General Dimension Formula

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Given a Reeb chord, linearisation of Reeb flow defines a map

$$\xi_x \longrightarrow \xi_{\phi_t(x)}$$

which preserves $\omega = d\lambda$.

Given a trivialisation of ξ over Reeb chord/orbit we get a map

$$[0,1] \longrightarrow Sp(2n) \quad \text{chord}$$

$$S^1 \longrightarrow Sp(2n) \quad \text{orbit}$$

$Sp(2n)/U(n)$ contractible $\rightsquigarrow \pi_1(Sp(2n)) = \pi_1(U(n)) \cong \mathbb{Z}$.

$$\begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \longmapsto 1$$

μ

<u>Defn</u>: Maslov cycle $\{A \in Sp(2n) : \det(\mathbb{1} - A) = 0\}$

$Sp^*(2n) = Sp(2n) - M.c.$

has 2 components: $Sp^+(2n) := \{\det > 0\}$

$Sp^-(2n) := \{\det < 0\}$.

any loop in $Sp^*(2n)$ is contractible in $Sp(2n)$.

$\mu(A) := \frac{1}{2} \#$ of $A(t): S^1 \to Sp(2n) \pitchfork$ Maslov cycle.

<u>Defn</u>: $t \in S^1$ a crossing if $\det(\mathbb{1} - A(t)) = 0$.

Crossing form in a crossing t

$$\ker(\mathbb{1} - A(t)) \longrightarrow \mathbb{R}$$

$$v \overset{\Gamma}{\longmapsto} \omega(v, \dot{A}(t) v)$$

t is regular if Γ is nondeg, ~~if Γ nondeg~~ reg. crossings are isolated.

$$\mu(A(t)) := \sum_{t \in \text{crossings}} \operatorname{sign}(\Gamma, t)$$

A path $A(t) : [0,1] \longrightarrow Sp(2n)$ is admissible if $A(0) = \mathbb{1}$, $A(1) \in Sp^*(2n)$

$$Sp(2n) \longleftrightarrow S : [0,1] \longrightarrow Sym(2n)$$
$$A(t) \longmapsto -J\dot{A}(t) A^{-1}(t)$$
$$A(0) = \mathbb{1} \longleftarrow S(t)$$
$$\dot{A}(t) = J_0 S(t) A(t)$$

with these notations $\Gamma(t) = \langle v, Sv \rangle$

<u>Defn:</u> $$\mu_{CZ}(A(t)) = \tfrac{1}{2} \operatorname{sign}(S(0)) + \sum_{t \text{ crossing}} \operatorname{sign}(\Gamma, t).$$

<u>Thm:</u> μ_{CZ} is uniquely determined by:

1) $A(t) \sim B(t)$ through admissible matrices $\Rightarrow \mu_{CZ}(A(t)) = \mu_{CZ}(B(t))$

2) $A : S^1 \longrightarrow Sp(2n)$
$B : [0,1] \longrightarrow Sp(2n)$
$\mu_{CZ}(A(t) B(t)) = \mu_{CZ}(B(t)) + 2\mu(A(t))$

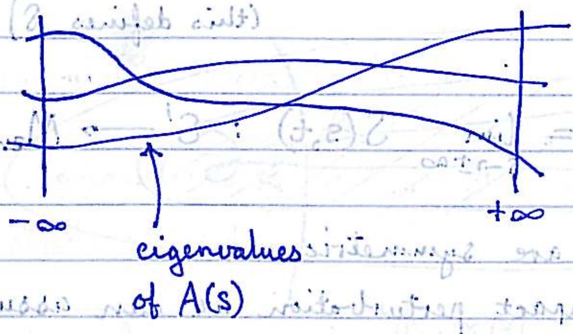
3) $S \in Sym(2n)$ $\mu_{CZ}\left(t \longmapsto e^{J_0 S t}\right) = \tfrac{1}{2} \operatorname{sign}(S).$

Spectral Flow

$A(s): W^{1,2}(S^1, \mathbb{R}^{2n}) \longrightarrow L^2(S^1, \mathbb{R}^{2n})$

smooth family of unbounded self-adjoint operators on $H = L^2(S^1, \mathbb{R}^{2n})$

Assume $A^{\pm} = \lim_{s \to \pm\infty} A(s)$ is invertible



Defn: $s \in \mathbb{R}$ is a crossing $\ker A(s) \neq \{0\}$

$\Gamma: \ker(A(s)) \longrightarrow \mathbb{R}$

$v \longmapsto \langle v, \dot{A}(s) v \rangle$

$$\mu_{spec}(A(t)) = \sum_{s \text{ crossings}} \operatorname{sign}(\Gamma(s))$$

Now let $D_A := \partial/\partial s + A(s)$

$$W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

Thm D_A is Fredholm with index

$\operatorname{ind} D_A = \mu_{spec}(A)$

E.g. ~~$\mu(\delta)$~~ $u : \underset{t}{S^1} \times \underset{s}{\mathbb{R}} \longrightarrow (\mathbb{R} \times M, d(e^s \alpha))$

Using trivialization of $u^* T(\mathbb{R} \times M)$ the linearised $\bar\partial$ operator has the form

$$D\xi = \partial_s \xi + J_0 \partial_t \xi + S\xi$$

(this defines S)

Now, let

$$S_\pm(t) := \lim_{s \to \pm\infty} S(s,t) : S^1 \longrightarrow M_{2n}(\mathbb{R})$$

<u>Lem:</u> $S_\pm(t)$ are symmetric.
By a compact perturbation we can assume $S(s,t)$ symmetric $\forall\, s, t$.

<u>Defn:</u> $\Psi(s,t) :=$ $\begin{cases} \Psi(s,0) = \mathbb{1} \\ \frac{\partial}{\partial t}\Psi = -J_0 S \Psi . \end{cases}$

$$\Psi_\pm : S^1 \longrightarrow Sp(2n)$$

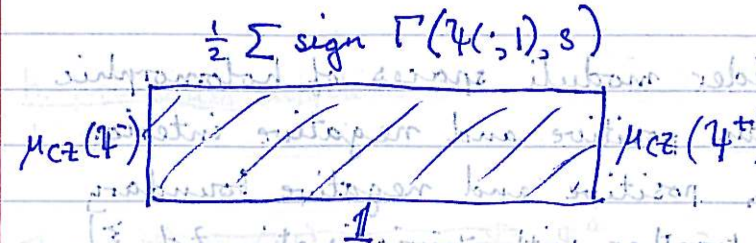
$$\Psi_\pm = \lim_{s \to \pm\infty} \Psi(s,t)$$

<u>Lem:</u> $\Psi_\pm$ is linearised Reeb flow at limiting ~~orbits~~ Reeb orbits $\gamma_\pm$.

$$\text{ind } D = \mu_{spec} A = \sum_{\substack{\text{crossings} \\ s}} \text{sign } \Gamma(\Psi(\cdot,1), s)$$

$$= \mu_{CZ}(\Psi^+) - \mu_{CZ}(\Psi^-)$$

$$= \mu_{CZ}(\gamma^+) - \mu_{CZ}(\gamma^-)$$



General Dimension Formula

(X) X = symp. mfd with cylindrical ends.

(Y) $\bar{Y}$ cpct $\subset X$ s.t. $S \longrightarrow X$ J-holo.

$$\partial S \subset \bar{Y} \Rightarrow S \subset Y = \bar{Y} \cup \mathbb{R}_+ \times (\bar{Y} \cap M)$$

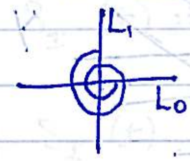
(L) $L \subset Y \subset X$ proper immersed Lag. with clean intersection $\forall\ x \in K = L$.

$$T_x K \subset T_x L_0 \cap T_x L_1$$

Reeb orbits are non-degenerate and avoid Λ.

Now consider moduli spaces of holomorphic curves with positive and negative interior punctures, positive and negative boundary punctures asymptotic to Reeb chords, together with trivialisation Z of $\xi|_{\text{Reeb chord}}$ in which linearised Reeb flow appears to be identity,

~~pos~~ boundary intersection points, ~~in~~ with



$w \in \mathbb{Z}$ if doesn't change sheets
$\mathbb{Z} + \frac{1}{2}$ if changes sheets

trivialise ξ over boundary

$$\dim \mathcal{M}(u) = (n-3)(2-2g-r) + (s+t+l)$$
$$+ \sum_{\gamma \text{ pos Reeb orbit}} \left(\mu_{CZ}(\gamma) - n - 3\right)$$
$$+ \sum_{\gamma \text{ neg Reeb orbit}} \left(\mu_{CZ}(\gamma) + n - 3\right)$$
$$+ 2c_1(\xi, Z)$$
$$+ \sum \mu(\partial u)$$

boundary components