\hat{w}^{\text{in}}$ to a sufficiently high order. However, the primary goal of this article was to demonstrate the importance of "cutting off" the advection term in addition to the reaction kinetics; specifically, Theorem 2 implies that failure to do so can result in a different leading-order correction to the propagation speed of the front.

Numerical verification of Theorem 2 suggests that the second-order term in the expansions for both $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$ is of the order $\mathscr{O}(\varepsilon^3)$, recall Figure 5. That suggestion is consistent with [9, Theorem 1.2], where it was shown that for Equations (1.8) and (1.9) with $k = 0$, the second-order correction to $c(\varepsilon)$ is indeed $\mathscr{O}(\varepsilon^3)$.

Section 4 introduces a family of scalar advection-reaction-diffusion equations which generalise Equations (1.8) and (1.9). Our findings are summarised in Table 1. We conclude that for $n = 1$, it is sufficient to consider a cut-off in the reaction kinetics only. However, for $n = 2$ and $n = 3$, the leading-order asymptotics of $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$ depends strongly on the advection term due to the effect of the kinetics being weaker. Specifically, for $n = 2$, the leading-order correction to $c^{(1)}(\varepsilon)$ and $c^{(2)}(\varepsilon)$ is of the same order in ε, while the corresponding coefficients differ. For $n = 3$, front propagation is dominated by advection; hence, failure to include a cut-off in the advection term results in a leading-order correction to the speed of propagation which is of the wrong order. While the generalisation in Section 4 illustrates the significance of cutting off advection for the front propagation dynamics, we focused on (1.6) in our analysis due to the corresponding critical speed c_{crit} being known explicitly.

A Particle system for Burgers equation

In this section, we consider the N-particle system underlying the (viscous) Burgers equation, solutions of which converge to those of the following initial boundary value problem as $N \to \infty$:

$$u_t + kuu_x = \sigma u_{xx}, \quad \text{with } \lim_{x\to-\infty} u(x,t) = 1 \text{ and } \lim_{x\to\infty} u(x,t) = 0, \tag{A.1}$$

where $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, and $t \geq 0$.

In our numerical simulation of the stochastic particle system, we applied the algorithm proposed in [2], as it is simple to implement and computationally efficient; the basic details are given below for completeness. We begin by fixing N points (particles) in $\mathbb{R}$, with corresponding positions $\{y_0^i\}_{i=1}^N$, which allows us to write the initial particle density as

$$u_0(x) = \frac{1}{N}\sum_{i=1}^{N} H(x - y_0^i).$$

Fixing T, we can discretise the time interval $[0,T]$ by writing $T = M\Delta_t$, with step size Δ_t and time steps given by $t_m = m\Delta_t$ for $m = 0,\dots,M$. We then proceed with the following iteration,

$$\begin{aligned} Y^i_{t_{m+1}} &= Y^i_{t_m} + \frac{k}{N}\sum_{j=1}^{N} H(Y^i_{t_m} - Y^j_{t_m})\Delta_t + \sigma\sqrt{\Delta_t}\Delta w^i_{m+1} \quad \text{for } m = 0,\dots,M-1, \\ Y^i_0 &= y^i_0, \end{aligned} \tag{A.2}$$

where $\Delta w^i_{m+1} \sim \mathcal{N}(0,1)$ is normally distributed with mean 0 and variance 1. The numerical approximation $u(x,t_m)$ for u at time t_m ($m = 1,\dots,M$), with $x \in \mathbb{R}$, is then recovered from

$$u(x,t_m) = \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_m}), \tag{A.3}$$

see [2]. In particular, we only require for the spatial domain to be discretised in order to plot the solution, see Figure 1.

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