

## Algebraic Structures on $\mathcal{SH}^*$

Sheet

- Outline:
- I. TQFT structure (product, unit)
  - II. BV operator
  - III. More about BV algebras.

I. Ref: Ritter's paper.

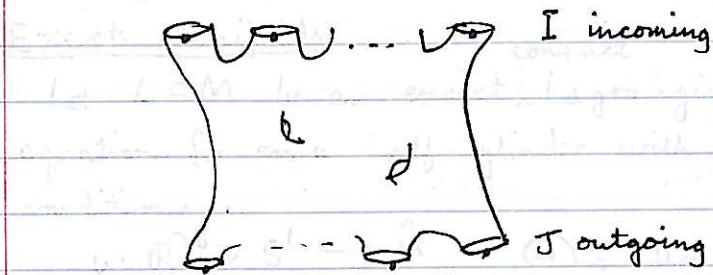
generators of  $\mathcal{SH}^*$  are parametrised orbits of a  $t$ -indep't Hamiltonian, for our purposes.

Differential counts maps

$$(S^1 \times M)^* \mathcal{H}B \xrightarrow{x^\pm} (M)^* \mathcal{H}B$$


$$|x_-| - |x_+| = 1$$

Why not count moduli spaces



want this to give map  $(\mathcal{SH}^*)^{\otimes I} \rightarrow (\mathcal{SH}^*)^{\otimes J}$

fixed  $\Sigma$ , asymptotic markers,  $H$  linear on the symplectisation part of boundary.

Fix weights  $h_1, \dots, h_I > 0$   
 $k_1, \dots, k_J > 0$

On  $\Sigma$ , fix cylindrical ends  $\cong S^1 \times [0, \infty)$  for pos. ends  
 $S^1 \times (-\infty, 0]$  for neg. ends  
 $(J \in S' = \text{asymptotic marker})$

Fix 1-form  $\beta$  on  $\Sigma$  s.t.

$$(i) d\beta \leq 0$$

(ii) pull back of  $\beta$  to cylindrical end of weight  $w$  is wdt.

By Stokes' theorem

$$0 \geq \int_{\Sigma} d\beta = \sum h_i - \sum k_j$$

In particular there's at least one outgoing end. (a)

Solve  $(du - X \otimes \beta)^{0,1} = 0$  (\*\*) with usual asymptotics

Claim: pullback of (\*\*) with weight  $w$  gives

$$\partial_s u + J(\partial_t u - wX) = 0$$

$\Rightarrow$  get maps  $HF^*(h, H) \otimes \dots \otimes HF^*(h_I, H) \rightarrow HF^*(k_1, H) \otimes \dots \otimes HF^*(k_J, H)$

$$\Phi_{\Sigma, I, J, h_1, \dots, h_I, k_1, \dots, k_J}$$

$$HF^*(k_1, H) \otimes \dots \otimes HF^*(k_J, H)$$

this is natural w.r.t. continuation

$$\Rightarrow \text{get } SH^{\otimes I} \xrightarrow{\Phi_{\Sigma, I, J}} SH^{\otimes J} \quad J > 1.$$

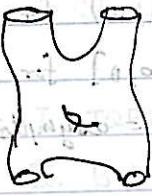
Claim: these maps glue nicely: let  $\Sigma_1 \# \Sigma_2 := \Sigma$ , glued

by  $\Sigma_1$  to  $\Sigma_2$  with parameter  $\lambda$   $\Rightarrow$  ends  $h_i$  of  $\Sigma_1$

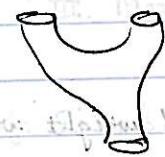
and  $k_j$  of  $\Sigma_2$  for minimization of energy

$$(M)^*H_2 \rightarrow (A)^*H$$

the map  $(-1)^{\# \Sigma} : \Sigma_1 \times \Sigma_2 \rightarrow \text{the moduli space } \Sigma$   
 where  $\# \Sigma = (\text{genus}) \times 2$



$\Sigma_1$



$\Sigma_2$

then  $\# \Sigma = 2$  about  $BV$  along  $\Sigma_2$  is a map with  
 $\# \Sigma = 2$  (ii)

this is from the moduli space of a closed string (ii)

restriction of  $S\mathcal{H}^*$  are non-trivial orbits of

$$\Phi_{\Sigma_2} \circ \Phi_{\Sigma_1} = \Phi_{\Sigma_1 \#_2 \Sigma_2} \quad \text{for more components, write } \Phi$$

$$\Phi_{\Sigma_1 \#_2 \Sigma_2} \text{ counts } \# \Sigma - \# \Sigma = 2b - 2b = 0$$

This gives

a) Ring structure  $\alpha$  and  $\beta$  are relations involving all

$$\text{restrictions of } \alpha = \#(\text{genus} - 1) \text{ and } \beta = \#(\text{genus} - 1)$$

product  $\alpha \# \beta$  (i.e. for shrinking manifold)

$$\alpha = (Xw - n\beta) \cdot G + n\beta$$



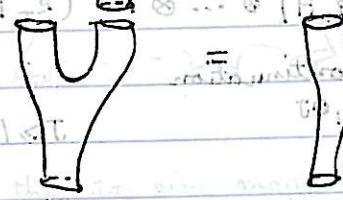
b) Unit  $i : H^*(M) \otimes k \xrightarrow{\cong} S\mathcal{H}^*(\hat{M})$



$1 \mapsto e$

check unitarity:

$$(H^*)^* \otimes \dots \otimes (H^*)^* \circ H^* = \text{id}$$



= identity or in words

Locally  $H^* \cong \mathbb{C}^{\oplus \# \Sigma}$   $\cong$  plane with  $\# \Sigma$  marked points

in fact these are all constant disks  $\Rightarrow$  unit

corresponds to minimum of  $H$ , or

$$H^*(\hat{M}) \xrightarrow{\cong} S\mathcal{H}^*(\hat{M})$$

$1 \mapsto e$

stationary station, stable.

Consider the 1-parameter family of moduli spaces

$$M_{BV}(x^-, x^+) = \bigcup_t$$


$(\text{H}_2\text{O})\text{H}_2 + (\text{H}_2\text{O})\text{H}_2$  undergoes net evaporation  $x + (1)^{\theta=0}$  at the drying point.

$$\dim M_{BV}(x^-, x^+) = 1 + \dim M(x^-, x^+) \text{ and } (1) \text{ is true}$$

this is rigid if  $|x_+| \geq |x_-| = -1$ . (1)  $\rightarrow$  12

Get an operator  $\Delta: \mathcal{SH}^*(\hat{M}) \rightarrow \mathcal{SH}^{*-1}(\hat{M})$

this is the BV operator.

Defn: A BV algebra is a graded commutative  
(dg) algebra  $(V, *)$  with a map

$$\Delta : V^* \rightarrow V^{*-1}$$

s.t.  $H_2 \leftarrow H_2 \otimes ((S), \cdot)$ ;  $H$  contains  $p_0$

$$(i) \Delta^2 = 0$$

$$\text{iii) } \Delta(xyz) = \Delta(xy)z + (-1)^{|x|} x \Delta(yz)$$

$$+ (-1)^{(|x|+1)(|y|)} y \Delta(xz)$$

$$-\Delta(x)yz - (-1)^{|x|} x \Delta(y)z$$

$$= (-1)^{|x|+|y|} xy \Delta(z).$$

Why is  $\Delta$  non-trivial?

Rm

Rmk:   $\approx$

The first predictor of mortality

Defn:  $\text{Fr}(1) = \left\{ \begin{array}{c} \text{dilate, rotate, translate} \\ \text{at radius } r \\ \theta=0 \\ \text{fixed size} \end{array} \right\}$

Each point in  $\text{Fr}(1)$  gives an operator  $\text{SH}(\hat{M}) \rightarrow \text{SH}(\hat{M})$

Point:  $\text{Fr}(1)$  has non-trivial topology  $\Rightarrow M$  with

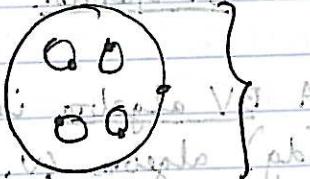
$$H_0(\text{Fr}(1)) = H_0(S^1)$$

$S^1 \rightarrow \text{Fr}(1)$  representing  $[S^1]$  which is circle

gives a map

$$(\hat{M}) \text{SH}^*(\hat{M}) \rightarrow \text{SH}^{*-1}(\hat{M})$$

operators are in fact



In general,  $\text{Fr}(k) = \left\{ \begin{array}{c} \text{disjoint disks} \\ \text{in } V \text{ with } A \text{ : int} \\ \text{give a disk} \end{array} \right\}$

$$\text{operators } H_1(\text{Fr}(k)) \otimes \text{SH}^{\otimes k} \rightarrow \text{SH}^{*-k}$$

$$o = \Delta^k(j)$$

Say that  $\text{SH}^*$  is an algebra over  $H_1(k)$  (framed little disks operad).

E.g. gives a map  $a \otimes b \otimes c \mapsto a \cdot b \cdot c$ .

1-param family



$$a \otimes b \otimes c \mapsto a \cdot \Delta(b \cdot c)$$

continuously rotate by  $\frac{2\pi}{3}$

