Intro to Contact Homology   Russell

Motivation

1) Does the Weinstein conjecture hold in general?
   (2007 - Taubes for dim $Y = 3$)
2) Define alg. invariants to distinguish contact mfds

Approach: Let (Y, ξ) be our contact mfd.

1) Define an action functional on $Loop(Y)$ s.t. critical locus is exactly the Reeb orbits.

$$\mathcal{A}(\gamma) = \int_\gamma \lambda$$

$\hat{J}$ = a.c. structure on ξ, $\lambda^{\otimes 2} + d\lambda(J\cdot, \cdot)$ is metric

$$\nabla\gamma = J \cdot \pi_\xi(\dot{\gamma})$$

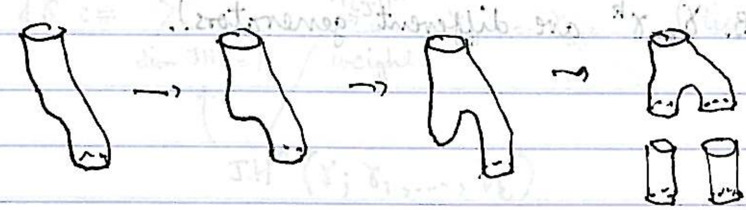
2) Define flowlines
   J = a.c. structure on $Y \times \mathbb{R}$ by extending $\hat{J}$ as before

3) Count cylinders between orbits in symplectisation

4) $d^2 = 0$ by counting rigid broken cylinders - this is ∂ of 1-dim space of cylinders $\Rightarrow$ signed count is 0.

5) Invariance.

Problem with (4): (Hofer studying Eliashberg's filling by J-hol disks)



cylindrical contact homology only works for very restricted class of (Y, ξ)s, s.t. this degeneration can't happen.

More generally we need to count curves with an arbitrary number of negative ends.

<u>Lem</u>: If $(a, f): \Sigma \to \mathbb{R} \times Y$ is J-hol then a has no local maxima.

~~Pf: $\Delta e^u = -d(da \circ j) = d(\lambda) \geq 0$.~~

<u>Pf</u>: $\Delta e^a = -* d(e^a \, da \circ j) = * d(e^a \lambda) \geq 0$.

<u>Defn of Contact Homology</u>:

$\mathbb{Z}_2$ graded algebra

Reeb orbits

Define 'bad' orbits: CZ_2 : Conley-Zehnder mod 2

1) $CZ_2(\gamma^{2k}) \neq CZ_2(\gamma^{2n+1})$ ← iterate γ $(2n+1)$ times

2) $CZ_2(\gamma^m)$ is constant function of m

<u>Defn</u>: Even multiple covers of type 1 are "bad". All others are good.

<u>Defn</u>: $|\gamma| = (CZ(\gamma) + n - 3)_2$. A is the unital algebra over $\mathbb{Q}[H_2(Y, \mathbb{Z})]$ freely generated by good Reeb orbits.

$$\gamma_1 \gamma_2 = (-1)^{|\gamma_1||\gamma_2|} \gamma_2 \gamma_1.$$

(NB. γ, γ^k are different generators).

Pick free generators $c_1 \ldots c_k$ of $H_1(Y;\mathbb{Z})$

For every γ pick a surface S_γ realizing a cobordism between γ and some collection of $n_i c_i$

For every holomorphic curve in $\mathbb{R}\times Y$, glue the fixed surfaces to the curve to obtain a cobordism between $\sum n_i c_i$ and $\sum m_i c_i$

By freeness, $n_i = m_i \Rightarrow$ we get a class in $H_2(Y;\mathbb{Z})$

<u>Defn:</u> $JH^A(\gamma;\gamma_1,\ldots,\gamma_k) = \{$space of holo curves w/ asymptotics γ @ $+\infty$, $\gamma_1,\ldots,\gamma_k$ @ $-\infty$, ~~total~~ homology class $A \in H_2(Y;\mathbb{Z})\}$

$$\mathbb{Q}[H_2(Y,\mathbb{Z})] = \left\{\sum_{i=1}^{n} c_i e^{A_i} : A_i \in H_2(Y,\mathbb{Z})\right\}$$

where $e^A \cdot e^B := e^{A+B}$.

Define differential:

$$d\gamma := \sum_{\dim JH^A = 1} \frac{n_{JH^A}}{\text{weight}} \gamma_1 \ldots \gamma_k \quad (\text{times } e^A\ ?)$$

$JH^A(\gamma;\gamma_1,\ldots,\gamma_k)$

weight = $k!\,\prod_j i_j!\; k_{\gamma_j}$

k_{γ_j} is multiplicity of γ_j

i_j = # times γ_j appears in product

n_{JH^A} := signed count of 0-dim components of $JH^A/\mathbb{R}$-action.

Prop: $d(\gamma_1 \cdot \gamma_2) = (d\gamma_1)\cdot\gamma_2 - (-1)^{|\gamma_1|}\gamma_1 \cdot (d\gamma_2)$.

Defn: $CH(Y,\xi) = \ker \partial / \operatorname{im} \partial$

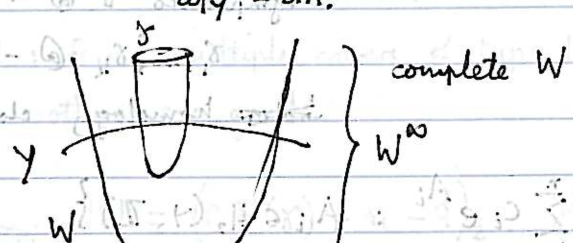
Prop: ① $CH(Y,\xi) = 0$ if ξ overtwisted.

② If Y has a filling, there's a "coefficient system" s.t. $CH \neq 0$.

Exact strong filling:

$(Y,\xi) = \partial(W,\omega)$

$\omega|_Y = d\lambda$.



$\Phi : CH(Y) \longrightarrow \mathbb{Q}$.

$\Phi\gamma \longmapsto$ count of holomorphic planes in W which go to γ at ∞.

$d\Phi = 0 \overset{?}{=} \Phi d.$

$\uparrow$

$d_Q = 0$

Because Φd counts boundary of 1-d moduli space of holomorphic planes asymptotic to γ.