<u>h-principles</u> supplement

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"almost symplectic": non-deg 2-forms

<u>Thm</u>: ~~$\mathcal{J}_{symp}(V) \hookrightarrow \mathcal{J}$~~ $\mathcal{S}^{a}_{symp}(V) \hookrightarrow \mathcal{J}_{symp}(V)$

is a homotopy equivalence if V open.

$a \in H^2(V)$.

<u>h-principles for partial differential relations on open mfds</u>

V open, $X \to V$ natural fibre bundle. Then any open

Diff V-invariant differential relation $\mathcal{R} \subset X^{(r)}$

(E.g. Laplace's equation $\sum_i \partial_i^2 f = 0$, for $f: \mathbb{R}^n \to \mathbb{R}$

$\Rightarrow X = \mathbb{R}^{n+1}$, $V = \mathbb{R}^n$

$X^{(2)}$ has coords x_i, y, z_i, w_{ij}

$$\begin{array}{cccc} x_i & y & z_i & w_{ij} \\ \uparrow & \uparrow & \uparrow & \uparrow \\ V & f & \dfrac{\partial f}{\partial x_i} & \dfrac{\partial^2 f}{\partial x_i \partial x_j} \end{array}$$

although this is <u>not</u> Diff V-invariant and <u>not</u> open - just an example of a PDE.

A PDR might be $\sum_i \partial_i^2 f > 0$)

satisfies a parametric h-principle. Relative h-princ. also holds for $B \subset V$ closed s.t. every component of $V \setminus B$ has an exit to ∞.

What does "satisfies a parametric h-principle" mean?

Given a smooth section $s: V \to X$, we obtain a section $J^r_s : V \to X^{(r)}$. These sections of $X^{(r)}$ are called holonomic.

~~satisfying~~ A "formal solution" of our PDR is a section of $X^{(r)}$ satisfying the PDR.

"satisfies an h-principle" means any formal solution can be deformed to a holonomic solution, ~~i.e.~~ without changing holonomic ~~sec~~ solutions (i.e., holonomic solutions $\xhookrightarrow[h.e.]{}$ formal solutions).

A "relative h-principle" means you start with a formal solution which is holonomic on a nbhd of $B \subset V$, and can deform to a holonomic solution without changing the solution on some (possibly smaller) nbhd of B.

<u>Pf of thm:</u> We will see that every $\omega \in \mathcal{J}_{symp}$ can be deformed in $\mathcal{J}_{symp}$ to $\tilde{\omega} \in \mathcal{S}^a_{symp}$ $\forall a \in H^2(V)$.

Assume ~~WLOG~~ $a = 0$ (why?)

We have

$$\begin{tikzcd} \Lambda^1 V \arrow[r, "J^1"] \arrow[rr, bend right, "d"'] & (\Lambda^1 V)^{(1)} \arrow[r, "\tilde{D}"] & \Lambda^2 V \end{tikzcd}$$

Now note: any 2-form $\omega \in \mathcal{J}_{symp} \rightsquigarrow F_\omega \in \Gamma((\Lambda^1 V)^{(1)})$ ← global sec.
s.t. $\tilde{D} F_\omega = \omega$

a "formal primitive" of ω.

The non-degeneracy condition in $\Omega^2(V)$ gives an open PDR on $(\Lambda^1 V)^{(1)}$.

The h-principle gives $F_t \in \mathrm{Sec}(\Lambda^1 V)^{(1)}$ s.t. $F_1 = F_\omega$
$F_0 =: J^1_\alpha$ for some $\alpha \in \Omega^1 V$

Take $\omega_t = \tilde{D} \circ F_t$

$\Rightarrow \omega_0 = \tilde{D} \circ J^1_\alpha = d\alpha$

$\omega_1 = \omega$

This gives you your deformation.

Thm: $\mathcal{S}^+_{cont} \hookrightarrow \mathcal{J}^+_{cont}$ is h.e. for V open

$\mathcal{S}^+_{cont}$: cont. str. in V co-oriented

$\mathcal{J}^+_{cont}$: $\{(\xi_+, \omega)\}$ co-oriented hyperplane distribution in TV positive conformal symp. str. in ξ
(need conf. symp. struc. given by $d\alpha$ to be cont.)

Rmk: For closed manifolds, almost everything is unknown.

Thm: V^n, (W^{2n}, ω). All forms of h-principle hold for Lagrangian immersions $f: V^n \longrightarrow W^{2n}$ if we include the condition $[f^*\omega] = 0 \in H^2(V)$.

What is a "formal Lagrangian immersion"?

In particular, every isotropic monomorphism

$TV \longrightarrow TW$ (i.e. map $V \xrightarrow{i} W$ with maps $T_x V \to T_{i(x)} W$ $\forall x \in V$, which are isotropic)

can be deformed to a Lagrangian immersion $V \to W$.

Cor: If $TV \longrightarrow TW$ is isotropic, $\dim V < \frac{1}{2} \dim W$, you can deform to an isotropic <u>embedding</u>.